

Business Calculus, Summer 2004

Homework #2

Due: Tuesday, July 13th, 2004 by end of class

1. Find the equation for the line

- (a) through the points $(-3, 2)$ and $(1, 0)$
- (b) with slope $m = 4/3$ and passing through the point $(4, 1)$
- (c) with slope $m = -2$ and y-intercept π
- (d) with slope $m = -3/2$ and x-intercept 4

Solutions:

(a) Let $(x_1, y_1) = (-3, 2)$, $(x_2, y_2) = (1, 0)$. Then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 2}{1 - (-3)} = \frac{-2}{4} = \frac{-1}{2}.$$

Plugging into the equation $y - y_1 = m(x - x_1)$ we get

$$y - 2 = \frac{-1}{2}(x - (-3))$$

$$y - 2 = \frac{-1}{2}(x + 3)$$

$$y - 2 = \frac{-1}{2}x - \frac{3}{2}$$

$$y = \frac{-1}{2}x + \frac{1}{2}$$

(b) Plugging into $y - y_1 = m(x - x_1)$ we have $y - 1 = \frac{4}{3}(x - 4)$ which rearranges to $y = \frac{4}{3}x - \frac{13}{3}$.

(c) Plugging into $y = mx + b$ we have $y = -2x + \pi$.

(d) Since the x-intercept is 4, the line goes through $(4, 0)$. Plugging into $y - y_1 = m(x - x_1)$ and rearranging we get $y = \frac{-3}{2}x + 6$.

2. (a) A hot dog vendor knows that he can sell 750 hot dogs a day for \$1.50 each, but only 500 per day at \$2.00 each. Let x be the number of hot dogs per day he can sell at p dollars per hot dog. Given that there's a straight-line relationship between p and x find the demand function $p = p(x)$ for the hot dogs.

- (b) Each hot dog costs the vendor \$0.80. On top of that he faces fixed costs of \$50 per day. Find a straight-line formula for the cost C of selling x hot dogs per day.
- (c) Find the revenue and profit functions for the vendor. How many hot dogs does he have to sell each day in order to break even?

Solutions:

- (a) Since p is the variable chosen we should use p instead of y in our straight line formula. We're given that $(x_1, p_1) = (750, 1.50)$ and $(x_2, p_2) = (500, 2.00)$ are points on the straight line, so

$$m = \frac{p_2 - p_1}{x_2 - x_1} = \frac{2.00 - 1.50}{500 - 750} = \frac{0.50}{-250} = -0.002$$

Subbing into $p - p_1 = m(x - x_1)$ we have $p - 1.5 = -0.002(x - 750)$ which rearranges to $p = p(x) = -0.002x + 3.0$.

- (b) Fixed costs are \$50 per day and variable costs are \$0.80 per hot dog. Hence $C(x) = FC + (VC)x = 50 + 0.8x$.
- (c) We can sell x hot dogs at $p(x)$ dollars each, so the revenue function is

$$R(x) = xp(x) = x(-0.002x + 3.0) = -0.002x^2 + 3.0x,$$

and from this we get the profit function $P(x) = R(x) - C(x)$ is

$$P(x) = -0.002x^2 + 3.0x - (50 + 0.8x) = -0.002x^2 + 2.2x - 50.$$

The vendor breaks even when $P(x) = 0$, which, by quadratic formula, happens at

$$x = \frac{-2.2 \pm \sqrt{2.2^2 - 4 * (-0.002) * (-50)}}{2 * (-0.002)} = \frac{-2.2 \pm 2.1071}{-0.004} = 23.225, 1076.775$$

So once the vendor sells 24 hot dogs he's broken even for the day. Anything more than this and he makes up a profit, up until 1077 hot dogs. To sell 1077 hot dogs he has to lower the price per dog to $p(1077) = -0.002 * 1077 + 3 = 0.84$, at which price he's no longer making a profit.

3. On a road trip the distance covered after t hours is given by the function $d = d(t) = \frac{1}{3}t^2 + 2t$. Find the average velocity between times
- (a) $t = 3$ and $t = 4$, (b) $t = 3$ and $t = 4$
- (c) $t = 3$ and $t = 3.5$, (d) $t = 3$ and $t = 3.1$

Can you guess the instantaneous velocity at time $t = 3$?

