

Business Calculus, Summer 2004

Homework #5

Due: Tuesday, August 3rd, 2004 by end of class

1. Find the partial derivatives of

$$(a) f(x, y) = \sqrt{x^2 + e^{y^3}}$$

$$(b) g(x, y) = 17x^{2/3}y^{-1/6} + x^{5/4}$$

Solutions:

$$(a) f(x, y) = (x^2 + e^{y^3})^{1/2}, \text{ so}$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{2}(x^2 + e^{y^3})^{-1/2} \frac{\partial}{\partial x} (x^2 + e^{y^3}) \\ &= \frac{1}{2\sqrt{x^2 + e^{y^3}}}(2x) \\ &= \frac{x}{\sqrt{x^2 + e^{y^3}}} \end{aligned}$$

Also

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{1}{2}(x^2 + e^{y^3})^{-1/2} \frac{\partial}{\partial y} (x^2 + e^{y^3}) \\ &= \frac{3y^2 e^{y^3}}{2\sqrt{x^2 + e^{y^3}}} \end{aligned}$$

(b) We have

$$\begin{aligned} \frac{\partial g}{\partial x} &= 17y^{-1/6} \frac{\partial}{\partial x} (x^{2/3}) + \frac{\partial}{\partial x} (x^{5/4}) \\ &= 17 \left(\frac{2}{3} \right) y^{-1/6} x^{-1/3} + \frac{5}{4} x^{1/4} \\ &= \frac{34}{3} x^{-1/3} y^{-1/6} + \frac{5}{4} x^{1/4} \end{aligned}$$

and

$$\frac{\partial g}{\partial y} = \frac{-17}{6} x^{2/3} y^{-7/6}$$

2. A fencing manufacturer produces two grades of fence. His profit function is

$$P(x, y) = -700 + 0.5x + 0.3y - 0.0004xy$$

where x and y are the number of yards of heavy-duty fencing and standard fencing sold per week. Suppose that currently the production level is $x = 400$ and $y = 4000$. By calculating $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ at this production level, tell me how he should change his production (i.e. should he increase one and decrease the other, etc.).

Solution: The partial derivatives are

$$\begin{aligned}\frac{\partial P}{\partial x}(x, y) &= 0.5 - 0.0004y \\ \frac{\partial P}{\partial y}(x, y) &= 0.3 - 0.0004x,\end{aligned}$$

so therefore

$$\begin{aligned}\frac{\partial P}{\partial x}(400, 4000) &= 0.5 - 0.0004(4000) = -1.1 \\ \frac{\partial P}{\partial y}(400, 4000) &= 0.3 - 0.0004(400) = 0.14\end{aligned}$$

So since the derivative with respect to x is negative it would be best to decrease the amount of heavy-duty fence being produced. Similarly the amount of standard fencing should be increased since the derivative with respect to y is positive.

3. Find all second partial derivatives of

(a) $g(x, y) = (x^3 + 3xy^2 + y)e^{xy}$

(b) $h(x, y) = \frac{1+\ln x}{1+y}$

Solutions:

(a) The first derivatives are

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x} (x^3 + 3xy^2 + y) e^{xy} + (x^3 + 3xy^2 + y) \frac{\partial}{\partial x} (e^{xy}) \\ &= (3x^2 + 3y^2)e^{xy} + (x^3 + 3xy^2 + y)ye^{xy} \\ &= (3x^2 + 4y^2 + x^3y + 3xy^3)e^{xy}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{\partial}{\partial y} (x^3 + 3xy^2 + y) e^{xy} + (x^3 + 3xy^2 + y) \frac{\partial}{\partial y} (e^{xy}) \\ &= (6xy + 1)e^{xy} + (x^3 + 3xy^2 + y)xe^{xy} \\ &= (x^4 + 3x^2y^2 + 7xy + 1)e^{xy}.\end{aligned}$$

So the second derivatives are

$$\begin{aligned}\frac{\partial^2 g}{\partial x^2} &= (6x + 3x^2y + 3y^3)e^{xy} + (3x^2 + 4y^2 + x^3y + 3xy^3)ye^{xy} \\ &= (6x + 6x^2y + 7y^3 + x^3y^2 + 3xy^4)e^{xy}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 g}{\partial y \partial x} &= (8y + x^3 + 9xy^2)e^{xy} + (3x^2 + 4y^2 + x^3y + 3xy^3)xe^{xy} \\ &= (8y + 4x^3 + 13xy^2 + x^4y + 3x^2y^3)e^{xy}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 g}{\partial y^2} &= (6x^2y + 7x)e^{xy} + (x^4 + 3x^2y^2 + 7xy + 1)xe^{xy} \\ &= (13x^2y + 8x + x^5 + 3x^3y^2)e^{xy}\end{aligned}$$

