Midterm 2 - SOLUTIONS
Business Calculus - V63.0017

1. (10pts.) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ if $y = x^2e^x$.
   SOL'N: By the product rule,
   $$\frac{d}{dx}(x^2e^x) = \left(\frac{d}{dx}(x^2)\right)(e^x) + (x^2)\left(\frac{d}{dx}(e^x)\right) = 2xe^x + x^2e^x$$
   and differentiating again using the product rule on each term gives
   $$\frac{d^2}{dx^2}(x^2e^x) = \frac{d}{dx}(2xe^x) + \frac{d}{dx}(x^2e^x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = e^x(2 + 4x + x^2).$$

2. (10pts.) Find all points where the tangent line to the graph of $y = \dfrac{x}{x^2+1}$ is horizontal.
   SOL'N: The tangent line is horizontal when the derivative is equal to zero. Using the quotient rule:
   $$\frac{dy}{dx} = \frac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$
   which equals zero when the numerator is zero at $x = \pm 1$.

3. (10pts.) Find the derivative $u'(t)$ if $u(t) = e^{-(t^2/2)}$.
   SOL'N: Using the chain rule
   $$u'(t) = e^{-(t^2/2)} \cdot \frac{d}{dt}(-(t^2/2)) = -te^{-(t^2/2)}.$$

4. (10pts.) Find the absolute maximum and minimum values of $y = (x-3)^2 - 1$ on the interval $0 \le x \le 5$.
   SOL'N: Checking critical points and endpoints; $\frac{dy}{dx} = 2(x-3)$, so $x = 3$ is only critical point. Comparing values at $x = 0$, $x = 3$ and $x = 5$ shows that the minimum value is $y = -1$ (at $x = 3$) and the maximum value is $y = 8$ (at $x = 0$).

5. (10pts.) A manufacturer's costs C depend on warehouse capacity x according to the formula
   $$C(x) = 5 + 4x + \frac{1600}{x} \text{ thousand dollars per year}$$
   where x is measured in thousands of cubic feet. Which capacity will minimize costs? SOL'N: The "feasible" set, or domain of $C(x)$ is $x > 0$. Checking for critical points, $C'(x) = 4 - \frac{1600}{x^2}$ equals zero only when $x = 20$, which gives the (only) minimum of $C(x)$ since $C''(x) = 2\frac{1600}{x^3} > 0$ for all $x > 0$.

6. (10pts.) Among all pairs of numbers whose sum is 100, find the pair whose product is largest. SOL'N: If $x + y = 100$, then we want maximum of $xy = x(100-x) = 100x - x^2$. (Here x and y can be any numbers.) Checking critical points, $\frac{d}{dx}(100x - x^2) = 100 - 2x$ so $x = 50$ is only critical point and gives the maximal pair $x = y = 50$, since $\frac{d^2}{dx^2}(100x - x^2) = -2 < 0$ (concave down).

7. (5pts.) Find the amount accumulated after 5 years on a principal of \$1000 if interest is compounded quarterly at an annual rate of 8 percent. SOL'N: Using formula $A(t) = P(1 + \frac{r}{N})^{Nt}$, $A(5) = 1000(1.02)^{20} = 1485.95$

8. (5pts.) Find the amount accumulated after 50 years on a principal of \$100 if interest is compounded continuously at an annual rate of 4 percent. SOL'N: Using formula $A(t) = Pe^{rt}$, $A(50) = 100e^2 = 738.91$.

9. (10pts.) At what annual interest rate, compounded continuously, would an initial investment double in 10 years?. SOL'N: Solving for r in $2P = Pe^{10r}$ gives $r = \frac{\ln 2}{10} = 6.93\%$.

10. (10pts.) What is the present value of \$2000 to be paid 5 years from now, if the alternative is to invest at 5 percent annual interest, compounded continuously? SOL'N: $P = 2000e^{-0.25} = 1557.60$.

11. (10pts.) If $g(t)$ is a differentiable function, write down the *definition* of the derivative $g'(0)$ at $t = 0$. SOL'N:
    $$g'(0) = \lim_{\Delta t \to 0} \frac{g(0+\Delta t) - g(0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{g(\Delta t) - g(0)}{\Delta t}.$$