# 3. Girsanov, numeraires, and all that

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## Contents



## 1 Arbitrage asset pricing in a nutshell

This is a technical intermezzo in preparation for next two themes: CMS based instruments and term structure modeling. We start by reviewing briefly some basic concepts of arbitrage pricing theory, just to cover our upcoming needs. For a full account of this theory, I encourage you to take the course in continuous time finance offered in this program. In particular, I will be skipping over a lot of technicalities while discussing the probabilistic concepts underlying this framework, and, again, I recommend further study for a more indebt understanding of these concepts. Next, we will discuss the technique of *change of numeraire*, which will play a key role in the following lectures.

## 1.1 Self-financing portfolios

We consider a financial market which consists of a number of assets which we shall denote by $I_0, I_1, \ldots, I_N$. We model the price processes of these assets by $S_0(t), S_1(t), \ldots, S_N(t)$, i.e. $S_i(t)$ denotes the price of asset $I_i$ at time $t$. Each price process is a *diffusion process*, i.e. there is an underlying multidimensional Wiener process $W_1(t), W_1(t), \ldots, W_d(t)$, $d \leq N+1$, and the price process follows a stochastic differential equation (SDE) of the form:

$$dS_j(t) = \Delta_j(S(t), t)\, dt + \sum_{k=1}^{d} C_{jk}(S(t), t)\, dW_k(t)\,. \tag{1}$$

The coefficient $\Delta_j(S(t), t)$ is called the *drift coefficients*, while the coefficients $C_{jk}(S(t), t)$ are referred to as the *diffusion coefficients*.

For example, in the classic Black-Scholes model, $S_0(t) = B(t)$ is the riskless bond, and $S_1(t) = S(t)$ is a (risky) stock, with the dynamics given by

$$\begin{aligned} dB(t) &= rB(t)\, dt, \\ dS(t) &= \mu S(t)\, dt + \sigma S(t)\, dW(t)\,. \end{aligned} \tag{2}$$

A *portfolio* is specified by the weights $w_0(t), w_1(t), \ldots, w_N(t)$, of the assets at time $t$. We assume, of course, that the weights are non-negative, and they add up to one. The value process of the portfolio is given by

$$V(t) = \sum_{0 \leq i \leq N} w_i(t)\, S_i(t)\,. \tag{3}$$

A portfolio is *self-financing*, if

$$dV(t) = \sum_{i=0}^{N} w_i(t)\, dS_i(t)\,, \tag{4}$$

or, equivalently,

$$V(t) = V(0) + \int_0^t \sum_{i=0}^{N} w_i(s)\, dS_i(s)\,. \tag{5}$$

In other words, the price process of a self-financing portfolio does not allow for infusion or withdrawal of capital.

A fundamental assumption of arbitrage pricing theory is that financial markets (or at least, their models) are free of arbitrage opportunities[1]. An *arbitrage opportunity* arises if one can construct a self-financing portfolio such that:

[1] This assumption is, mercifully, violated frequently enough so that the entire hedge fund industry can sustain itself exploiting the market's lack of respect for arbitrage freeness.

(a) The initial value of the portfolio is zero, $V(0) = 0$.

(b) With probability one, the portfolio has a non-negative value at maturity, $\mathsf{P}(V(T) \geq 0) = 1$.

(c) With a positive probability, the value of the portfolio at maturity is positive, $\mathsf{P}(V(T) > 0) > 0$.

We say the model is *arbitrage free* if it does not allow arbitrage opportunities. Requiring arbitrage freeness has important consequences for price dynamics.

## 1.2 The fundamental theorem

A key concept in modern asset pricing theory is that of a *numeraire*. A numeraire is any asset with price process $\mathfrak{N}(t)$ such that $\mathfrak{N}(t) > 0$, for all times $t$. The *relative price* process of asset $I_i$ is defined by

$$S_i^{\mathfrak{N}}(t) = \frac{S_i(t)}{\mathfrak{N}(t)}\,. \tag{6}$$

In other words, the relative price of an asset is its price expressed in the units of the numeraire.

A probability measure $\mathsf{Q}$ is called an *equivalent martingale measure* for the above market, with numeraire $\mathfrak{N}(t)$, if it has the following properties:

(a) $\mathsf{Q}$ is equivalent to $\mathsf{P}$, i.e.

$$d\mathsf{P}(\omega) = D_{\mathsf{PQ}}(\omega)\, d\mathsf{Q}(\omega)\,,$$

and

$$d\mathsf{P}(\omega) = D_{\mathsf{QP}}(\omega)\, d\mathsf{Q}(\omega)\,,$$

with some $D_{\mathsf{PQ}}(\omega) > 0$ and $D_{\mathsf{QP}}(\omega) > 0$.

