Previously Asked Oral Questions
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Contents

1 Introduction

The bulk of this document was created by June-Yup Lee and others at the end of 1995, and was later edited by Matt Wilkins, and later Chris Wendl.

It is precisely what its title indicates, a list of previously asked questions from the oral exams, nothing more and nothing less. In other words, you shouldn't treat this as a syllabus for the orals, but on the other hand each of these questions has actually been *asked* at some student's oral exam. Some of the older questions in here will probably look rather difficult, as there's a tendency for people to remember mainly the questions they had trouble with and add those in particular to the list (In recent years, as questions have been collected immediately after the exams, this is less true.) Don't be daunted: in all likelihood you will be asked mostly the basics of your general subjects and only a few tricky questions.

Often in the orals you will be asked a bunch of questions all on a similar theme—most instances of this in the following have not been broken up into separate questions.

2 Real analysis

2.1 Measure and Integral

October 2006

1. State the ergodic theorem. Can you give a more general version? What conditions should you impose on the transformation in addition to its being measure-preserving?
2. Prove the Riemann-Lebesgue lemma.
3. How do you know $C^\infty([0,1], dm)$ is dense in $L^1([0,1], dm)$?
4. What is the Vitali covering lemma?
5. Prove the Holder inequality.
6. What is an L^p space? What is L^2 special? Give an inner-product on L^2? Why is it good? Give an example of an orthonormal basis? What type of convergence is there for an expansion of a function in terms of an o.n. basis?

October 2005

1. What is uniform integrability?
2. What is the Radon-Nikodym Theorem? Does it hold if one measure is absolutely continuous w.r.t. another measure on merely a field (not necessarily a σ-field? (Ans: No, take your measure to have the Cantor Ternary function as its distribution)

3. What is the Dominated Convergence Theorem? Can you give an example of a function which converges point-wise but not in L^1? State and sketch a proof of the Monotone Convergence theorem and Fatou's Lemma.

April 2005

1. What is a Borel set? Can you give a set that isn't Borel? Is it possible to construct a non-Borel set without using the axiom of choice?

2. Given some σ-algebra of sets in a non-locally-compact space (e.g. the unit sphere in an infinite dimensional Hilbert space) and a finite subadditive measure, can you still prove the big three convergence theorems for integration?

3. What results from real/functional analysis relate the integration theory back to the topology of the space? (Ans: Riesz-Markov on $C(X)$. The topology of X (locally compact, compact, or neither) determines when the functionals on $C(X)$ are given by integrating against a measure.)

4. Why does Fatou's thorem only hold for positive functions? (Ans: counterexample)

5. What are L^p spaces? Do you have inclusions of L^p to L^q? When do you have them? (Ans: finite measures) Prove this and find the embedding constant (i.e. $\| f \|_p < C \cdot \| f \|_q$)

6. What is a measure? What is an L^p space? When can you embed L^p into L^q? What is Riesz representation?

February 2005

1. State Fubini's Theorem. Given these conditions, how do you know that the integral of $f(x, y)$ with respect to y is x-measurable?

2. Give an example of an uncountable set of Lebesgue-measure zero; prove it has these properties.

3. Provide a natural probability measure supported on the Cantor set (Hint: can you use the "middle-thirds" construction to come up with an increasing function with derivative zero a.e.?) How do you prove the aforementioned function is continuous?

October 2004

1. State the dominated convergence theorem. How do you prove it?

2. Why is L^2 distinguished among L^p? (Ans: L^2 is a Hilbert space). Can you show they are different? (Ans: Give an example of function that is L^2 but not L^1). Can you give an example that does not blow up at 0?

3. Consider $f : [0,1] \to \mathbb{R}$, $\frac{df}{dx} \in L^2$. What can you say about f? (Ans: f is Holder continuous; apply Holder inequality to the derivative). For what other L^p spaces does this work? What if f is defined on $[0,1]^2$?

April 2004

1. What's a measurable set? (Ans: give Carathéodory condition)

2. Are there any nonmeasurable sets?

3. Are uncountable unions of measurable sets measurable? (Ans: no, you could get any subset this way.)

4. Suppose f is integrable. Is $\frac{d}{dx}\int_a^x f(t)\, dt$ equal to f? (Ans: yes, almost everywhere.) Is the integral of f' what you think it should be? (Ans: iff f is absolutely continuous.)

October 2003

1. Tell us about L^p spaces: why not $p < 1$? Show that you can approximate by continuous functions.

2. Define Lebesgue measure

3. State the dominated convergence theorem. Can you give an example of when it does not apply? (Ans: $f_n = n\chi_{[0,1/n]}$)

October 2002

1. Radon Nikodym theorem: how to prove it.

2. What are Lebesgue points? How do you prove the fundamental theorem of Lebesgue integration?

April 2002

1. State the dominated convergence theorem. Use it to show that if a random variable X is integrable, the derivative of its characteristic function at 0 is $iE(X)$. (this person's outline also included probability)

2. Construct the Lebesgue measure. Define measure, etc.. How do you show that the Lebesgue measurable sets include the Borel sets? (Ans: show that intervals are measurable; key fact is compactness of $[a,b]$.)

February 2002

1. Does monotone convergence hold if you drop the positivity assumption, but add the assumptions of integrability of each function in the sequence, as well as the limit?

October 2001

1. How are the Lebesgue measurable sets in $\mathbb{R}$ related to the Borel sets? (Ans: any measurable set is Borel up to a set of measure 0.)

February 2001

1. Lusin theorem for non-compact space. Is it true? (Ans: Yes) Prove it.

April 2000

1. Measure of Cantor set. What if you delete middle thirds sometimes, but middle quarters other times. (Look at the ratio of new set measure to old set measure.)

February 2000

1. Do you know an example of an uncountable set with zero measure? [Cantor set.] How do you know that it has zero measure? How do you know that it is uncountable? Would it make any difference if, instead of the middle third, the middle fifth of each interval were dropped at each step? [No.]

