Problem Set 9

Answers to the underlined problems are written up on the following pages.
Section 5.1: 13, 25, 32, 34, 35; 2, 6, 8, 11, 14, 20, 26, 28, 30
Section 5.2: 6, 7, 11, 14, 20, 27; 2, 4, 8, 16, 18, 24
Section 5.3: 22, 25, 28, 37; 6, 8, 10, 12, 16, 24, 34, 35, 36

Section 5.1

13 $\int \frac{1}{s^3}\,ds = \int s^{-3}\,ds = \frac{s^{-3+1}}{-3+1} = \frac{s^{-2}}{-2} + c = -\frac{1}{2s^2} + c$

25

$$\begin{aligned}\int x^7 - 4x^5 + \frac{1}{2}x^2 - 8\,dx &= \int x^7\,dx - 4\int x^5\,dx + \frac{1}{2}\int x^2\,dx - 8\int 1\,dx \\ &= \frac{x^8}{8} - 4\left(\frac{x^6}{6}\right) + \frac{1}{2}\left(\frac{x^3}{3}\right) - 8x + c = \frac{x^8}{8} - \frac{2x^6}{3} + \frac{x^3}{6} - 8x + c\end{aligned}$$

32 Since $\frac{x^3-1}{x^2} = \frac{x^3}{x^2} - \frac{1}{x^2} = x - x^{-2}$, we can rewrite the problem as $\int (x - x^{-2})dx = \int x\,dx - \int x^{-2}\,dx = \frac{x^2}{2} - \frac{x^{-2+1}}{-2+1} + c = \frac{x^2}{2} - \frac{x^{-1}}{-1} + c = \frac{x^2}{2} + \frac{1}{x} + c = \frac{x^3+2}{2x} + c$

34 Note that $5\sqrt{3}$ is just a constant, unlike $\sqrt{x}$. Thus
$\int \left(\sqrt{x} - 5\sqrt{3}\right) dx = \int x^{1/2}\,dx - 5\sqrt{3}\int 1\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 5\sqrt{3}\,x + c = \frac{2}{3}x^{3/2} - 5\sqrt{3}\,x + c$

35 $\int \left(0.9x^2 - 120x + 1600\right) dx = 0.9\int x^2\,dx - 120\int x\,dx + 1600\int 1\,dx$
$= 0.9\left(\frac{x^3}{3}\right) - 120\left(\frac{x^2}{2}\right) + 1600x + c = 0.3x^3 - 60x^2 + 1600x + c$

Section 5.2

6 If $\frac{df}{dx} = e^{3x}$, then f must be an antiderivative of $y = e^{3x}$, so $f(x) = \int e^{3x}dx = \frac{1}{3}e^{3x} + c$. Since we also want $f(0) = 0$, we have $0 = \frac{1}{3}e^{3(0)} + c = \frac{1}{3}(1) + c = \frac{1}{3} + c \implies c = -\frac{1}{3}$. Thus $f(x) = \frac{1}{3}e^{3x} - \frac{1}{3}$.

7 If $C'(x) = 18 - 0.012x$, then $C(x)$ is an antiderivative of $f(x) = 18 - 0.012x$. So $C(x) = \int(18 - 0.012x)dx = 18\int 1\,dx - 0.012\int x\,dx = 18x - 0.012\left(\frac{x^2}{2}\right) + c = 18x - 0.006x^2 + c$. Since $C(0) = 8500$, we can evaluate c: $8500 = 18(0) - 0.006(0)^2 + c \implies c = 8500$, so $C(x) = 18x - 0.006x^2 + 8500$.

11 If $h(x)$ = cumulative heat loss up to day x, then $\frac{dh}{dx}$ = rate of heat loss (per day) = $r(x) = 0.72x - 0.003x^2$ therms/day. Thus $h(x)$ is an antiderivative of the given function $r(x)$:

$$\begin{aligned}h(x) = \int r(x)dx &= \int (0.72x - 0.003x^2)dx = .072\int x\,dx - 0.003\int x^2 dx \\ &= 0.72\left(\frac{x^2}{2}\right) - 0.003\left(\frac{x^3}{3}\right) + c = 0.36x^2 - 0.001x^3 + c\end{aligned}$$

Since $h(0) = 0$ ($x = 0$ is start of season), we get $0 = h(0) = 0 + 0 + c$, $c = 0$ and $h(x) = 0.36x^2 - 0.001x^3$ for all $0 \le x \le 240$ ($x = 240$ = end of season). One-third of the way into the season ($x = \frac{240}{3} = 80$), the cumulative heat loss is $h(80) = 0.36(80)^2 - 0.001(80)^3 = 1792$ therms. At the end of the season ($x = 240$) the cumulative heat loss is $h(240) = 0.36(240)^2 - 0.001(240)^3 = 6912$ therms.

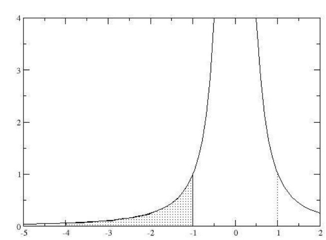
14 Net rate of inflow = (rate of flow in)−(rate of flow out) = $(7000-90t^2)-(10,000+180t)$ gallons per hour = $-3000-180t-90t^2$. Let $V(t)$ = volume in reservoir at time t; we are given that $V(0)=1.5\times10^7$. Now $\frac{dV}{dt}$ = net rate of inflow = $-3000-180t-90t^2$. So $V(t)$ is an antiderivative of this known function: $V(t)=\int(-3000-180t-90t^2)dt=-3000t-90t^2-30t^3+c$. Since $V(0)=1.5\times10^7$, we find c: $1.5\times10^7=v(0)=0-0-0+c \implies c=1.5\times10^7$; thus $V(t)=-3000t-90t^2-30t^3+1.5\times10^7$ gallons after t hours.

