

# Business Calculus, Summer 2004

## Homework #1

Due: Tuesday, July 6th 2004 by end of class

1. Suppose we deposit \$150 into a bank account. How many years will it take for the value of the account to double under each of the following interest schemes:

- (a) 8.5% compounded annually
- (b) 7.5% compounded semi-annually
- (c) 7% compounded monthly
- (d) 6.75% compounded continuously

Does the amount of money originally deposited into the account affect the doubling time?

**Solutions:** Throughout this problem we have  $P = 150$ ,  $A = 300$ ,  $r$  varies and  $t$  is the unknown

- (a) Here  $N = 1$ ,  $r = .085$ . Using the discrete compounding formula we have

$$\begin{aligned}300 &= (1 + .085)^t 150 \\2 &= 1.085^t \\ \log 2 &= \log 1.085^t \\ \log 2 &= t \log 1.085 \\ t &= \frac{\log 2}{\log 1.085} \approx 8.5 \text{ years}\end{aligned}$$

- (b) Here  $N = 2$ ,  $r = .075$ . Using the discrete compounding formula we have

$$\begin{aligned}300 &= (1 + .075/2)^{2t} 150 \\2 &= 1.0375^{2t} \\ \log 2 &= \log 1.0375^{2t} \\ \log 2 &= 2t \log 1.0375 \\ t &= \frac{\log 2}{2 \log 1.0375} \approx 9.41 \text{ years}\end{aligned}$$

- (c) Here  $N = 12$ ,  $r = .07$ . Again using the discrete compounding formula we have

$$\begin{aligned}300 &= (1 + .07/12)^{12t} 150 \\2 &= 1.0058333^{12t} \\ \log 2 &= \log 1.0058333^{12t} \\ \log 2 &= 12t \log 1.0058333 \\ t &= \frac{\log 2}{12 \log 1.0058333} \approx 9.93 \text{ years}\end{aligned}$$

(d) Here  $r = .0675$  and we use the continuous compounding formula

$$\begin{aligned}300 &= 150e^{.0675t} \\2 &= e^{.0675t} \\ \ln 2 &= \ln e^{.0675t} \\ \ln 2 &= .0675t \ln e \\ t &= \frac{\ln 2}{.0675} \approx 10.27 \text{ years}\end{aligned}$$

Here we used  $\ln$  because we conveniently get that  $\ln e = 1$ . However we could use  $\log$  just as well.

The doubling time is not affected by the amount of money initially invested, since the speed of interest growth is the same on every dollar.

2. First determine how much \$1000 compounded annually at 9% will grow to in 12 years. Once you've done this, determine the rate of interest that would be necessary to grow the same \$1000 to the same final amount in the same 12 years, but assuming the money is compounded continuously instead of just annually.

**Solution:** To figure out how much the 1000 will grow to in 12 years, use the discrete compounding formula with  $P = 1000, r = .09, N = 1, t = 12$ . Then

$$A = P \left(1 + \frac{r}{N}\right)^{Nt} = 1000 \left(1 + \frac{.09}{1}\right)^{12} = 1000(1.09)^{12} = 2812.66$$

Before computing the second part we should note that the rate necessary under continuous compounding will be less than the rate used for annual compounding since continuous compounding grows our money faster. We use the continuous compounding formula with  $P = 1000, r = .09, A = 2812.66$ , and  $r$  unknown. We get

$$\begin{aligned}2812.66 &= 1000e^{12r} \\2.81266 &= e^{12r} \\ \ln 2.81266 &= \ln e^{12r} \\ \ln 2.81266 &= 12r \ln e \\ r &= \frac{\ln 2.81266}{12} \approx .0862\end{aligned}$$

Hence under continuous compounding we would only need an 8.62% interest rate.

3. A corporate bond has a redemption value of \$100,000, but does not come due until after 20 years. Also, at the end of every second year, starting with the end of year 2 and ending at the end of year 20, the bond makes coupon payments of \$5000. Assuming that the market rate for money is 4.5% compounded annually, what is the value of the bond right now?