2. What does the Dominated Convergence Theorem state? Give a counter example for the case when the dominated hypothesis is violated.

October 1999

1. Define Lebesgue measure. Integrate a characteristic function.

April 1999

1. Why do we need Lebegue integral? Give an example of L intergable but not R intergable.

2. Is a set of zero measure nessarily countable? Give an example. Prove contor set is not countable.

3. What is Lebesgue measure? What condition is equivalent to the countable additivity? What properties the space must have to ensure that a measure constructed in this way have nice properties (= is countably additive)?

4. What are the main properties of Lebesgue integral? When does the integral of a series of non-negative integrable functions converges to the sum of the integrals? (use of monotonic convergence) What happens with monotonic convergence theorem when sequence of functions is decreasing?

October 1998

1. Why do they say that the rationals are countable? I answered: because one can count them! How? Are reals countable? Why not? (the diagonal trick...)

2. Prove Borel-Cantelli Lemma (both versions)

3. State and prove Egoroff's theorem
4. Give an example of a function which is continous ae but not differential ae

Pre 1996

1. Define measure. Prove measure of rationals = 0.
2. Actually calculate $\int \chi_Q$.
3. Is there any uncountable set of measure 0? A: Yes, Contor set.
4. What is a Borel set? Give an example.
5. Given set in plane, define its measure. A: area.
6. Define integral.
7. Difference between Riemann and Lebesgue integrals?
8. Why do you use Lebesgue integral instead of Riemann? A: completeness of integrable function.
9. State Bounded convergence theorem, Fatou's Lemma, Monotonic and Dominated convergence theorem. A: Bounded – extension of integral for measurable function from simple function. Fatou's – integral of positive function is supremum of bounded function h. Monotonic – Positive measurable function. Dominated – $g + f$ and $g - f$ are positive.
10. State Fatou's lemma. A: $\int \liminf f_n \leq \liminf \int f_n$
11. Which functions can be applied for dominated convergence thm, Monotone convergence thm in order to interchange of limits of integral.
12. $f_n(x) \to f(x)$ uniformly and $\int_0^\infty f_n(x)dx = 1, \forall_n$, does $\int_0^\infty f(x)dx = \int_0^\infty f_n(x)dx$? A: No, e.g. $f_n = \begin{cases} 1/n, & 0 \leq x \leq n \\ 0, & x > n \end{cases}$
13. If $\int_0^\infty f(x)dx \leq \infty$, is $f(x)$ necessary bounded as $x \to \infty$. A: No.
14. Use Lebesgue domain convergence thm to evaluate $\lim_{n\to\infty} \int_1^\infty \frac{e^{-nx}}{x} dx = 0$
15. Differentiability of monotone function. When is a function an integral of another?
16. Does every measurable function have a derivatives?
17. What conditions does a integral of integrable function satisfy?
18. Bounded variation, continuous, absolute continuous: What implies what? Riemann & Lebesgue integral.

19. What is a function of bounded variation? How can it be represented? Is the Lebesgue integral differentiable?

20. What is absolutely continuous function? 2 definitions. What are its properties? Is it differentiable? Is every differentiable function absolutely continuous?

21. If differentiate then integrate, do we get the same function? – iff absolute continuous.

2.2 Topology of Functions

April 2007

1. Give an example of weakly convergent sequence which is not strongly convergent in L^2.

October 2006

1. What is weak convergence (in L^2 or L^p)? Does it imply a.e. convergence? Example. Egoroff's Theorem?

2. Consider a sequence of continuous function that map $[0, 1]$ to R. What can you say about relations among the following modes of convergence: a.e., L^1, and in (Lebesgue) measure/probability? Can you give an example that a sequence converges in measure but does not converge a.e.? Can you give a condition such that a.e. conv implies L^1 convergence? What is uniform integrability?

April 2005

1. What is Fourier series? How can you check that the formula you gave for the coefficients is correct? Why is $\{e^{2\pi inx}\}$ a basis in $L^2[0, 1]$?

2. Can you give applications of Fourier series in probability?

3. Can you say something about the asymptotic behavior of Fourier series coefficients? Why is the Riemann-Lebesgue lemma true? How and for what type of function do Fourier series converge (I said uniformly for C^2 and in L^2 for L^2). Can you assume less than C^2 for uniform convergence? If your function is C^k, what can you say about the decay of coefficients?

February 2005

1. What are L^p spaces? For which p are these Banach spaces? What is L^∞? Are the L^p spaces separable? How about L^∞? What is the dual of $L^{5/3}$? Is L^1 the dual of a space?

2. What theorems do you know about Fourier series convergence? What does smoothness of a function say about its Fourier coefficients? Are there similar results for the Fourier transform?

3. What is the relationship between pointwise convergence and convergence in L^p? What about weak convergence in L^p?

4. Why is L^2 complete? Suppose you have a closed linear subspace C of L^2 and a point f outside of it. How do you prove the existence of a unique projection of f onto C? Identify a countable dense subset of L^2.

5. Define Fourier series and state what conditions are needed for its absolute convergence. When does the series converge to the function (Ans: Holder with constant $a > 0$)?

October 2004

1. What is the decay of Fourier coefficients for $f \in C^m$? In what sense does the Fourier series converge to the function? What is special about the Fourier series of an L^2 function?

April 2004

1. Suppose f_n are integrable and $f_n \to g$ a.e.: then $\int f_n \to \int g$? (Ans: no, state dominated convergence theorem)

October 2003

1. What is a Fourier series? What conditions do you need for a Fourier series for f to converge to f (Ans: $f \in C^2$ will do it)

April 2003

1. What is a Fourier series? When does the Fourier series converge pointwise?

2. What is the bounded convergence theorem? Give an intuitive explanation for why it is true.

October 2002

1. Describe the L^p spaces in the unit interval. When are these spaces Banach spaces?

2. What's the dual of L^p for $p \geq 1$. How do you prove that?

3. What's the dual of L^∞? Show that this space is not separable.

4. Modes of convergence (a.e., measure, L^1), relations between then in general and under special conditions.

5. Do you know the Weierstrass approximation theorem? State it precisely. Convergence in what sense? What about non-compact sets?

