

Business Calculus, Summer 2004

Homework #4

Due: Tuesday, July 27th, 2004 by end of class

1. Find all the critical points of

(a) $f(x) = \frac{x}{x^2+1}$

(b) $g(x) = \frac{16}{x} - x^2$

(c) $h(x) = 2x\sqrt{3x^2+1}$

Solutions:

(a) Let $a(x) = x$ and $b(x) = x^2 + 1$ so that

$$f'(x) = \frac{a'(x)b(x) - a(x)b'(x)}{b(x)^2} = \frac{(1)(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

For critical points we need $f'(x) = 0$. But a fraction is zero only if the numerator is zero, which means $1 - x^2 = 0$, or $x^2 = 1$. Hence $x = \pm 1$ are the two critical points.

(b) Differentiating and setting g' equal to zero we have

$$\begin{aligned}g'(x) &= \frac{-16}{x^2} - 2x = 0 \\ \frac{-16}{x^2} &= 2x \\ -8 &= x^3 \\ -2 &= x\end{aligned}$$

so the only critical point is $x = -2$.

(c) Let $a(x) = 2x$, $b(x) = \sqrt{3x^2+1}$. Then $a'(x) = 2$, and

$$\begin{aligned}b'(x) &= \frac{d}{dx} ((3x^2+1)^{1/2}) = \frac{1}{2}(3x^2+1)^{-1/2} \frac{d}{dx} (3x^2+1) \\ &= \frac{6x}{2}(3x^2+1)^{-1/2} \\ &= \frac{3x}{\sqrt{3x^2+1}}.\end{aligned}$$

Now we can use the product rule to find $f'(x)$, and then set $f'(x) = 0$:

$$\begin{aligned}f'(x) &= a'(x)b(x) + a(x)b'(x) = 2\sqrt{3x^2 + 1} + 2x \frac{3x}{\sqrt{3x^2 + 1}} = 0 \\-2\sqrt{3x^2 + 1} &= \frac{6x^2}{\sqrt{3x^2 + 1}} \\(\sqrt{3x^2 + 1})^2 &= \frac{6x^2}{-2} \\3x^2 + 1 &= -3x^2 \\6x^2 + 1 &= 0\end{aligned}$$

Notice that the $6x^2 + 1$ on the right hand side of the equation is always positive and certainly never zero, so this function has no critical points.

2. Find the second derivatives $f''(x)$ of

(a) $f(x) = x^2 e^{-x}$

(b) $f(x) = \sqrt{1 - x^2}$

(c) $f(x) = x \ln x$

Solutions:

(a) The first derivative is

$$\begin{aligned}f'(x) &= \frac{d}{dx}(x^2)e^{-x} + x^2 \frac{d}{dx}(e^{-x}) \\&= 2xe^{-x} - x^2 e^{-x} \\&= (2x - x^2)e^{-x}.\end{aligned}$$

So the second derivative is

$$\begin{aligned}f''(x) &= \frac{d}{dx}((2x - x^2)e^{-x}) \\&= \frac{d}{dx}(2x - x^2)e^{-x} + (2x - x^2) \frac{d}{dx}(e^{-x}) \\&= (2 - 2x)e^{-x} - (2x - x^2)e^{-x} \\&= (x^2 - 4x + 2)e^{-x}.\end{aligned}$$

(b) The first derivative is

$$\begin{aligned}f'(x) &= \frac{d}{dx} ((1-x^2)^{1/2}) \\&= \frac{1}{2}(1-x^2)^{-1/2} \frac{d}{dx}(1-x^2) \\&= \frac{-2x}{2\sqrt{1-x^2}} \\&= \frac{-x}{\sqrt{1-x^2}}\end{aligned}$$

so the second derivative is

$$\begin{aligned}f''(x) &= \frac{d}{dx} \left(\frac{-x}{\sqrt{1-x^2}} \right) \\&= \frac{\frac{d}{dx}(-x)\sqrt{1-x^2} - (-x)\frac{d}{dx}(\sqrt{1-x^2})}{(\sqrt{1-x^2})^2} \\&= \frac{-\sqrt{1-x^2} + x\frac{-x}{\sqrt{1-x^2}}}{1-x^2} \\&= \frac{\frac{-(1-x^2)-x^2}{\sqrt{1-x^2}}}{1-x^2} \\&= \frac{-1}{(1-x^2)^{3/2}}\end{aligned}$$

(c) The first derivative is

$$\begin{aligned}f'(x) &= \frac{d}{dx}(x) \ln x + x \frac{d}{dx}(\ln x) \\&= \ln x + x \left(\frac{1}{x} \right) \\&= \ln x + 1\end{aligned}$$

so the second derivative is

$$f''(x) = \frac{d}{dx}(\ln x + 1) = \frac{1}{x}.$$

3. Use the first derivative test to find all the critical points of the following functions. Then use the first derivative sign test or the second derivative test to classify them as maxes, mins or “nothing” points:

(a) $f(x) = \frac{4-x}{x^2+1}$

(b) $g(x) = 3x^2 - x^3$

(c) $h(x) = (x^2 + 1)e^{-x}$

Solutions:

(a) Using the quotient rule we have

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(4-x)(x^2+1) - (4-x)\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\ &= \frac{-(x^2+1) - (4-x)(2x)}{(x^2+1)^2} \\ &= \frac{x^2 - 8x - 1}{(x^2+1)^2} \end{aligned}$$

