Business Calculus, Summer 2004

Homework #3

Due: Tuesday, July 20th, 2004 by end of class

1. Use any rule(s) you like to find the derivatives of

   (a) $\frac{x^2-4}{x^4+5x+1}$

   (b) $(3x^3 - 7x^2 + 2)e^x$

   (c) $\frac{1}{17-28x}$

   (d) $x^2 \ln x$

   Solutions:

   (a) Use the quotient rule for this one with $f(x) = x^2 - 4, g(x) = x^4 + 5x + 1$. Then $f'(x) = 2x, g'(x) = 4x^3 + 5$, and

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{x^2-4}{x^4+5x+1}\right) &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\
&= \frac{2x(x^4+5x+1) - (x^2-4)(4x^3+5)}{(x^4+5x+1)^2} \\
&= \frac{2x^5 + 10x^2 + 2x - (4x^5 - 16x^3 + 5x^2 - 20)}{(x^4+5x+1)^2} \\
&= \frac{-2x^5 + 16x^3 + 5x^2 + 2x + 20}{(x^4+5x+1)^2}
\end{aligned}
$$

   (b) Use the product rule with $f(x) = 3x^3 - 7x^2 + 2, g(x) = e^x$. Then $f'(x) = 9x^2 - 14x$ and $g'(x) = e^x$. So

$$
\begin{aligned}
\frac{d}{dx}((3x^3 - 7x^2 + 2)e^x) &= f'(x)g(x) + f(x)g'(x) \\
&= (9x^2 - 14x)e^x + (3x^3 - 7x^2 + 2)e^x \\
&= (3x^3 + 9x^2 - 7x^2 - 14x + 2)e^x \\
&= (3x^3 + 2x^2 - 14x + 2)e^x
\end{aligned}
$$

(c) Use the quotient rule for this with $f(x) = 1, g(x) = 17 - 28x$. Then $f'(x) = 0$, $g'(x) = -28$, so

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{17-28x}\right) &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{0(17-28x) - 1(-28)}{(17-28x)^2} \\ &= \frac{28}{(17-28x)^2} \end{aligned}$$

(d) Use the product rule with $f(x) = x^2, g(x) = \ln x$. Then $f'(x) = 2x$, $g'(x) = \frac{1}{x}$. Therefore

$$\begin{aligned} \frac{d}{dx}(x^2 \ln x) &= f'(x)g(x) + f(x)g'(x) \\ &= 2x \ln x + x^2 \left(\frac{1}{x}\right) \\ &= 2x \ln x + x \\ &= x(2 \ln x + 1) \end{aligned}$$

2. Find the equation of the tangent line through the point $(2, -21)$ on the graph of $y = -3x^4 + 4x^2 + 6x - 1$.

<u>**Solution**</u>: Since the point $(2, -21)$ is a point on the graph we know $f(2) = -21$. Also we get that

$$f'(x) = -12x^3 + 8x + 6$$

so therefore $f'(2) = -12(2)^3 + 8(2) + 6 = -74$. Subbing into the tangent line equation, which is really just point-slope form, we get

$$\begin{aligned} y - f(x_0) &= f'(x_0)(x - x_0) \\ y - (-21) &= -74(x - 2) \\ y &= -74x + 127 \end{aligned}$$

3. Find all points on the graph of $y = x^3 - x^2 - 5x$ where the tangent line has slope 2.

<u>**Solution**</u>: The slope of the tangent line is given by the derivative. Hence we want to find all points x such that $f'(x) = 2$. The derivative is

$$f'(x) = 3x^2 - 2x - 5$$

so we need to solve

$$\begin{aligned} 3x^2 - 2x - 5 &= 2 \\ 3x^2 - 2x - 7 &= 0 \\ x &= \frac{2 \pm \sqrt{2^2 - 4(3)(-7)}}{2(3)} \\ &= \frac{2 \pm \sqrt{88}}{6} \\ &= 1.8968, -1.2301 \end{aligned}$$

4. A penny is dropped off the 86th floor observation deck of the Empire State Building, which is 1050 feet above street level. Ignoring the effects of air friction, the distance (s, in feet) the penny has fallen after t seconds is governed by the formula $s = 16t^2$. How long does it take for the penny to hit the ground, and when it does at what speed is it travelling?

   Solution: To find the time it takes for the penny to hit we solve $1050 = 16t^2$ which gives solutions $t = \pm 8.1$. Of course we use $t = 8.1$ since we're dealing with time. The function $s = s(t) = 16t^2$ tells us the distance travelled up to time t, so its derivative is the speed at time t. Hence $s'(t) = 32t$, and $s'(8.1) = 32 \times 8.1 = 259.2$, so the penny is travelling at 259.2 feet/second when it hits the ground.

5. Use the approximation principle to estimate the value of $\sqrt{53}$. No calculators allowed on this question except to check your answer.