(b) The relative price processes $S_i^{\mathfrak{N}}(t)$ are martingales under $\mathsf{Q}$,

$$S_i^{\mathfrak{N}}(s) = \mathsf{E}^{\mathsf{Q}}\left[S_i^{\mathfrak{N}}(t)\,\middle|\mathcal{F}_s\right]. \tag{7}$$

**The Fundamental Theorem** of arbitrage free pricing states that *the model is arbitrage free if and only if there exists an equivalent martingale measure* $\mathsf{Q}$.

An important consequence of this theorem is the arbitrage pricing law:

$$\frac{V(s)}{\mathfrak{N}(s)} = \mathsf{E}^{\mathsf{Q}}\left[\frac{V(T)}{\mathfrak{N}(T)}\,\middle|\mathcal{F}_s\right]. \tag{8}$$

One is free to change numeraire $\mathfrak{N}(t) \to \mathfrak{N}'(t)$. Girsanov's theorem (see the Appendix for a summary) implies that there exists a martingale measure $\mathsf{Q}'$ such that

$$\frac{V(s)}{\mathfrak{N}'(s)} = \mathsf{E}^{\mathsf{Q}'}\left[\frac{V(T)}{\mathfrak{N}'(T)}\,\middle|\mathcal{F}_s\right], \tag{9}$$

and thus the radon-Nikodym derivative is given by the ratio of the numeraires:

$$\begin{aligned}\left.\frac{d\mathsf{Q}'}{d\mathsf{Q}}\right|_s &= \frac{\dfrac{\mathfrak{N}(s)}{\mathfrak{N}(T)}}{\dfrac{\mathfrak{N}'(s)}{\mathfrak{N}'(T)}} \\ &= \frac{\mathfrak{N}(s)}{\mathfrak{N}(T)}\frac{\mathfrak{N}'(T)}{\mathfrak{N}'(s)}\,.\end{aligned} \tag{10}$$

We will have more to say about this important fact. In the meantime, let us review some of the most important numeraires encountered in interest rates modeling.

# 2 Examples of numeraires

We shall now revisit the numeraires that we have encountered in Lecture 2 in the context of valuation of vanilla interest rate options.

## 2.1 Spot numeraire

The *spot numeraire* (or *rolling numeraire*) is simply a \$1 deposited in a bank and accruing the (riskless) instantaneous rate. Its value at time $t$ is

$$\mathfrak{N}(t) = \exp\left(\int_0^t f(s)\,ds\right). \tag{11}$$

The special case of a constant riskless rate $f(t) = r$ plays a key role in the Black-Scholes model, and the rolling numeraire is the riskless bond $B(t)$ mentioned before.

## 2.2 Forward numeraire

The $T$*-forward numeraire* is simply the zero coupon bond for maturity $T$. Its price at $t < T$ is given by

$$\mathfrak{N}_T(t) = P(t, T)\,. \tag{12}$$

As explained in Lecture 2, the $T$-forward numeraire arises naturally in pricing instruments based of forwards maturing at $T$. Forward rates for maturity at $T$ are martingales under the measure associated with this numeraire.

### 2.3 Annuity numeraire

The *annuity numeraire* is associated with a (forward starting) swap. The annuity pays \$1 on each coupon day of the swap, accrued according to the swap's day count day conventions. Its PV for the value day $t$ is given by the forward level function:

$$\begin{aligned}\mathfrak{N}_{T_{\text{start}},T_{\text{mat}}}(t) &= L(t, T_{\text{start}}, T_{\text{mat}}) \\ &= \sum_{j=1}^{n} \alpha_j P(t, T_j),\end{aligned} \tag{13}$$

where the summation runs over the coupon dates of the annuity.

The annuity numeraire arises as the natural numeraire when valuing swaptions. As explained in Lecture 2, the swap rate $S(T_{\text{start}}, T_{\text{mat}})$ is a martingale under the measure associated with the annuity numeraire.

## 3 Change of numeraire technique

Choice of a numeraire is a matter of convenience and is dictated by the valuation problem at hand. Asset valuation leads frequently to complicated stochastic processes, and one way of making the problem easier to eliminate the drift term from the stochastic differential equation defining the process. The change of numeraire technique allows us to achieve precisely this: modify the probability law (the measure) of the process so that, under this new measure, the process is driftless, i.e. it is a martingale.

