Homework 6

Section 11.1

(4) $\sum_{k=0}^{3} (-1)^k 2^{k+1} = 2 + (-1)2^2 + (-1)^2 2^3 + (-1)^3 2^4$

$= 2 - 4 + 8 - 16$

$= -10$

(14) $\frac{3^3}{3!} + \frac{4^4}{4!} + \dots + \frac{10^{10}}{10!} = \sum_{k=3}^{10} \frac{k^k}{k!}$

or $\frac{3^3}{3!} + \frac{4^4}{4!} + \dots + \frac{10^{10}}{10!} = \sum_{i=0}^{7} \frac{(i+3)^{i+3}}{(i+3)!}$

(26) Check that $\frac{1}{k^2-k} = \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$

$$\Rightarrow \sum_{k=3}^{\infty} \frac{1}{k^2-k} = \sum_{k=3}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k}\right)$$

$$= \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{k=3} + \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{k=4} + \underbrace{\left(\frac{1}{4} - \frac{1}{5}\right)}_{k=5} + \underbrace{\left(\frac{1}{5} - \frac{1}{6}\right)}_{k=6} + \dots$$

$$= \frac{1}{2}$$

Since the series telescopes and all that's left after cancellation is the first term.

(30) $\sum_{k=0}^{\infty} \frac{(-1)^k}{5^k} = \sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k = \frac{1}{1-(-1/5)} = \frac{1}{6/5} = \frac{5}{6}$

Since the sum is just a geometric series with $r = -1/5$.

(34) $\sum_{k=2}^{\infty} \frac{3^{k-1}}{4^{3k+1}} = \frac{3^{-1}}{4} \sum_{k=2}^{\infty} \frac{3^k}{4^{3k}} = \frac{1}{3\cdot 4} \sum_{k=2}^{\infty} \frac{3^k}{(4^3)^k}$

$$= \frac{1}{12} \sum_{k=2}^{\infty} \frac{3^k}{64^k} \blacksquare$$

$$= \frac{1}{12}\left(\left(\frac{3}{64}\right)^2 + \left(\frac{3}{64}\right)^3 + \left(\frac{3}{64}\right)^4 + \left(\frac{3}{64}\right)^5 + \dots \right)$$

$$= \frac{1}{12}\left(\frac{3}{64}\right)^2 \underbrace{\left(1 + \frac{3}{64} + \left(\frac{3}{64}\right)^2 + \left(\frac{3}{64}\right)^3 + \dots \right)}_{\text{geometric series with } r = \frac{3}{64}}$$

$$= \frac{1}{12}\left(\frac{3}{64}\right)^2 \frac{1}{1 - 3/64}$$

$$= \frac{1}{12}\left(\frac{3}{64}\right)^2 \frac{64}{61}$$

Section 11.2

(4) $\ln k > 1$ for all $k > 3$ $\Rightarrow$ $\frac{\ln k}{k} > \frac{1}{k} > 0$

But $\sum \frac{1}{k}$ diverges (by the p-test)

$\Rightarrow \sum \frac{\ln k}{k}$ diverges.

(8) $\sum \left(\frac{5}{2}\right)^{-k} = \sum \left(\frac{2}{5}\right)^k$ converges since it's a geometric series with $r = \frac{2}{5}$, & $2/5 < 1$.

(12) $k(k+1)(k+2) > k \cdot k \cdot k = k^3$

$\Rightarrow \frac{1}{k(k+1)(k+2)} < \frac{1}{k^3}$

$\Rightarrow \sum \frac{1}{k(k+1)(k+2)} < \sum \frac{1}{k^3}$

but $\sum \frac{1}{k^3}$ converges by the p-test

$\Rightarrow \sum \frac{1}{k(k+1)(k+2)}$ converges.

(16) $\sum \frac{2}{k(\ln k)^2}$. Use the integral test. Let $f(x) = \cancel{\frac{2}{k(\ln k}} = \frac{2}{x(\ln x)^2}$ and note

$f(x)$ is positive and decreasing for $x \geq 2$

Hence $\sum_{k=2}^{\infty} \frac{2}{k(\ln k)^2}$ & $\int_2^{\infty} \frac{dx}{x(\ln x)^2}$ always either ~~converge~~ ~~or~~ converge or diverge together.

But

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^\infty \frac{du}{u^2}$$

$u = \ln x$
$\Rightarrow du = \frac{dx}{x}$

$< \infty$ by the p-test.

Hence

$\int_2^\infty \frac{dx}{x(\ln x)^2}$ converges ~~for~~ $\Rightarrow \sum_{k=2}^\infty \frac{1}{k(\ln k)^2}$ converges by the integral test.

(22) $\sum \frac{1}{2^{k+1}-1}$ The terms $\frac{1}{2^{k+1}-1}$ are like the geometric series $\frac{1}{2^{k+1}} = \left(\frac{1}{2}\right)^{k+1}$ which we know converges since it has $r = \frac{1}{2} < 1$.

So use the limit comparison test. Note

$$\lim_{k\to\infty} \frac{2^{k+1}}{2^{k+1}-1} = 1$$

Thus ~~both~~ $\sum \frac{1}{2^{k+1}}$ & $\sum \frac{1}{2^{k+1}-1}$ either converge or diverge together, and we know that $\sum \frac{1}{2^{k+1}}$ converges, so $\sum \frac{1}{2^{k+1}-1}$ does as well.