Business Calculus, Spring 2004

Quiz #9

Name: ____________________
Section Time: ____________________
TA's Name: ____________________

1. Find all the critical points of the function

$$f(x, y) = xe^{-(x^2+y^3)/2}$$

(*Hint:* Recall that $e^a \neq 0$ for any value of a).

2. The profit function for a facility that manufactures two products is given in terms of the production levels x and y by

$$P(x, y) = -180,000 + 600x - 0.1x^2 + 800y - 0.2y^2 - 0.015xy$$

Find the combination of production levels x and y that yields maximum profit. Give a convincing reason why the production level you get corresponds to a maximum and not a minimum.

Solutions:

1. We need to find the partial derivatives and set them both equal to zero. First by using the product rule we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(xe^{-(x^2+y^3)/2}\right) = \frac{\partial}{\partial x}(x)e^{-(x^2+y^3)/2} + x\frac{\partial}{\partial x}\left(e^{-(x^2+y^3)/2}\right)\\ &= e^{-(x^2+y^3)/2} + x\left(\frac{-2x}{2}e^{-(x^2+y^3)/2}\right)\\ &= e^{-(x^2+y^3)/2} - x^2e^{-(x^2+y^3)/2}\\ &= (1-x^2)e^{-(x^2+y^3)/2}\end{aligned}$$

and similarly

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(xe^{-(x^2+y^3)/2}\right) = x\frac{\partial}{\partial y}\left(e^{-(x^2+y^3)/2}\right)\\ &= x\left(\frac{-3y^2}{2}e^{-(x^2+y^3)/2}\right)\\ &= \frac{-3xy^2}{2}e^{-(x^2+y^3)/2}\end{aligned}$$

Now we set both equations equal to zero which gives

$$\frac{\partial f}{\partial x} = (1-x^2)e^{-(x^2+y^3)/2} = 0$$
$$\frac{\partial f}{\partial y} = \frac{-3xy^2}{2}e^{-(x^2+y^3)/2} = 0$$

But since $e^{-(x^2+y^3)/2} \neq 0$ we can conclude that

$$1 - x^2 = 0$$
$$\frac{-3xy^2}{2} = 0$$

The first equation tells us we must have $x = \pm 1$. The second equation tells us either $x = 0$ or $y = 0$, but $x = 0$ is impossible since the first equation already told us we must have $x = \pm 1$. Therefore the only useful solution to the second equation is $y = 0$. In summary then we have $x = \pm 1$ and $y = 0$, so there are two critical points: $(1, 0)$ and $(-1, 0)$.

2. Since we're manufacturing some quantities x and y we should first note that the feasible set is $x \geq 0, y \geq 0$. To find the maximum of the function we need to find the critical points, so

$$\frac{\partial P}{\partial x} = 600 - 0.2x - 0.015y = 0$$
$$\frac{\partial P}{\partial y} = 800 - 0.4y - 0.015x = 0$$

Rearranging the first equation we get

$$0.2x = 600 - 0.015y$$
$$x = 3000 - 0.075y$$

Now we plug this into the second equation to get

$$\begin{aligned} 0 = 800 - 0.4y - 0.1x &= 800 - 0.4y - 0.015(3000 - 0.075y) \\ &= 800 - 45 - 0.4y + 0.001125y \\ &= 755 - 0.398875y \end{aligned}$$

so $y = 755/0.398875 = 1892.82$. But then we plug this into the equation $x = 3000 - 0.075y$ to get that $x = 3000 - 0.075(1892.82) = 2858.04$. Therefore $(2858.04, 1898.82)$ is a critical point. If we simply inspect the function P we can clearly tell it has a maximum and no minimum, so this critical point must correspond to the maximum. We can also convince ourselves of this since we're dealing with a profit function, which, since we can always lose as much money as we want but can't make an unlimited amount, has a maximum but no minimum.