20 If $A(t)$ = total amount consumed after t years from 1960, we know that $\frac{dA}{dt}=r(t)=2e^{0.06t}$. Thus $A(t)$ is an antiderivative of the given function $r(t)$:
$A(t)=\int r(t)dt=\int(2e^{0.06t})dt=2\int e^{0.06t}dt=\frac{2}{0.06}e^{0.06t}+c=\frac{100}{3}e^{0.06t}+c$. When $t=0$, $A(0)=0$, so we get c: $0=\frac{100}{3}e^{0.06(0)}+c=\frac{100}{3}+c \implies c=-\frac{100}{3}$. So $A(t)=\frac{100}{3}e^{0.06t}-\frac{100}{3}=\frac{100}{3}\left(e^{0.06t}-1\right)$ for all $t\geq0$.
From 1960 to 1967 ($t=0$ to $t=7$), $A=A(7)=\frac{100}{3}(e^{0.06(7)}-1)\approx17.399$
From 1960 to 1970 ($t=0$ to $t=10$), $A=A(10)=\frac{100}{3}(e^{0.06(10)}-1)\approx27.404$
From 1962 to 1968 ($t=2$ to $t=8$), amount consumed = $A(8)-A(2)=\frac{100}{3}\int(e^{0.06(8)}-1)-\frac{100}{3}(e^{0.06(2)}-1)\approx16.286$.

27 Acceleration is $a=\frac{dv}{dt}=300-2t$; then v is the antiderivative of the given function $a(t)$: $v=\int a(t)dt=\int(300-2t)dt=300t-t^2+c_1$ (we put a subscript on c because we will have another such constant in this problem). Since $v=0$ when $t=0$, we get $c_1=0$ and $v=300t-t^2$. At burnout ($t=90$), velocity is $v(90)=300(90)-(90)^2=18,900$ ft/sec. As for height $h(t)$, we have $\frac{dh}{dt}=v(t)=300t-t^2$, so we find h as an antiderivative, $h(t)=\int v(t)dt=\int(300t-t^2)dt=150t^2-\frac{1}{3}t^3+c_2$; since $h(0)=0$, we have $c_2=0$. At burnout ($t=90$), $h=h(t)=150(90)^2-\frac{1}{3}(90)^3=972,000$ ft.

Section 5.3

22 First find the indefinite integral $F(x)=\int(e^{x/2}-\frac{3}{x}+4)dx=\frac{1}{1/2}e^{x/2}-3\ln x+4x=2e^{x/2}-3\ln x+4x$. Then, by definition, the definite integral is
$\int_1^2(e^{x/2}-\frac{3}{x}+4)dx=[F(2)-F(1)]=[2e^{2/2}-3\ln2+4(2)]-[2e^{1/2}-3\ln1+4]=2(e-e^{1/2})-3\ln2+4\approx4.0597$ since $\ln1=0$.

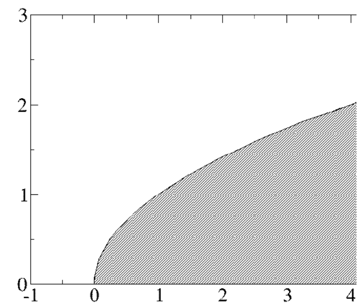


25 The definite integral $\int_{-4}^{-1}\frac{1}{x^2}dx$ gives the area of the shaded region shown in the figure at right. To evaluate the definite integral, first find the indefinite integral $F(x)=\int\frac{1}{x^2}dx=\int x^{-2}dx=\frac{x^{-2+1}}{-2+1}=-\frac{1}{x}$. Then, by the definition of the definite integral,

$$\int_{-4}^{-1}\frac{1}{x^2}dx=F(-1)-F(-4)=\left[-\frac{1}{x}\Big|_{-4}^{-1}\right]=\left[-\frac{1}{-1}-\left(-\frac{1}{-4}\right)\right]=1-\frac{1}{4}=\frac{3}{4}.$$

This is the area of R.

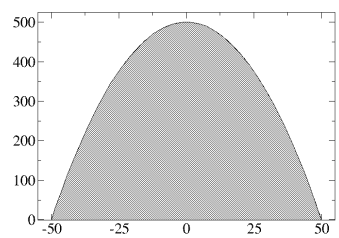
28 The definite integral $\int_0^4 \sqrt{x}\,dx = \int_0^4 x^{1/2}dx$ gives the area of the shaded region shown at right. First find $F(x) = \displaystyle\int x^{1/2}dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \dfrac{2}{3}x^{3/2}$. Then, by definition of the definite integral,



$$\int_0^4 \sqrt{x}\,dx = \left[F(x)\Big|_0^4\right] = F(4)-F(0) = \frac{2}{3}(4)^{3/2}-\frac{2}{3}(0)^{3/2} = \frac{2}{3}(8)-$$

This is the area R.

37 We want the area of shaded region in the figure on the right. It is given by the definite integral $\left(\displaystyle\int_{-50}^{50} 500 - \frac{x^2}{5}\right) dx$. First we find $F(x) = 500x - \frac{x^3}{15}$. We may then find the definite integral (and the area):



$$\text{Area} = \int_{-50}^{50}\left(500 - \frac{x^2}{5}\right) dx = \left[F(x)\Big|_{-50}^{50}\right] = 500(50)-\frac{50^3}{15}-\left[500(-50) - \frac{(-50)^3}{15}\right] \approx 66,666 \text{ square feet.}$$