6. Discuss trigonometric polynomial approximations to continuous functions. What condition do you need on a function on the interval $[a, b]$ in order to get a uniformly converging Fourier series? What if (i) $b - a < 2\pi$, (ii) $b - a > 2\pi$? How can you use Stone-Weierstrass to prove this result? Why doesn't it matter that the trig. polynomials don't separate 0 and 2π?

April 2002

1. Define weak convergence. Give an example. What conditions can be added to weak convergence in order to get strong convergence? How does it relate to pointwise convergence? Give examples.

February 2002

1. Let f_n be a sequence of continuous functions on [0,1] that are not only uniformly bounded (in the sup norm) but also have uniformly bounded continuous derivatives. What can you say about convergence?

2. Can you prove Arzela-Ascoli?

3. Consider a sequence of L^p functions uniformly bounded in the p-norm, all supported in some compact set, and also equicontinuous in the sense of L^p translation. What can you say about convergence?

4. Let ϕ be a C^∞ compactly supported function and the sequence $\phi_n(x) = \phi(x - n)$ defined on the line. Can you apply Arzela-Ascoli?

5. How do you define the Fourier transform of an L^2-function? Can you define it on a subset of $\mathbb{R}^n$ other than $\mathbb{R}^n$ itself? (Give an example.)

October 2001

1. State rigorously the different types of convergence. Discuss the relationship between convergence in L^p, almost everywhere and weak convergence.

April 2000

1. What is a Fourier series? What functions does it converge for? When does it converge uniformly? (Answers: L2(0,1) (Use Stone Weierstrass, if you have a smooth enough function, you get estimates for the derivatives).

2. Discuss and give examples of compact self-adjoint operators (Hilbert Space question). (Ans: $K \star f$) when is this self adjoint? (Ans: K s.a. - show!) Discuss compactness (use Cauchy-Schwarz & Ascoli Arzela).

3. Definition of weakly convergence in L2.

4. If a series of functions (fn) converges weakly to f and the L2 norm of fn converges to the L2 norm of f. Does this implies that fn converges to f in L2. What about convergence in a general Banach Space.

5. The definition of Fourier series, what can you say about the convergence. (I mentioned proved Holder implies uniformly convergence.)

6. Open mapping theorem and the proof.

7. Definition of Hardy space, and the difference of Hardy space and Lp theory.

8. Given $f : R^n\ toR^n$ smooth and $(f(x) - f(y), x - y) \geq c|x - y|^2$ Prove the uniqueness of $x' = -f(x)$

9. What is the story about duality between L_p and L_q?

10. Converse of Holder inequality, sketch of the proof.

February 2000

1. What is the Fourier Series of a function? [Hilbert-space basis for the space $L^2([0, 2\pi])$.] What is the inner product in this space? How do you show that the closure of the span is actually the whole space? What kind of convergence does one have? [From Hilbert space theory, L^2 convergence. But indeed also pointwise at points of continuity of f.] What about the end-points of the interval, if f is smooth but $f(0) \neq f(2\pi)$? [Convergence to $\frac{f(0)+f(2\pi)}{2}$.]

2. what are the different ways that a sequence of functions $f_n \to f$? (a.e., in measure, and L^p). which type of convergence implies which?

3. can you give an example of a convergence thm that gaurantees convergence of $f_n \to f$ in L^p? does $f_n \to f$ in L^p imply $f_n \to f$ a.e.? does $f_n \to f$ in L^p imply subsequence of $f_{n_k} \to f$ a.e.?

April 1999

1. Under what condition does the fourier series of a fn converge uniformly. Prove it.

October 1998

1. What is a compact set? What are compact sets in R^n? (closed and bounded) Is it always the case? What about the continuous functions on $[0, 1]$? Is this a normed space? Why is the sup norm nice? (completeness) What about the compact sets there? (arzela-ascoli) What about $L^2(0, 1)$? (same as arzela-ascoli, but equicontinuity of the shift operator..)

Pre 1996

1. Prove Holder and Minkowski inequalities.

2. Prove L^p is complete. A: linear normed space is complete iff all absolutely summable is summable.

3. Prove dual space of L^p is L^q where $1/p+1/q = 1$. A: $\int |fg| \leq M\|f\|_p, \forall f \in L^p$ iff $g \in L^q$.

4. Arzela-Ascoli thm.

5. Define equicontinuous of a set of functions. State a thm about it. A: Ascoli Arzela. Application? Cauchy-Peano existence thm.

6. Consider family of functions : what is condition for equicontinuous? – that derivatives be uniform bounded.

7. If sequence of continuous functions converge uniformly to continuous function, is the sequence equicontinuous? A: Yes.

8. What is countable, separable metric space?

9. Is a metric space always normable so norm gives back metric? A: No, $d(cx, cy) = cd(x, y); d(x + a, y + a) = d(x, y)$

10. Unit sphere in Hilbert space: compact $\Leftrightarrow$ finite dimension.

11. Topology of function space.

12. Necessary & sufficient condition for a subset of $C[0, 1]$ to be compact in maximum norm?

13. If sequence of L_1 functions $\{f_n\}$ whose $\int f_n \to 0$, Is there a subsequence $\{f_{n'}\}$ which $f_{n'} \to 0$ a.e. ?

14. Norms for continuous functions. Under which norm are they complete? A: maximum norm since uniform convergence.

15. Does set of all harmonic functions $\{u(x, y)\}$ in a region D with inner product $(u, v) = \int_D (u_x v_x + u_y v_y) dxdy$ form a complete Hilbert space?