(b) The first derivatives are

$$\begin{aligned}\frac{\partial h}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1 + \ln x}{1 + y} \right) \\ &= \frac{1}{1 + y} \frac{\partial}{\partial x} (1 + \ln x) \\ &= \frac{1}{1 + y} \frac{1}{x} \\ &= \frac{1}{x(1 + y)}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial h}{\partial y} &= \frac{\partial}{\partial y} ((1 + \ln x)(1 + y)^{-1}) \\ &= (1 + \ln x) \frac{\partial}{\partial y} ((1 + y)^{-1}) \\ &= (1 + \ln x) - (1 + y)^{-2} \frac{\partial}{\partial y} (1 + y) \\ &= -\frac{1 + \ln x}{(1 + y)^2}.\end{aligned}$$

So the second derivatives are

$$\begin{aligned}\frac{\partial^2 h}{\partial x^2} &= \frac{1}{1 + y} \frac{\partial}{\partial x} \left(\frac{1}{x} \right) \\ &= \frac{1}{1 + y} \left(\frac{-1}{x^2} \right) \\ &= \frac{-1}{x^2(1 + y)}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 h}{\partial y \partial x} &= \frac{1}{x} \frac{\partial}{\partial y} \left(\frac{1}{1+y} \right) \\ &= \frac{1}{x} \left(\frac{-1}{(1+y)^2} \right) \\ &= \frac{-1}{x(1+y)^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 h}{\partial y^2} &= \frac{\partial}{\partial y} \left(-(1 + \ln x)(1+y)^{-2} \right) \\ &= -(1 + \ln x) \frac{\partial}{\partial y} (1+y)^{-2} \\ &= -(1 + \ln x)(-2)(1+y)^{-3} \\ &= \frac{2(1 + \ln x)}{(1+y)^3}\end{aligned}$$

4. Find all critical points and test for local maxima and minima using the second derivative test (page 364):

(a) $f(x, y) = x^3 - xy - y^3$

(b) $g(x, y) = (x^2 - 2xy)e^{x+2y}$

Solutions:

(a) The partial derivatives are

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3x^2 - y \\ \frac{\partial f}{\partial y} &= -x - 3y^2\end{aligned}$$

so to find the critical points we set both equations to zero. The first equation equalling zero tells us that $y = 3x^2$, so subbing this into the second equation we have

$$\begin{aligned}-x - 3(3x^2)^2 &= -x - 27x^4 = 0 \\ -x(1 + 27x^3) &= 0 \\ x = 0 \text{ or } 1 + 27x^3 &= 0 \\ x = 0 \text{ or } x^3 &= \frac{-1}{27} \\ x = 0 \text{ or } x &= \frac{-1}{3}\end{aligned}$$

Therefore $y = 3(0)^2 = 0$ and $y = 3(-1/3)^2 = 1/3$, so the critical points are $(0, 0)$ and $(-1/3, 1/3)$. To tell if they're maxes or mins we need to know the second derivatives. We have

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= 6x \\ \frac{\partial^2 f}{\partial x \partial y} &= -1 \\ \frac{\partial^2 f}{\partial y^2} &= -6y\end{aligned}$$

so the discriminant is

$$\begin{aligned}D(x, y) &= \left(\frac{\partial^2 f}{\partial x^2}\right) \left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\ &= (6x)(-6y) - (-1)^2 = -36xy - 1\end{aligned}$$

Then $D(0, 0) = -1 < 0$ so there's a saddle point at $(0, 0)$. Moreover $D(-1/3, 1/3) = -36(-1/3)(1/3) - 1 = 8 > 0$, and $\frac{\partial^2 f}{\partial x^2}(-1/3, 1/3) = 6(-1/3) = -2 < 0$, so there's a max at $(-1/3, 1/3)$.