   Solution: We know the value of $\sqrt{49}$ exactly, so we can use this to help us with the approximation principle. Let $f(x) = \sqrt{x}$ and $x_0 = 49$. We use the second form of the approximation principle, namely

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

   Obviously we want to set $x = 49$. We have $f(49) = 7$, and

$$f'(x) = \frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}$$

   so $f'(49) = \frac{1}{2}49^{-1/2} = \frac{1}{2\sqrt{49}} = \frac{1}{14}$. Therefore we get

$$\sqrt{53} = f(53) \approx f(49) + f'(49)(53 - 49) = 7 + \frac{1}{14}(4) = 7 + \frac{2}{7} = 7.2857.$$

   The actual value is 7.2801, so we're not far off.

6. Find the derivatives of

   (a) $\sqrt[4]{x^3 - 3x^2 + 2x - 2}$

   (b) $\left(\frac{x^2+1}{x-1}\right)^{17}$

(c) $\frac{e^{-4x}}{(x^2-3)^2}$

(d) $\sqrt{x}e^{x^2-3x}$

<u>**Solutions**</u>:

(a) Let $f(x) = x^{1/4}$, $g(x) = x^3 - 3x^2 + 2x - 2$. Then $f(g(x)) = (x^3 - 3x^2 + 2x - 2)^{1/4}$. We see that $f'(x) = \frac{1}{4}x^{-3/4}$, $g'(x) = 3x^2 - 6x + 2$, so

$$\begin{aligned}\frac{d}{dx}\left(\sqrt[4]{x^3 - 3x^2 + 2x - 2}\right) &= \frac{d}{dx}\left(f(g(x))\right)\\ &= f'(g(x))g'(x)\\ &= \frac{1}{4}(x^3 - 3x^2 + 2x - 2)^{-3/4}(3x^2 - 6x + 2)\\ &= \frac{3x^2 - 6x + 2}{4(x^3 - 3x^2 + 2x - 2)^{3/4}}\end{aligned}$$

(b) Let $f(x) = x^{17}$, $g(x) = \frac{x^2+1}{x-1}$. Then $f(g(x))$ is exactly what we're trying to differentiate. We have $f'(x) = 17x^{16}$, and by the quotient rule

$$g'(x) = \frac{2x(x-1) - (x^2+1)(1)}{(x-1)^2} = \frac{x^2 - 2x - 1}{(x-1)^2}.$$

Therefore

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2+1}{x-1}\right)^{17} &= f'(g(x))g'(x)\\ &= 17\left(\frac{x^2+1}{x-1}\right)^{16}\left(\frac{x^2-2x-1}{(x-1)^2}\right)\end{aligned}$$

(c) The first thing we do is a quotient rule with $f(x) = e^{-4x}$ and $g(x) = (x^2-3)^2$. However we use the chain rule to find $f'(x)$ and $g'(x)$. We have

$$f'(x) = \frac{d}{dx}\left(e^{-4x}\right) = e^{-4x}\frac{d}{dx}(-4x) = -4e^{-4x},$$

and similarly

$$g'(x) = \frac{d}{dx}\left((x^2-3)^2\right) = 2(x^2-3)\frac{d}{dx}(x^2-3) = 4x(x^2-3).$$

Therefore

$$\begin{aligned}\frac{d}{dx}\frac{e^{-4x}}{(x^2-3)^2} = \frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\\ &= \frac{-4e^{-4x}(x^2-3)^2 - e^{-4x}4x(x^2-3)}{((x^2-3)^2)^2}\\ &= \frac{-4(x^2-3)e^{-4x}((x^2-3)+x)}{(x^2-3)^4}\\ &= \frac{-4e^{-4x}(x^2+x-3)}{(x^2-3)^3}\end{aligned}$$

(d) Here we do a product rule with $f(x) = \sqrt{x}$, $g(x) = e^{x^2-3x}$. We know $f'(x) = \frac{1}{2}x^{-1/2}$, but we need the chain rule to evaluate $g'(x)$. We have

$$g'(x) = e^{x^2-3x}\frac{d}{dx}(x^2 - 3x) = (2x-3)e^{x^2-3x}.$$

Therefore

$$\begin{aligned}
\frac{d}{dx}\left(\sqrt{x}e^{x^2-3x}\right) &= f'(x)g(x) + f(x)g'(x) \\
&= \frac{1}{2}x^{-1/2}e^{x^2-3x} + \sqrt{x}(2x-3)e^{x^2-3x} \\
&= e^{x^2-3x}\left(\frac{1}{2\sqrt{x}} + (2x-3)\sqrt{x}\right) \\
&= \frac{e^{x^2-3x}}{2\sqrt{x}}\left(1 + (2x-3)2x\right) \\
&= \frac{e^{x^2-3x}}{2\sqrt{x}}(4x^2 - 6x + 1)
\end{aligned}$$

The last simplifying steps weren't necessary, but were looked upon favorably.