To solve $f'(x) = 0$ note that a fraction is only zero if the numerator is zero, so we only have to solve $x^2 - 8x - 1 = 0$. By quadratic formula we have

$$x = \frac{8 \pm \sqrt{64+4}}{2} = \frac{8 \pm 2\sqrt{17}}{2} = 4 \pm \sqrt{17} \approx -0.123, 8.123$$

as our critical points. Now we'll use the first derivative test to classify them. Note $f'(-1) = \frac{(-1)^2 - 8(-1) - 1}{2} = 4 > 0$ and $f'(0) = \frac{-1}{1} = -1 < 0$, so f is increasing to the left of -0.123 and decreasing to the right of it, so f has a max at -0.123. Similarly $f'(8) = \frac{8^2 - 8(8) - 1}{(8^2+1)^2} = \frac{-1}{65^2} < 0$ and $f'(9) = \frac{9^2 - 8(9) - 1}{(9^2+1)^2} = \frac{8}{82^2} > 0$, so f is decreasing to the left of 8.123 and increasing to the right of it. Hence f has a min at 8.123.

(b) We have $g'(x) = 6x - 3x^2$, so $g'(x) = 0$ means

$$6x - 3x^2 = 3x(2 - x) = 0$$

and the critical points are $x = 0$ and $x = 2$. The second derivative is $g''(x) = 6 - 6x$. Then $g''(0) = 6 > 0$ so there's a min at $x = 0$, and $g''(2) = 6 - 12 = -6 < 0$ so there's a max at $x = 2$.

(c) We have $h'(x) = 2xe^{-x} - (x^2 + 1)e^{-x} = -(x^2 - 2x + 1)e^{-x}$, so to solve $h'(x) = 0$ we do

$$\begin{aligned} -(x^2 - 2x + 1)e^{-x} &= 0 \\ x^2 - 2x + 1 &= 0 \\ (x - 1)^2 &= 0 \\ x &= 1 \end{aligned}$$

We can drop the e^{-x} term since it's never zero. So $x = 1$ is our only critical point. The second derivative is

$$h''(x) = -(2x - 2)e^{-x} + (x^2 - 2x + 1)e^{-x} = (x^2 - 4x + 3)e^{-x},$$

which gives $h''(1) = (1^2 - 4 + 3)e^{-1} = 0$. Hence $x = 1$ is neither a max nor a min, it's a nothing point.

4. A private jetliner travels at a constant speed of v miles per hour. Fuel charges are $0.01v^2$ dollars per hour. Salary for the crew and the cost of renting the jet amount to \$1600 per hour. Find the speed v that minimizes the total cost (fuel plus salaries and rental) of a 3000-mile trip.

Solution: If the plane flies at v miles per hour then it takes $3000/v$ hours to complete the trip. Hence the total cost for fuel for the trip is

$$0.01v^2 \times \frac{3000}{v} = 30v,$$

and the cost for salary and plane rental is

$$1600 \times \frac{3000}{v} = \frac{4,800,000}{v}.$$

So the total cost C as a function of the speed v is

$$C(v) = 30v + \frac{4,800,000}{v}.$$

Differentiating and setting this equal to zero we get

$$\begin{aligned} C'(v) &= 30 - \frac{4,800,000}{v^2} = 0 \\ 30 &= \frac{4,800,000}{v^2} \\ v^2 &= \frac{4,800,000}{30} = 160,000 \\ v &= \pm 400 \end{aligned}$$

Since we're only interested in $v \geq 0$ the optimal speed is 400 miles/hour. It's clear that this problem has a speed which minimizes the cost and there is no speed which maximizes the cost (since, for example, if the speed is zero the cost will be infinite because of salary), and since there's only one critical point it must be the one which minimizes costs. Hence there's no need for the second derivative test.

5. A store offering a certain brand of lawnmower for p dollars can sell $500 - 4p$ of them. Each lawnmower costs the store \$125. What price yields maximum profit?

Solution: The revenue as a function of the price is $R(p) = (500 - 4p)p = 500p - 4p^2$ (quantity times price per lawnmower). The cost as a function of the price is $C(p) = (500 - 4p) \times 125 = 62500 - 500p$ (quantity times cost per lawnmower). Hence the profit as a function of the price is

$$P(p) = R(p) - C(p) = 500p - 4p^2 - (62500 - 500p) = -2p^2 + 1000p - 62500.$$

To maximize this we take the derivative and solve for it equal to zero.

$$\begin{aligned}P'(p) &= -4p + 750 = 0 \\4p &= 750 \\p &= \frac{750}{4} = 175\end{aligned}$$

So selling each lawnmower for \$175 will maximize the profit.

6. A rancher wants to build four adjacent cattle stalls in one large rectangle that encloses a total area of 50,000 square feet. How should the length x and the width y of the large rectangle be chosen to minimize the total amount of fencing required?

Solution: We're given that $xy = 50,000$. The quantity we want to minimize is the perimeter of the stalls which, since there's 5 x lengths and 2 y lengths, is $5x + 2y$. But the first equation tells us that $y = \frac{50,000}{x}$, so we substitute this into the perimeter formula to get

$$5x + 2\frac{50,000}{x} = 5x + \frac{100,000}{x}.$$

Differentiating and setting equal to zero we have

$$\begin{aligned}\frac{d}{dx} \left(5x + \frac{100,000}{x} \right) &= 5 - \frac{100,000}{x^2} = 0 \\x^2 &= \frac{100,000}{5} = 20,000 \\x &= \pm 141.42\end{aligned}$$

Since $x \geq 0$ we only use $x = 141.42$. Then $y = \frac{50,000}{141.42} = 353.56$, so the large rectangle should be 141.42 feet by 353.56 feet to minimize the necessary fencing.