**Solutions:** We know the future value of the bond at various times, what we need is the present value. However, there are 11 payments with this bond (one each at years 2, 4, ..., 18, and 2 at year 20), so we have to figure out the present value of each one separately. Since the interest is compounded annually at 4.5% we'll use the formula

$$A = P(1.045)^t,$$

but because we want the present value  $P$  we rearrange it as

$$P = A(1.045)^{-t}.$$

The payment schedule for the bond looks like this

Year of Payment	Payment	Present Value
2	5000	$5000(1.045)^{-2}$
4	5000	$5000(1.045)^{-4}$
6	5000	$5000(1.045)^{-6}$
$\vdots$	$\vdots$	$\vdots$
18	5000	$5000(1.045)^{-18}$
20	5000	$5000(1.045)^{-20}$
20	100,000	$100,000(1.045)^{-20}$

So we know the present value of each payment, to find the total present value we only need to add them up. This could be done by hand, but of course it would take a very long time, so instead we use the geometric sum formula. The sum of the payments of 5000 appears to follow a nice pattern, in fact:

$$\begin{aligned}
 & 5000(1.045)^{-2} + 5000(1.045)^{-4} + \dots + 5000(1.045)^{-18} + 5000(1.045)^{-20} \\
 &= 5000(1.045)^{-20}[(1.045)^{18} + (1.045)^{16} + \dots + (1.045)^2 + 1] \\
 &= 5000(1.045)^{-20}[(1.045^2)^9 + ((1.045^2)^8 + \dots + ((1.045^2)^1 + 1) \\
 &= 5000(1.045)^{-20} \frac{((1.045^2)^{10} - 1)}{(1.045^2 - 1)} \\
 &= 5000(1.045)^{-20} \frac{(1.045)^{20} - 1}{(1.045)^2 - 1} \\
 &= 31,804.25
 \end{aligned}$$

The only payment we haven't taken account of is the final \$100,000. It's present value is

$$100,000(1.045)^{-20} = 41,464.29$$

Therefore the total present value of the bond is

$$\$31,804.25 + \$41,464.29 = \$73,268.54$$

So even though this bond will pay out a total of \$150,000 over its lifetime, investors are only willing to pay \$73,268.53 for it. The difference between the two represents the premium the bond issuer has to pay to the bondholder for the use of the bondholder's money over the 20 years. The market rate for money, which represents a realistic return the bondholder could achieve by investing in an alternate security such as Treasury certificates, is what ultimately determines the value of the bond.

4. A young couple decides to take out a 30-year mortgage on a home they want to buy for \$225,000. The lending company requires that the couple makes a 5% down payment, and the company will lend them the rest. They charge 3% interest, compounded monthly. What will the couple's monthly mortgage payment have to be?

**Solution:** First note that the couple is only borrowing 95% of the \$225,000, which is \$213,750. Let  $Z$  be the amount of the monthly payment. We need to figure out the present value of each of these payments. Since they occur at different times they have different present values, for example the first payment has a much larger present value than the last one does. Because the interest is 3% compounded monthly we should use the formula

$$A = P(1 + .03/12)^{12t} = P(1.0025)^{12t}$$

which we can rearrange as

$$P = A(1.0025)^{-12t}$$

to solve for  $P$ . The present values of the different payments of  $Z$  is given by the following table:

Time of Payment (in Years)	Payment	Present Value
1/12 (Month 1)	$Z$	$Z(1.0025)^{-12*1/12} = Z(1.0025)^{-1}$
2/12 (Month 2)	$Z$	$Z(1.0025)^{-12*2/12} = Z(1.0025)^{-2}$
3/12 (Month 3)	$Z$	$Z(1.0025)^{-12*3/12} = Z(1.0025)^{-3}$
$\vdots$	$\vdots$	$\vdots$
359/12 (Month 359)	$Z$	$Z(1.0025)^{-12*359/12} = Z(1.0025)^{-359}$
360/12 (Month 360)	$Z$	$Z(1.0025)^{-12*360/12} = Z(1.0025)^{-360}$