Consider a financial asset whose dynamics is given in terms of the state variable $X(t)$. Under the measure $\mathsf{P}$ this dynamics reads:

$$dX(t) = \Delta^{\mathsf{P}}(t)\,dt + C(t)\,dW^{\mathsf{P}}(t). \tag{14}$$

Our goal is to relate this dynamics to the dynamics of the same asset under an equivalent measure $\mathsf{Q}$:

$$dX(t) = \Delta^{\mathsf{Q}}(t)\,dt + C(t)\,dW^{\mathsf{Q}}(t). \tag{15}$$

Remember that the diffusion coefficients in these equations are the unaffected by the change of measure! We assume that $\mathsf{P}$ is associated with the numeraire $\mathfrak{N}(t)$ whose dynamics is given by:

$$d\mathfrak{N}(t) = A_{\mathfrak{N}}(t)\,dt + B_{\mathfrak{N}}(t)\,dW^{\mathsf{P}}(t), \tag{16}$$

while $\mathsf{Q}$ is associated with the numeraire $\mathfrak{M}(t)$ whose dynamics is given by:

$$d\mathfrak{M}(t) = A_{\mathfrak{M}}(t)\,dt + B_{\mathfrak{M}}(t)\,dW^{\mathsf{P}}(t). \tag{17}$$

According to Girsanov's theorem, the Radon-Nikodym derivative

$$D(t) = \left.\frac{d\mathsf{Q}}{d\mathsf{P}}\right|_t \tag{18}$$

is a martingale under $\mathsf{P}$, and

$$dD(t) = \theta(t)\, D(t)\, dW^{\mathsf{P}}(t), \tag{19}$$

with

$$\theta(t) = -\frac{\Delta^{\mathsf{Q}}(t) - \Delta^{\mathsf{P}}(t)}{C(t)}\,. \tag{20}$$

From the fundamental theorem of asset pricing we infer that

$$D(t) = \frac{\mathfrak{N}(0)}{\mathfrak{M}(0)}\,\frac{\mathfrak{M}(t)}{\mathfrak{N}(t)}\,. \tag{21}$$

Since $D(t)$ is a martingale,

$$dD(t) = \frac{\mathfrak{N}(0)}{\mathfrak{M}(0)}\,\beta(t)\, dW^{\mathsf{P}}(t), \tag{22}$$

with some process $\beta(t)$ (note that the factor $\mathfrak{N}(0)/\mathfrak{M}(0)$ is just a convenient normalization). Consequently,

$$\begin{aligned}\frac{\mathfrak{N}(0)}{\mathfrak{M}(0)}\,\beta(t) &= \theta(t)\, D(t)\\ &= \theta(t)\,\frac{\mathfrak{N}(0)}{\mathfrak{M}(0)}\,\frac{\mathfrak{M}(t)}{\mathfrak{N}(t)}\,,\end{aligned} \tag{23}$$

and so

$$\theta(t)\,\frac{\mathfrak{N}(t)}{\mathfrak{M}(t)} = \beta(t)\,. \tag{24}$$

As a result we obtain the following formula:

$$\Delta^{\mathsf{Q}}(t) - \Delta^{\mathsf{P}}(t) = -\frac{\mathfrak{N}(t)}{\mathfrak{M}(t)}\, C(t)\,\beta(t), \tag{25}$$

relating the change in drift to the, yet unspecified, process $\beta(t)$.

We will transform this result into a more useful form by expressing $\beta(t)$ in terms of the coefficients of the price dynamics of the numeraires $\mathfrak{N}(t)$ and $\mathfrak{M}(t)$. Note that

$$d\left(\frac{\mathfrak{M}(t)}{\mathfrak{N}(t)}\right) = A_{\mathfrak{M}/\mathfrak{N}}(t)\, dt + \frac{\mathfrak{N}(t)}{\mathfrak{M}(t)}\left(\frac{B_{\mathfrak{M}}(t)}{\mathfrak{M}(t)} - \frac{B_{\mathfrak{N}}(t)}{\mathfrak{N}(t)}\right) dW^{\mathsf{P}}(t), \tag{26}$$

with some drift coefficient $A_{\mathfrak{M}/\mathfrak{N}}(t)$, whose exact form is not important to us. As a consequence,

$$\beta(t) = \frac{\mathfrak{N}(t)}{\mathfrak{M}(t)} \left( \frac{B_{\mathfrak{N}}(t)}{\mathfrak{N}(t)} - \frac{B_{\mathfrak{M}}(t)}{\mathfrak{M}(t)} \right). \tag{27}$$