Solutions: Recall the average rate of change between points t_1 and t_2 is

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

(a) We have $f(3) = 9, f(5) = 55/3$, so the average rate of change is

$$\frac{55/3 - 9}{5 - 3} = \frac{28/3}{2} = \frac{28}{6} = \frac{14}{3} = 4.6666.$$

(b) We have $f(3) = 9, f(4) = 40/3$, so the average rate of change is

$$\frac{40/3 - 9}{4 - 3} = \frac{13/3}{1} = \frac{13}{3} = 4.3333.$$

(c) We have $f(3) = 9, f(3.5) = 11.0833$, so the average rate of change is

$$\frac{11.0833 - 9}{3.5 - 3} = \frac{2.0833}{.5} = 4.1666.$$

(d) We have $f(3) = 9, f(3.1) = 9.4033$, so the average rate of change is

$$\frac{9.4033 - 9}{3.1 - 3} = \frac{.4033}{.1} = 4.033.$$

Based on this it seems reasonable to guess that the instantaneous rate of change at $t = 3$ is 4. The derivative of d is $d'(t) = \frac{2}{3}t + 2$, and $d'(3) = \frac{2}{3}3 + 2 = 4$, which confirms our guess. Note in this problem the average rate of change is our average speed, and the instantaneous rate of change is the speed at exactly time 3.

4. Use the “Delta Process” from Section 2.2 to find the instantaneous rate of change of the function $f(t) = t^2 + 3t + 4$ at any time t .

Solution: Section 2.2, page 64 tells us how to do the Delta Process. The average rate of change going from t to $t + \Delta t$ is

$$\begin{aligned} \frac{f(t + \Delta t) - f(t)}{\Delta t} &= \frac{(t + \Delta t)^2 + 3(t + \Delta t) + 4 - (t^2 + 3t + 4)}{\Delta t} \\ &= \frac{t^2 + 2t\Delta t + (\Delta t)^2 + 3t + 3\Delta t + 4 - t^2 - 3t - 4}{\Delta t} \\ &= \frac{2t\Delta t + (\Delta t)^2 + 3\Delta t}{\Delta t} \\ &= \frac{\Delta t(2t + \Delta t + 3)}{\Delta t} \\ &= 2t + 3 + \Delta t. \end{aligned}$$

Now we can take the limit since there are no Δt in the denominator. We get

$$\lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} 2t + 3 + \Delta t = 2t + 3$$

which is the instantaneous rate of change at any time t .

5. Use the rule $\frac{d}{dx}(x^n) = nx^{n-1}$ to find the derivative of $\frac{75}{4}x^4 + 19x^3 - \frac{14}{3}x^2 + 2x - 7$.

Solution:

$$\begin{aligned}\frac{d}{dx} \left(\frac{75}{4}x^4 + 19x^3 - \frac{14}{3}x^2 + 2x - 7 \right) &= \frac{75}{4} \frac{d}{dx}x^4 + 19 \frac{d}{dx}x^3 - \frac{14}{3} \frac{d}{dx}x^2 + 2 \frac{d}{dx}x - 7 \\ &= \frac{75}{4}(4x^3) + 19(3x^2) - \frac{14}{3}(2x) + 2(1) - 0 \\ &= 75x^3 + 57x^2 - \frac{28}{3}x + 2\end{aligned}$$

6. Find the derivative of $(x^4 - x^2 + 2x)(12x^3 + 6x^2 - 1)$ in two ways:

- (a) Multiply out the expression above then take the derivative.
(b) Use the product rule from Section 3.1, and then multiply out your final answer and collect all like terms.

Solutions:

- (a) Multiplying out we get

$$(x^4 - x^2 + 2x)(12x^3 + 6x^2 - 1) = 12x^7 + 6x^6 - 12x^5 + 17x^4 + 12x^3 + x^2 - 2x$$

so the derivative is

$$\begin{aligned}\frac{d}{dx}((x^4 - x^2 + 2x)(12x^3 + 6x^2 - 1)) &= \frac{d}{dx}(12x^7 + 6x^6 - 12x^5 + 17x^4 + 12x^3 + x^2 - 2x) \\ &= 84x^6 + 36x^5 - 60x^4 + 68x^3 + 36x^2 + 2x - 2\end{aligned}$$

- (b) Let $f(x) = x^4 - x^2 + 2x$, $g(x) = 12x^3 + 6x^2 - 1$. Then $f'(x) = 4x^3 - 2x + 2$ and $g'(x) = 36x^2 + 12x$. The product rule tells us that

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

so therefore

$$\begin{aligned}\frac{d}{dx}((x^4 - x^2 + 2x)(12x^3 + 6x^2 - 1)) &= (4x^3 - 2x + 2)(12x^3 + 6x^2 - 1) + (x^4 - x^2 + 2x)(36x^2 + 12x) \\ &= 48x^6 + 24x^5 - 4x^3 - 24x^4 - 12x^3 + 2x + 24x^3 + 12x^2 - 2 \\ &\quad + 36x^6 + 24x^5 - 36x^4 - 12x^3 + 72x^3 + 24x^2 \\ &= 84x^6 + 36x^5 - 60x^4 + 68x^3 + 36x^2 + 2x - 2\end{aligned}$$