16. What is a modulus of continuous function? Find it for $y = \sqrt{x}$ in interval $0 \leq x \leq 1$. A: $w(\delta) = sup_{|x-y|\leq\delta}\{|f(x) - f(y)|\}$

17. If a function has k continuous derivatives, is there a sequence of polynomial converging uniformly to it and whose 1st derivatives converging uniformly to that of the function? A: Yes.

2.3 Functional Analysis

April 2007

1. Hilbert spaces. What condition is there for an operator to be bounded? Give an example of unbounded operator.

2. Is Laplacian invertible on $L^2([0, 1])$ (defined on a dense subset) (no it has kernel) how about laplacian - Identity? (yes) .

April 2000

1. Radon-Nikodym Theorem

Pre 1996

1. Fubini's thm. Covering thm.

2. Baire category thm with 3 applications. (i) Principle of uniform boundness (ii) existence of continuous nowhere differentiable function (iii) rationals is not complete.
3. Prove Brouwer fixed point thm in 2 dimension.
4. Example of non-measurable set, axiom of choice, Zermelo's principle?
5. State Riesz Fischer thm. State and prove Hahn Banach thm.
6. Radon-Nikodym thm.

3 Harmonic Analysis

April 2007

1. Fourier transform in L^1. Why is it well defined? Any properties, for example, continuous. How to prove it? (This related to dominated convergence theorem).

October 2006

1. Do you know Calderon-Zygmund-Decomposition? (A rough idea was OK).

April 2005

1. What is a Fourier series? What is your favorite convergence theorem? If the Fourier series converge, how do you know they converge to the function?
2. Name some equivalent ways of showing the completeness of $e^{2\pi inx}$ in $L^2(T)$.

February 2005

1. State the heat equation. How do you find solutions?

October 2004

1. What is a Fourier series? Why is it good? (Ans: It converges to the function if the function is "good enough"). Elaborate.

October 2002

1. Consider a continuous function f on $[a, b]$ (don't assume this to be $[0, 1]$). What's a necessary condition for pointwise convergence of a series of the form $\sum a_n \sin nx + \sum b_n \cos nx$ to f on $[a, b]$? Now let $f(x) = x$. If the interval $[a, b]$ is small enough, can you get uniform convergence of such a series to f on $[a, b]$?

2. Consider $f(x) = x$ on $[0, 2\pi]$, extended by periodicity to the real line. Does the Fourier series converge uniformly? Why not? Does it converge at 2π? Does it converge uniformly on a compact subinterval of $(0, 2\pi)$?

3. Solve the boundary value problem for the heat equation $u_t = u_{xx}$ on $[0, 2\pi] \times [0, \infty)$ with boundary condition $u(x, 0) = f(x)$.

April 2000

1. What is the Fourier Series all about? (find a complete orthonormal system of $L^2[0, 1]$, and do an expansion for smooth function. and approximate L^2 function by smooth ones)

2. When does the Fourier Series converge? (e.g if f is C^1, it converges uniformly.)

3. Gibb's phenomenon

4 Hilbert Spaces

April 2000

1. What is the spectral theory all about?

2. Give an example of a self-adjoint operator on $L_2[0, 1]$? (I gave an operator defined as an integration against a continuous kernel)

5 Functional Analysis

People taking real analysis, but not functional analysis, have been asked functional analysis questions (see §2.3). However the questions in here are from people taking functional analysis as an entire subject.
October 2006

1. Construct a continuous function ϕ from the real line to an infinite dimensional Hilbert space such that for $a < b < c < d$, $\phi(b) - \phi(a) \perp \phi(d) - \phi(c)$.

2. Consider a bounded linear operator from a Banach space to another with a range that has finite codimension. Can you conclude that the range is closed? (Ans: No, consider $f \mapsto \int_0^x f(x)dx$ from $L^1([0, 1])$ to itself, the range contains all absolutely continuous functions which are dense in L^1, hence codim is 0). What if the range has finite algebraic codimension, i.e. every vector can be written as a unique linear combination of the space and a finite dimensional complement? (Answer: Yes) Can the space have a countable Hamel basis?

3. Is $L^\infty([0, 1], dm)$ the dual of any space? Same question for $L^1([0, 1], dm)$?

4. What is C_0? (Sequences converging to 0)?
5. What is the dual of L^p, $0 < p < \infty$?

6. What is weak convergence? Weak star? Do you know criteria in a Banach space for weak star convergence?
7. Do you know conditions in L^p for a weakly convergent sequence to converge strongly?
8. What is the relation of weak star convergence to convergence of probability measures? What is tightness? Give an example. Give an example to show that tightness is necessary.

October 2005

1. Using Hahn-Banach, prove that $\overline{Ran(A)} = Ker(A')^{\perp}$ for any bounded linear operator A.
2. Talk about the inverse of the Laplacian. What nice properties does it have?
3. What is a separable space? Give examples. What is a countable dense subset of $L^2([0,1])$?
4. Why is L^∞ not separable?
5. What's the dual of $(C([0,1]), \|\cdot\|_\infty)$? In what sense is the unit ball of the dual compact?
6. State the Uniform Boundedness Principle and the Closed Graph Theorem. What's a closed operator?