(b) The first derivatives are

$$\begin{aligned}\frac{\partial g}{\partial x} &= (2x - 2y)e^{x+2y} + (x^2 - 2xy)e^{x+2y}(1) \\ &= (x^2 - 2xy + 2x - 2y)e^{x+2y}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial g}{\partial y} &= (-2x)e^{x+2y} + (x^2 - 2xy)e^{x+2y}(2) \\ &= (2x^2 - 4xy - 2x)e^{x+2y}.\end{aligned}$$

To find the critical points we need to solve

$$\begin{aligned}(x^2 - 2xy + 2x - 2y)e^{x+2y} &= 0 \\ (2x^2 - 4xy - 2x)e^{x+2y} &= 0\end{aligned}$$

simultaneously, and since the e^{x+2y} is never zero we can get rid of it. We therefore need to solve

$$\begin{aligned}x^2 - 2xy + 2x - 2y &= 0 \\ 2x^2 - 4xy - 2x &= 0\end{aligned}$$

which tell us that $x^2 = 2xy - 2x + 2y$, and $x^2 = 2xy + x$. Therefore $2xy - 2x + 2y = 2xy + x$ which gives $-3x + 2y = 0$. This, however, is a whole line of solutions which means there's infinitely many critical points. In this class we don't have the tools to deal with this, so there's nothing more we can do.

5. A rectangular box without a top is to contain 108 cubic feet. The material for the base costs \$8 per square foot but the material for the sides only costs \$1 per square foot. Find the height, width, and length that yield the cheapest box.

Solution: Denote the height, width and length as h, w and l . We're told we must have $lwh = 108$. The box has 5 faces, one on the bottom and four on the sides. The area of the bottom is lw , and it costs \$8 per square foot, so the cost of the bottom is $8lw$. The area of two of the sides is lh , and the area of the other two sides is wh , and all four sides cost \$1 per square foot, so the cost of the sides is $2lh + 2wh$. The total cost is therefore $8lw + 2lh + 2wh$. Subbing in $h = \frac{108}{lw}$ we have

$$C(l, w) = 8lw + 2l\frac{108}{lw} + 2w\frac{108}{wh} = 8lw + \frac{216}{w} + \frac{216}{l}$$

Then to minimize the cost we must solve

$$\begin{aligned}\frac{\partial C}{\partial l} &= 8w - \frac{216}{l^2} = 0 \\ \frac{\partial C}{\partial w} &= 8l - \frac{216}{w^2} = 0\end{aligned}$$

The first equation tells us that $w = \frac{216}{8l^2} = \frac{27}{l^2}$, and subbing this into the second we get

$$\begin{aligned}8l - \frac{216}{\left(\frac{27}{l^2}\right)^2} &= 0 \\ 8l - \frac{216l^4}{27^2} &= 0 \\ 8l \left(1 - \frac{27l^3}{27^2}\right) &= 0 \\ 1 - \frac{l^3}{27} &= 0 \\ l^3 &= 27 \\ l &= 3\end{aligned}$$

From this we get that $w = \frac{27}{l^2} = \frac{27}{3^2} = 3$, and then $h = \frac{108}{lw} = \frac{108}{9} = 12$. So the dimensions of 3 inches long and wide and 9 inches tall minimize the cost. We can be assured this a minimum because no other critical points come up, and clearly there is a minimum cost for this box but no maximum (since, for example, we can make the box as expensive as we want by making the bottom really big).

6. A furniture store has found that its earnings can be predicted by the formula

$$E(x, y) = -55,700 + 250y + 200x + xy - 0.5y^2 - 2x^2$$

where x is it's investment in inventory and y is how much it spends on advertising. Find the combination of inventory and advertising spending that maximizes the earnings.

Solution: We need to solve

$$\begin{aligned}\frac{\partial E}{\partial x} &= 200 + y - 4x = 0 \\ \frac{\partial E}{\partial y} &= 250 + x - y = 0\end{aligned}$$

The first equation says $y = 4x - 200$, and subbing this into the second we have

$$\begin{aligned}250 + x - (4x - 200) &= 0 \\ 450 - 3x &= 0 \\ x &= 150\end{aligned}$$

so then $y = 4(150) - 200 = 200$. So it would seem that $x = 150, y = 200$ maximizes the earnings but we should verify this. We have

$$\begin{aligned}\frac{\partial^2 E}{\partial x^2} &= -4 \\ \frac{\partial^2 E}{\partial x \partial y} &= 1 \\ \frac{\partial^2 E}{\partial y^2} &= -1\end{aligned}$$

so the discriminant is

$$D(x, y) = \frac{\partial^2 E}{\partial x^2} \frac{\partial^2 E}{\partial y^2} - \left(\frac{\partial^2 E}{\partial x \partial y} \right)^2 = (-4)(-1) - 1^2 = 3 > 0,$$

and since $\frac{\partial^2 E}{\partial x^2} < 0$ we see that these combinations of production levels do produce a max.