This implies finally that

$$\begin{aligned} \Delta^{\mathsf{Q}}(t) - \Delta^{\mathsf{P}}(t) &= C(t) \left( \frac{B_{\mathfrak{M}}(t)}{\mathfrak{M}(t)} - \frac{B_{\mathfrak{N}}(t)}{\mathfrak{N}(t)} \right) \\ &= dX(t)\, d\left( \log \frac{\mathfrak{M}(t)}{\mathfrak{N}(t)} \right). \end{aligned} \tag{28}$$

The formula above expresses the change in the drift in the dynamics of the state variable, which accompanies a change of numeraire, in terms of the processes themselves.

In order to rewrite (28) in a more intrinsic form, let us establish a bit of notation. For two stochastic processes $X(t)$ and $Y(t)$ we define the following bracket operation:

$$\{X, Y\}(t) = dX(t)\, d\big(\log Y(t)\big). \tag{29}$$

Thus the change of numeraire formula can be stated in the elegant, easy to remember form:

$$\Delta^{\mathsf{Q}}(t) = \Delta^{\mathsf{P}}(t) + \{X, \mathfrak{M}/\mathfrak{N}\}(t). \tag{30}$$

# A Girsanov's theorem

We say that the measure Q is *absolutely continuous* with respect to P if there exists a positive function $D$ (called the *Radon-Nikodym derivative*) such that

$$\mathsf{Q}(A) = \int_A D(\omega)\, d\mathsf{P}(\omega) \tag{31}$$

or

$$\frac{d\mathsf{Q}}{d\mathsf{P}}(\omega) = D(\omega). \tag{32}$$

In other words, the "volume element" $d\mathsf{Q}$ is always proportional to the "volume element" $d\mathsf{P}$, with the proportionality factor being a positive function throughout the probability space. Two probability measures Q and P are called *equivalent*, if Q is absolutely continuous with respect to P and P is absolutely continuous with respect to Q.

Consider now a diffusion process:

$$dX(t) = \Delta(X(t), t)\, dt + C(X(t), t)\, dW(t). \tag{33}$$

Recall that $X(t)$ is called a *martingale* if the diffusion above is driftless, i.e. $\Delta(X(t), t) = 0$. An important property of a martingale is that

$$X(s) = \mathsf{E}[X(t) | \mathcal{F}_s]. \tag{34}$$

where $\mathsf{E}[\cdot | \mathcal{F}_s]$ denotes the conditional expected value. In other words, given all information up to time $s$, the expected value of future values of a martingale is $X(s)$.

A natural question arises: can we transform a diffusion process into a martingale by a simple change in the probability measure? One might proceed like this: Write

$$\begin{aligned} dX(t) &= C(t)\left(\frac{\Delta(t)}{C(t)}\, dt + dW(t)\right) \\ &= C(t)\, d\widetilde{W}(t), \end{aligned} \tag{35}$$

where

$$\widetilde{W}(t) = W(t) + \int_0^t \frac{\Delta(s)}{C(s)}\, ds. \tag{36}$$

This looks like a new Brownian motion! Girsanov's theorem asserts that $\widetilde{W}(t)$ is actually a Brownian motion if we modify the probability measure as follows. Define:

$$D\big|_t = \exp\left(-\int_0^t \frac{\Delta(s)}{C(s)}\, dW(s) - \frac{1}{2}\int_0^t \left(\frac{\Delta(s)}{C(s)}\right)^2 ds\right). \tag{37}$$

Define now the equivalent measure $\mathsf{Q}$ with

$$\frac{d\mathsf{Q}}{d\mathsf{P}}\Big|_t = D\big|_t. \tag{38}$$

Then

(a) The process $D(t) = D\big|_t$ is a martingale under $\mathsf{P}$.

(a) $\widetilde{W}(t)$ is a Wiener process under $\mathsf{Q}$.

We have stated Girsanov's theorem for a one-dimensional Brownian motion. This assumption is not essential and, using a bit of linear algebra, one can easily formulate a version of Girsanov's theorem for an arbitrary multidimensional Brownian motion.

## References

[1] Brigo, D., and Mercurio, F.: *Interest Rate Models - Theory and Practice*, Springer Verlag (2006).

[2] Oksendal, B.: *Stochastic Differential Equations: An Introduction with Applications*, Springer Verlag (2005).