April 2005

1. What is the Open Mapping Theorem? Can you give applications?
2. Assume that you have a bounded mapping from a Banach space to a Banach space with finite codimension of the image. Prove that the range is closed.
3. Define a compact operator and give an example (Ans: integration against a nice kernel). What is a Fredholm operator? What is the Fredholm alternative, and what is the range in the second case of the Fredholm alternative (Ans: annihilator of $Ker(I - T^*)$).
4. State Hahn-Banach. Can you prove Hahn-Banach for positive linear functionls as defined on a subspace $C(X)$ with X compact?
5. What are the positive linear functionals on $C(X)$ with X compact that are also multiplicative, i.e. $l(f \cdot g) = l(f) \cdot l(g)$? (Ans: Dirac masses at one point)

6. What is a Banach space? In the definition of L^p, why is it not a Banach space if you don't take equivalence classes?

7. What is the Hahn-Banach theorem? Is $L^1[0,1]$ the dual of a Banach space? How about l_1?

October 2002

1. Let U be an open bounded subset of $\mathbb{C}$ with smooth boundary. Do the analytic square integrable functions form a Hilbert space? (yes)

2. How do you know that if the dual of a Banach space is separable, the space itself is separable?

3. State the Hahn-Banach theorem.

4. Use Hahn Banach to prove there exists a bounded linear functional on the integers which is not a radon measure.

April 2002

1. State the Hahn-Banach theorem and give an application (to differential equations if possible).

2. How do you know that sines and cosines form a complete orthonormal basis in L^2?

3. State the Stone-Weierstrass theorem and show that sines and cosines on $[-\pi, \pi]$ satisfy the hypotheses (actually they don't, you have to consider S^1 instead of $[-\pi, \pi]0$.

4. Can you state the spectral theorem for bounded self-adjoint operators? (Ans: no, but I can state it for compact self-adjoint operators.) Go ahead, and define all the words involved in your definition.

5. Give an example of a compact self-adjoint operator. (Ans: integration against a nice kernel.) How do you prove it's compact? (Arzela-Ascoli.) What can you say about its eigenvalues and eigenspaces? (Only accumulation point of eigenvalues is 0; eigenspaces are finite dimensional.) How could you find the largest eigenvalue? (Rayleigh quotient.)

February 2002

1. State your favorite theorem in functional analysis. (I gave Hahn-Banach for a general t.v.s.)

2. Look at the topological vector space $C[-1,1]$ and define $T(f) = f''(0)$ on the subspace $C^\infty[-1,1]$. Can you apply Hahn-Banach to this example?

October 2001

1. How can you prove that L^1 is not reflexive? (Ans: L^∞ is not separable, L^1 is. If L^1 is the dual of L^∞, the separability of the dual implies the separability of the original space.)

2. How are Banach spaces related to general normed linear spaces? (Ans: every normed linear space can be completed to a Banach space.)

3. Are all L^p spaces separable? Prove L^∞ is not.

October 1999

1. What is a Banach space?

2. What is a norm?

3. Give an example of bounded linear operator.

4. Given $\psi \in C_0^\infty$ and $f \in L^2$. Is the convolution between this two functions a bounded linear operator? Compute its norm.

October 1998

1. What conditions are needed to satisfy Open-Mapping Theorem?

6 Lie Groups

February 2000

1. What is SU(2) as a toplogical space

2. Provide a sujective homomorphism $f : SU(2) \to SO(3)$ that is a covering space map.

3. Say something about π_2(comapct lie group)

4. Describe the structure of a compact lie group

5. Mention a characterization of semisimplicity of a lie group

6. Relation between group character and fourier analysis

7 Algebraic Topology

April 2007

1. What is relation between homology groups and homotopy groups?

2. Can you define a natural map from π_n to H_n which will generalize to map from π_1 to H_1?

3. Can you give an example of space with finitely generated fundamental group and a countable collection of simple closed curves, none of which are homotopic to each other? (think of $\mathbb{R}^2$ minus 3 points).

4. Can you give an example of 2 spaces with same homology groups but not homotopy equivalent? ($S^2 \times S^2$ and $\mathbb{C}P^2 \wedge S^2$). What are homology groups in each case?(think of cell structure!) Why they arent homotopy equivalent? (different cohomology rings.)

5. What is ring structure on cohomology?

October 2005

1. What is RP2? Is it orientable? Define "orientable". Compute homology groups for RP2. What are its covering spaces? What is the general relation between the fundamental group of a covering space and of a space itself? What about the Klein bottle? What are its coverings and fundamental group?

2. Brouwer fixed point theorem.

April 2005

1. Give the relationship between homology and homotopy groups of a space and a covering space. Give a proof.

February 2005

1. What do you know about the fundamental group in relation to the first homology group (e.g. which one is larger, which way would a surjective homomorphism go?). Can you give an example of when the fundamental group is abelian and when it is not?

2. (Continuation) Suppose you have a compact orientable 2D manifold. How would you compute its fundamental group? Is it abelian?

3. What is the relationship between the second homotopy group and second homology group?

October 2004

1. Is there any space with a finite fundamental group? (Ans: the real projective plane). What is its fundamental group? (Ans: Z_2). How do you know that? (Because the 2-sphere is a two sheet cover, and it is simply connected). What is the relation between universal cover and fundamental group? Do you know of any finite fundamental group other than Z_2?

2. What is the fundamental group of a product of two spaces? (Ans: the product of fundamental groups). Is it true for homology groups? (Ans: no). Can you give an example? (Ans: the torus)

3. Can you map a torus injectively into a sphere? (Ans: no) Why not?

4. Define degree. Can you find a map from the torus to the sphere of degree 1?

April 2004

1. Can you tell me anything about higher homotopy groups? (Ans: they're abelian, and trivial if X has a contractible universal cover.) How do you know that? (Ans: apply algebraic lifting criterion to long exact sequence of homotopy groups)

2. Can you relate $\pi_k(\tilde{X})$ to $\pi_k(X)$ in general when $k \geq 2$? (Ans: p_* is an isomorphism.) Apply this to $S^1 \vee S^2$ (Ans: with much prompting, discussed how to show via universal cover and Hurewicz isomorphism that $\pi_2(S^1 \vee S^2)$ contains $\oplus_{i=1}^{\infty} \mathbb{Z}_i$)

October 2003

1. Tell me about 2-manifolds—homology, cohomology, fundamental group etc. (Ans: classify closed 2-manifolds into orientable/nonorientable with genus g)

2. What's the ring structure on cohomology?

3. What is a suspension?

4. What link is there between the homology of the suspension of X and the homology of X? (Ans: all except bottom homology gets shifted up a dimension)

October 2002

1. Give an example of a topological space with a noncommutative fundamental group. Provide proof of noncommutativity. [lift to covering space, use Seifert Van-Kampen theorem]

2. Calculate the homology of the Klein bottle by writing down its cell decomposition.

3. Can you give an example of two elements in a fundamental group that don't commute? [figure-eight.] How do you know they don't commute? [π_1 is a free group on two generators, which is nonabelian.] How do you know this? [Van Kampen.] Can you give a more elementary reason involving covering spaces why these elements don't commute? [if you lift to the universal cover you get different endpoints.]

