Score:

Name:

Business Calculus, Summer 2004
Midterm 1

Instructions: *This exam should be taken in 1 hour. Calculators and one hand-written page of notes are allowed. Use the back of the page if you need extra space.*

1. (20pts.) On January 1, 1996, Fred bought a bond with a maturity value of $250,000 that was to come due in 25 years. Eight-and-a-half years after buying it Fred loses everything else on one spin of the roulette wheel. He desperately needs some cash so he sells the bond to his neighbor Barney. On both transactions the price of the bond is its present value at the time of the transaction. If the market rate of money was 6% compounded monthly, how much did Fred make by holding the bond for those eight-and-a-half years?

   Solution: First note that Fred will be able to sell the bond for more than he bought it for, because Barney will only have to wait 25 - 8.5 = 16.5 years to collect the $250,000, while Fred had to wait 25 years when he bought it. Since this is discrete compounding we want to use the present-value formula

$$P = A\left(1+\frac{r}{N}\right)^{-Nt}$$

   with $A = 250,000, r = 0.06, N = 12$. When Fred buys the bond it's present value is

$$250,000\left(1+\frac{.06}{12}\right)^{-12*25} = 250,000(1.005)^{-300} = 55,991.42,$$

   and when he sells it since there's 16.5 years left the present value is

$$250,000\left(1+\frac{.06}{12}\right)^{-12*16.5} = 250,000(1.005)^{-198} = 93,123.61.$$

   Therefore by holding the bond for 8.5 years Fred makes $93,123.61 - $55,991.42 = $37,132.19 in profit.

2. (20pts.) Suppose that on Susie's 25th birthday she puts $1000 into a mutual fund that accumulates wealth at a rate of 10% compounded continuously. She does this on every birthday up to and including her 70th. If she doesn't make any other withdrawals or deposits, how much money does she have on her 75th birthday?

   Solution: Because this is continuous compounding we want to use the present value formula

$$A = Pe^{rt}$$

   with $P = 1000, r = .1$ and t varies with each payment. The value of each deposit on the 75th birthday is listed in the following table:

Birthday	Deposit	Years to Grow	Value on 75th Birthday
25	1000	50	$1000(e^{.1})^{50}$
26	1000	49	$1000(e^{.1})^{49}$
27	1000	48	$1000(e^{.1})^{48}$
⋮	⋮	⋮	⋮
69	1000	6	$1000(e^{.1})^{6}$
70	1000	5	$1000(e^{.1})^{5}$

Hence the total value on the 75th birthday is

$$\begin{aligned}
1000(e^{.1})^{50} + 1000(e^{.1})^{49} + 1000(e^{.1})^{48} + \ldots + 1000(e^{.1})^{6} + 1000(e^{.1})^{5} \\
&= 1000(e^{.1})^{5}\left[(e^{.1})^{45} + (e^{.1})^{44} + (e^{.1})^{43} + \ldots + (e^{.1})^{1} + 1\right] \\
&= 1000e^{.5}\frac{(e^{.1})^{46} - 1}{e^{.1} - 1} \\
&= 1,543,898.15
\end{aligned}$$

3. (10 pts.) Find the derivative of $x^{-3}(\sqrt{x} - 1)$.

Solution: Use the product rule with $f(x) = x^{-3}, g(x) = \sqrt{x} - 1$. Then $f'(x) = -3x^{-4}, g'(x) = \frac{1}{2}x^{-1/2}$, by the rule

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Hence

$$\begin{aligned}
\frac{d}{dx}(x^{-3}(\sqrt{x} - 1)) &= \frac{d}{dx}f(x)g(x) \\
&= f'(x)g(x) + f(x)g'(x) \\
&= -3x^{-4}(\sqrt{x} - 1) + x^{-3}\frac{1}{2}x^{-1/2} \\
&= -3x^{-7/2} + 3x^{-4} + \frac{1}{2}x^{-7/2} \\
&= -\frac{5}{2}x^{-7/2} + 3x^{-4}
\end{aligned}$$

The third equality was all that was required, although extra simplifications were greeted favorably.

4. (10 pts.) Let $f(x) = \frac{x^3+3x^2+5x-1}{x+1}$. Find the average rate of change of y with respect to x as x moves from $x = 1$ to $x = 3$.

Solution: We have

$$f(1) = \frac{1^3 + 3(1)^2 + 5(1) - 1}{1 + 1} = \frac{8}{2} = 4, f(3) = \frac{3^3 + 3(3)^2 + 5(3) - 1}{3 + 1} = \frac{68}{4} = 17.$$

Hence the average rate of change is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{17 - 4}{2} = \frac{13}{2}.$$

5. (10 pts.) Find the derivative of $\frac{3x-1}{2-x} + \frac{4}{x}$.