Summing up the present values in the last column we get

$$\begin{aligned} & Z(1.0025)^{-1} + Z(1.0025)^{-2} + Z(1.0025)^{-3} + \dots + Z(1.0025)^{-359} + Z(1.0025)^{-360} \\ &= Z(1.0025)^{-360}[(1.0025)^{359} + (1.0025)^{358} + (1.0025)^{357} + \dots + (1.0025)^1 + 1] \\ &= Z(1.0025)^{-360} \frac{(1.0025)^{360} - 1}{1.0025 - 1} \\ &= 237.189181 \times Z \end{aligned}$$

This is the present value of the series of payments of  $Z$  dollars the couple will have to make. In exchange for doing so, they're being given \$213,750 right now. Therefore, if this is any sort of fair deal, we must have that

$$213,750 = 237.189181 \times Z$$

from which we can solve to get

$$Z = \frac{213,750}{237.189181} = 901.18$$

So the monthly mortgage payment is \$901.18. In total then, the couple will be paying  $\$901.18 \times 360 = \$324,424.80$  to the lending company, meaning the total amount of interest paid will be almost \$100,000.

5. (a) Suppose that on a girl's 10<sup>th</sup> birthday her parents decide to open a college savings fund by depositing  $A$  dollars into an account. They deposit the same  $A$  dollars on every birthday up to and including her 18<sup>th</sup>, at which time they want the value of the fund to be \$75,000. If the money in the account compounds continuously at 7%, determine  $A$ . Once you've done this, figure out what percentage of the \$75,000 was deposited by the parents, and what percent comes from the interest.

- (b) Do part (a) again, including the percentage calculations, but assume the parents begin depositing money on the girl's 4<sup>th</sup> birthday.

**Solution:** First, I shouldn't have used the letter  $A$  for the yearly deposit because it conflicts with the  $A$  from our compounding formulas, so let's call the yearly deposit  $Z$  instead.

- (a) We need to figure out how much each deposit of  $Z$  grows to by the girl's 18<sup>th</sup> birthday, and the sum of these values should be \$75,000. Since the interest is 7% compounded continuously we want to use the formula

$$A = Pe^{.07t}.$$

From birthdays 10 to 18 there will be a total of 9 deposits, and the future value of each (on her 18<sup>th</sup> birthday) is summarized in the table:

Birthday	Deposit	Future Value
10	$Z$	$Ze^{.07*8} = Z(e^{.07})^8$
11	$Z$	$Ze^{.07*7} = Z(e^{.07})^7$
$\vdots$	$\vdots$	$\vdots$
17	$Z$	$Ze^{.07*1} = Ze^{.07}$
18	$Z$	$Ze^{.07*0} = Z$

The sum of the future values is therefore

$$\begin{aligned} Z(e^{.07})^8 + Z(e^{.07})^7 + \dots + Z(e^{.07})^1 + Z \\ &= Z[(e^{.07})^8 + (e^{.07})^7 + \dots + (e^{.07}) + 1] \\ &= Z \frac{(e^{.07})^9 - 1}{e^{.07} - 1} \\ &= 12.1036 \times Z. \end{aligned}$$

This is how much the deposited money is worth on the girl's 18<sup>th</sup> birthday. But we want it to be \$75,000. Hence we must have

$$12.1036 \times Z = 75,000$$

from which we get

$$Z = \frac{75,000}{12.1036} = 6196.50$$

So the yearly deposit has to be 6196.50. Since the parents make a total of 9 deposits, they will be contributing  $9 \times 6196.50 = 55,768.50$  towards the fund, or approximately 74.36%. The other 25.64% is the accumulated interest.

- (b) If the deposits start on the 4<sup>th</sup> birthday instead, we can write out the same table and do the same calculations to get the sum of the present values to be

$$Z \frac{(e^{.07})^{15} - 1}{e^{.07} - 1} = 25.6199 \times Z = 75,000$$

and solve to get  $Z = 2927.41$ . Since the parents are now making a total of 15 deposits, they will be contributing  $15 \times 2927.41 = 43,911.15$  towards the fund, or approximately 58.55%. The other 41.45% comes from interest. So the moral of the story is to start investing early.