4. Can you state Van Kampen's theorem? What is the necessary assumption on $A \cap B$ in the statement? [path-connectedness] Can you give an example where $A \cap B$ is not path-connected and the theorem fails? [S^1 decomposed as $D^1 \cup D^1$.]

5. Is there something analogous to Van Kampen's theorem in homology? [Mayer-Vietoris.] State Mayer-Vietoris. Can you describe the "boundary map"? Describe it in terms of maps arising in the long exact sequence of a pair.

6. What is Euler characteristic? [alternating sum of Betti numbers.] If you have a triangulated space, can you give an alternate description of Euler characteristic? Can you use Mayer-Vietoris or another method (e.g. triangulation) to describe the Euler characterstic of $A \cup B$ in terms of that of A, B and $A \cap B$? [$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$.]

February 2001

1. Relationship between the fundamental group and the first homology group? (Ans: the second is the abelianization of the first.) Find an example of a topological space that has A_5 (the group of even permutations of 5 elements) as its fundamental group. Is there any similar relationship between π_n and H_n?

April 2000

1. Fundamental group of torus

2. The relation between H_1 and π_1

8 Differential Geometry

April 2005

1. What can you tell me about Gauss-Bonnet? What is the Gaussian curvature of a surface? Draw surfaces of positive and negative curvatures. Does it depend on the particular way the surface is embedded in space? What is the Euler characteristic of the torus? Looking at the torus embedded in space as a donut, where are regions of positive and negative curvature?

February 2001

1. Curvature tensor. Gauss theorem.

2. Is there a metric on the sphere with negative curvature? (Ans: no, by Gauss-Bonnet)

February 2000

1. Describe the tensor product bundle over a manifold

2. Describe affine connection

9 Algebra

April 2007

1. State Sylows theorems. Is it true that any two p subgroups of a group have to be conjugates if they are not Sylow subgroups?(no). Give counterexample.(look at any abelian group).

2. What are all the finite fields and what are finite extentions of finite fields and their galios groups?(cyclic galois groups).

3. Any ideas about how to give a metric on a group which is finitely generated? (just mentioning maybe lengths of smallest words would be fine).

February 2005

1. Conjugacy classes of S_n. Subgroups of S_n. Show A_n is normal in S_n.
2. What is the Galois group of a finite extension of a finite field? What is the fundamental theorem of Galois? Give an example of an extension of the rationals that has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as its Galois group.

February 2001

1. Classify all finite groups with 12 elements.
2. Galois theory: find an extension that has $\mathbb{Z}_3$ as Galois group.

February 2000

1. Form the tensor product of 2 vector spaces and discuss its universal mapping property.
2. Whats the fundamental theorem of abelian groups
3. State the Sylow Theorems and inicate their proofs.

10 Complex variable

10.1 Cauchy Integral

April 2007

1. What is a winding number? How do you compute it? Prove it.
2. Give brief argument for proving the definition of index of a curve. What kind of singularities are there and how do you remove a removable singularity?(I gave a proof with moreras theorem).

3. What homological condition can you give for two curves in a domain(in $\mathbb{C}$) to have the same integral with respect to any holomorphic function in that region?(the index of both curves should be the same with respect to any point outside the domain).

October 2006

1. Compute $\int_0^\pi \frac{d\theta}{a+\cos\theta}$ for $a > 1$.
2. $H(\Omega)$ is the set of functions analytic in Ω. Define the L^2 norm on $H(\Omega)$. Is this a Hilbert space?
3. What is a function of a complex variable? What is an analytic function? Prove Cauchy's theorem.
4. Suppose an entire analytic function grows like a polynomial at infinity. What is it?
5. Evaluate the integral of $\sin(x^2)$ on the positive x-axis.

October 2005

1. Develop Taylor series for an analytic function.
2. Define $\arctan(z)$ analytically.
3. Argument principle. How can you develop it to prove the Fundamental Theorem of Algebra?
4. Prove Cauchy's Theorem with Green's formula and the Cauchy Riemann equations
5. What is the definition of an analytic function? Give a property that is equivalent to being analytic, and prove one direction of the equivalence (Student chose integration over closed contours, and proved Cauchy's theorem for a rectangle)
6. Give the different types of isolated singularities. For each type, if $f(z)$ has a singularity of this type at z_0, what about $1/f(z)$?
7. What does analytic mean? What are the Cauchy-Riemann equations and how do you derive them? Describe the path from the Cauchy-Riemann equations to concluding that an analytic function can be written as a power series.

February 2005

1. State Cauchy's theorem and integral formula. Is there a converse? (Ans: Morera's theorem). Outline a proof of this converse.

2. What is an analytic function? What is a harmonic function? ($\triangle u = 0$). Given u harmonic, is it always possible to find an analytic function with $Re(f) = u$? Do you know another characterization of harmonic functions? (Property of the mean). Can you show these definitions are equivalent?

3. Give a proof for Cauchy's integral theorem, both for $f \in C^1$ (complete) and without this assumption (sketch). Is there a special local property that arises as a consequence of the theorem, and can you prove it? Is it valid if γ is an arbitrary curve rather than a circle?

4. Integrate $\sqrt{x}/(1+x^2)$ on $[0, \infty)$.

5. What is the winding number? If you are given some curve, write down an integral that counts the winding number around, say, $z = 0$. Develop this idea to give a proof of the fundamental theorem of algebra without using Liouville's theorem.