Solution: Do the derivative of each part separately. For the first part we use the quotient rule, so let $f(x) = 3x - 1$, $g(x) = 2 - x$. Then

$$\begin{aligned}
\frac{d}{dx}\left(\frac{3x - 1}{2 - x}\right) &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \\
&= \frac{3(2 - x) - (3x - 1)(-1)}{(2 - x)^2} \\
&= \frac{6 - 3x + 3x - 1}{(2 - x)^2} \\
&= \frac{5}{(2 - x)^2}
\end{aligned}$$

Also $\frac{d}{dx}\frac{4}{x} = 4\frac{d}{dx}(x^{-1}) = -4x^{-2} = \frac{-4}{x^2}$. So our final answer is

$$\frac{d}{dx}\left(\frac{3x - 1}{2 - x} + \frac{4}{x}\right) = \frac{5}{(2 - x)^2} - \frac{4}{x^2}.$$

6. (15 pts.) Sam and Helga both put \$75 into their bank accounts at the same time. Three years later they have the same amount of money and neither has made any deposits or withdrawals. If Sam's account bore interest at a rate of 15% annually, and Helga's account accrued interest quarterly, then what was the rate on Helga's account?

   Solution: After three years Sam has $75(1.15)^3 = 114.07$ in his account. If r is the rate on Helga's account, then after 3 years she has $75\left(1+\frac{r}{4}\right)^{4\times 3}$, and since they have the same amount of money we can conclude that

$$
\begin{aligned}
114.07 &= 75\left(1+\frac{r}{4}\right)^{12} \\
1.520933 &= \left(1+\frac{r}{4}\right)^{12} \\
(1.520933)^{1/12} &= \left(\left(1+\frac{r}{4}\right)^{12}\right)^{1/12} \\
1.03556 &= 1+\frac{r}{4} \\
.14244 &= r
\end{aligned}
$$

   So Helga's account yields 14.22% compounded quarterly.

7. (15 pts.)

   (a) Apple believes there's a linear relationship between the price p (in dollars) they can charge per iPod and the number of units x they can sell at that price. Market research shows the relationship is $p = p(x) = -0.0005x + 400$. Given this, if they set the price at \$200, how many iPods can they expect to sell?

   (b) It costs \$16,500,000 to make 100,000 iPods and \$21,500,000 to make 150,000 iPods. Find the straight-line relationship between the number of units produced x and the total cost $C = C(x)$ of producing those x units.

   (c) Find the revenue and profit functions for the iPods. What number of iPods should be produced to maximize the profit? (*Hint:* The profit function is an upside-down parabola, so the maximum profit occurs at the vertex. The vertex of a parabola can be found by taking the average of the two roots of the equation describing the parabola.)

   Solutions:

   (a) If they set $p = 200$ then we have

$$
\begin{aligned}
200 &= -0.0005x + 400 \\
-200 &= -0.0005x \\
x &= \frac{-200}{-0.0005} = 400,000
\end{aligned}
$$

   so they can sell 400,000 iPods at that price.

   (b) Since it's a straight-line equation we need the slope. Using the formula with C acting as our y we get

$$m = \frac{21,500,000 - 16,500,000}{150,000 - 100,000} = \frac{5,000,000}{50,000} = 100.$$

   Plugging this into $C - C_1 = m(x - x_1)$ we get $C - 16,500,000 = 100(x - 100,000)$ which rearranges to $C = C(x) = 6,500,000 + 100x$.

   (c) The revenue we can get by selling x units at $p(x)$ dollars each is $xp(x)$, which is

$$xp(x) = x(-0.0005x + 400) = -0.0005x^2 + 400x.$$

   But then profit is just revenue minus costs, so

$$P(x) = R(x) - C(x) = -0.0005x^2 + 400x - (6,500,000 + 100x) = -0.0005x^2 + 300x - 6,500,000.$$

Using quadratic formula, the roots of this equation are

$$\begin{aligned} x &= \frac{-300 \pm \sqrt{300^2 - 4(-0.0005)(-6,500,000)}}{2(-0.0005)} \\ &= \frac{-300 \pm \sqrt{90,000 - 13,000}}{-0.001} \\ &= \frac{-300 \pm 277.4887}{-0.001} \\ &= 22,511.26 \text{ and } 577,488.7 \end{aligned}$$

The average of the two roots is about 300,000, so producing 300,000 iPods will yield the maximum profit.