6. Define an analytic function (Ans: complex differentiable). What are the Cauchy-Riemann equations and what is their connection with complex differentiable functions? State Cauchy's integral formula. State Liouville's theorem and how to prove it.

October 2004

1. State Cauchy's integral formula. What does it tell you about the behavior of an analytic function on a disk? Is the same true for a differentiable real function? For which real functions is it true?

2. Let u be harmonic on the plane, and suppose u is bounded. What is u? If u is harmonic on a closed disk, what can you say about ∇u? Suppose $u(0,0) = 0$. What does the zero set of u look like near $(0,0)$?

3. Consider a Jordan curve Γ in the complex plane. Consider $I = \int_\Gamma \frac{p(z)}{z - z_0}$, for p entire. What is $\lim_{z_0 \to a} I$, for $a \in \Gamma$.

April 2004

1. What does the residue theorem say? Can you find $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\, dx$? What about $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$?

2. What is Rouché's theorem and how would you use it? What do you need on the domain inside γ? (Ans: simply connected)

October 2003

1. What's the Maximum Modulus Theorem? What if $f(0) = 0$? (apply theorem to $f(z)/z$)

2. What if, in the disk, $f(z) \le M$ in the UHP, $f(z) \le m$ in LHP, for $m < M$? What do you know about $f(0)$? HINT: look at $g(z) = f(z)f(-z)$.

3. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx$.

4. Let γ_R by a semicircle of radius R in the upper half plane: how does $\displaystyle\int_{\gamma_R} \frac{e^{iz}}{1+z^2}\, |dz|$ die off?

5. Define an analytic function

6. What is Cauchy's theorem? Do you need f to have continuous second derivatives? Prove it in the case Ω is a rectangle (Goursat's proof)

7. If a function is analytic on $B_R(0)$, show that its Taylor series converges there.

April 2003

1. Find the Fourier transform of $1/(1+x^2)$. Wait, before you start, what is the decay like at ∞? (Let's throw in some functional analysis...) Prove it is continuous. If you put an arbitrary polynomial in the denominator, what can you say about the decay?

2. Write down some version of Cauchy's theorem.

3. How can you quickly prove Cauchy's theorem if you know that f is, say, entire?

October 2002

1. Classification of singularities. Coefficients of the Laurent series.

April 2002

1. Cauchy's theorem, integral formula, residues. Use them to compute $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x-i} dx$.

2. Classify singularities. Prove that an analytic function can be continued over an essential singularity.

3. Give an example of an essential singularity. What does Picard's theorem tell you?

4. Just out of curiosity, do you know how to prove that $\sum 1/n^2 = \pi/6$? (Ans: there are at least two proofs: one using a Fourier series, the other via contour integration.)

February 2002

1. When can one have an accumulation of singularities?

2. How do you develop the Laurent expansion?

3. Calculate $\int_{-\infty}^{\infty} \frac{\sin x \ dx}{x^2+1}$

4. What is the residue theorem?

5. What types of singularities are there? Classification of isolated singularities.

6. If a function has absolute value greater than 1 near a singularity, what kind of singularity can it be?

7. State Poisson's formula. How do you derive it? (any approach; I used separation of variables in polar coordinates.)

8. Prove that for a continuous boundary condition on the unit circle, the function defined by Poisson's formula is continuous onto the boundary (think about the approximate identity argument here).

9. Give an example of a function that cannot be continued analytically past its radius of convergence at any point on the circle of convergence. (One way is to use Poisson's formula with the right boundary condition.)

October 2001

1. Given a nice function in a punctured unit disk, what could happen at the origin? (Ans: removable singularity, pole or essential singularity.) How do you decide which one it is?

2. What can we say about a function near an essential singularity?

3. What can we say about a function in the punctured disk if $\lim_{z\to 0} |z|^{2/3}|f(z)| = 1$? (Ans: it never happens.) Why? (Ans: this implies a pole of order 2/3, but poles are always of integer order.) How do you prove that? (Ans: I don't remember.) Try Laurent expansion. How do you find the coefficient of $1/z$? How about the coefficient of $1/z^2$? Continuing in this manner, what can you conclude from the limit?

4. How does one remove a removable singularity? (assuming function is analytic and bounded in its neighborhood.) Ans: Use Cauchy integral formula to redefine the function, and show it is analytic everywhere and coincides with the original function everywhere.

5. Given f analytic in punctured disk and bounded near 0, prove it can be extended to 0.

April 2000

1. Cauchy's integral formula - what is it? what curves & domains?

2. Liouville's Thm -prove. Why is a function with 0 derivative constant?

3. What does the Cauchy-Goursat Theorem say. Prove it. (In a rectangle)
4. Name an application of Cauchy-Goursat Theorem. I named Cauchy's Integral Formula.
5. Prove Cauchy's Integral Formula.
6. Name an application of Cauchy's Integral Formula. I named the argument principle.
7. Prove the argument Principle.
8. If f=u+iv is analytic, what can you say about the curves u=constant and v=constant. (Ex, 0 is the zero of n-th order.)
9. angle between curves Re(f(z))=c1 Im(f(z))=c2
10. Weierstrass product expansion cos(z).

February 2000

1. What is your favourite theorem in Complex Variables? [Cauchy-Goursat.] Can you prove it? [As in Ahlfors, subdividing the rectangle into smaller ones.]
2. Consider a function f given by

$$f(z) = \int_{C_1} \frac{g(\xi)}{\xi - z} d\xi.$$

   Is f entire? [No. It is analytic inside and outside the circle, though.] And when would such an f be entire? [Silence...] Assume f is entire. What is the value of f when z goes to infinity? [Zero.] So? [So f must be a constant (by Liouville's thm.) and this constant must be zero.]
3. Can you prove the fundamental theorem of algebra? [I showed that there must be a root.] How do you know there are exactly n roots? [I quoted an elementary result in abstract algebra, that says that once one finds a root z_0, one can factor out $(z - z_0)$.]

October 1999

1. State and prove Cauchy's Thm for a rectangle.
2. Definition of an analytic function.
3. Can you state and prove Liouville's them?
4. Cauchy-Riemann equations, just write them down and say which one is mass conservation in 2- potential flow (person asked this was doing fluids)

April 1999

1. $f = u + iv$ analytic function. What is the angle between curves $u = const$ and $v = const$? What happens with the angle at the point where $f' = 0$? Consider example.

October 1998

1. How would you use analytic functions to find integrals? for example: $\int_{-\infty}^{\infty}(1 + x^2)^{-1}dx$. I mentioned and proved: cauchy formula.

2. What is a meromorphic function? What does it look like? What about meromorphic functions on the compactified plane?

Pre 1996

1. Prove Cauchy Theorem. A: With/without continuity of f'.

2. Cauchy-Riemann equations. Cauchy integral formula.

3. Is there a Weierstrass approximation thm for complex valued functions?

4. Algebraic functions and rational functions.

5. If Taylor series of analytic function has radius of convergence R, can the series converges at every point on boundary of convergence? A: Yes, e.g. $\sqrt{1 - z}$

6. Why do you introduce the Riemann surface? – Branch point.

7. What sort of singularity is a branch point? A: Can not be an interior singularity.

8. What is a singularity? A: expansion fails. Is a branch point a singular point? $\sqrt{z}$ about 1 stop at 0 so yes. A branch point is singular on certain Riemann surfaces and not on others. e.g. $\sqrt{\ln z - 2\pi i}$.

9. Riemann surface for $\sqrt{\log z - 2\pi i}$: $\sqrt{w}$ has boundary point at $w = 0, \infty$ so if define $\log 1 = 0 \Rightarrow \log z = \log r + iQ$ then boundary point at $z = exp 2\pi i$ but not at $z = 1$. So for $\sqrt{z}$ boundary point is same for all sheets, but for $\sqrt{\ln z - 2\pi i}$ varies from sheet 1 to sheet 2.

10. Requirement for analyticity?

11. In Cauchy integral formula, how do you integrate if boundary curve isn't recitifiable. A: you don't

12. $f(z)$ analytic in D. What can you say about f in D if (a) $f(z)$ pure real on ∂D (b) $Imf(z) =$ const. on ∂D. A: $f \equiv$ const in D.

13. Prove Liouville's thm by Schwarz' lemma.

14. Extend Schwarz' lemma to case where 1^{st} k derivatives vanish at $z = 0$

15. Prove Harnack's inequality for a non-negative function.

16. State and prove Morera's thm.

17. Prove Rouche's thm.

18. Residue thm; Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

19. How many zeros does $z^8 + 2e^z$ have in the unit disk? State Rouche's thm.

20. Mittag-Leffler. Discuss convergence. Sharper Result.

21. Weierstrass factor thm and application.

22. Weierstrass product representation of Γ. Relation of $\Gamma(z)$ to $sinz$.

23. Product representation for $sin(\pi z)$.

24. What's the Γ function. Continue analytic behave at ∞. Stirling's formula. Representation of its reciprocal.

25. Discuss $f(z) = \int_1^\infty \frac{\exp(tz)}{t} dt$ When analytic, integrable , continuable etc.

26. Max-Min modulus thm – Also for harmonic functions.

27. Hadamard 3-circle thm.

28. How do you get Green's function for arbitrary domain (if bounded by Jordan Curve) $G(z,\zeta) = -\frac{1}{2\pi} log|f(z,\zeta)|$ For rectangle? (map to 1/4 of double Riemann sheet by $\mathcal{P}$ function also if reflect about sides : function harm in plane 0 on lattice lines char singular at reflected points). What can you say about harmonic functions in general?

29. Given complex valued function on boundary of unit disk, can you find analytic function inside which takes on boundary values? A: Must be continuous & more.

30. Schwarz reflection principle.

10.2 Riemann Mapping

October 2006

1. State the Riemann Mapping Theorem. Can you write a formula for a map of a square to the unit circle? Explain how to continue a function analytically. Is the analytic continuation always possible? Can you prove that the Fourier transform of a continuous, compactly supported function is analytic? In what domain is it analytic? Can you prove it?

April 2005

1. State the Riemann mapping theorem. Is the map unique?

2. Does the set of conformal maps from the unit disc to the unit disc havs some sort of geometric structure? (Ans: Such maps can be identified with $SL(2, R)$, which is a manifold)

February 2005

1. What is a conformal map? How do you tell if a general map $R^N \to R^M$ preserves angles?
2. How would you use conformal mapping to solve Poisson's equation in a general domain in R^2?

October 2004

1. Why are conformal mappings useful?
2. What happens to angles when the derivative is zero, under an analytic map?

October 2003

1. What is the Riemann mapping theorem? What do we need to specify for the resulting analytic map to be unique?
2. How can you classify analytic maps from the unit disc to the unit disc? Prove it. (Schwartz lemma)

October 2002

1. What is a conformal mapping? [Ans: an analytic map whose derivative never vanishes.] What is an alternative definition? [Preserving angles and orientation.] What happens when the derivative is 0? [Use z^2 as an example.] What about $z^2 + z^3$?

April 2002

1. Map an open strip conformally to the open unit disc.
2. Define a conformal map. (Ans: angle-preserving.)
3. Riemann mapping theorem: state the theorem, and prove that the complex plane (without infinity) cannot be mapped to the unit disk.

April 2000

1. State Riemann Mapping Theorem.

October 1999

1. What are the conformal maps of the plane onto the plane? Moebius transformations? Group property.
2. Joukowski transformation and connection to 2d flow past airfoil (person was also taking fluids)

October 1998

1. What is a conformal mapping? Give an example. Is $\sqrt{z}$ conformal?

Pre 1996

1. What's the definition of conformal mapping?
2. Map a horizontal strip and half disk onto disk.
3. Map $\{(x, y)|0 \leq y \leq \pi/2$ or $\pi/2 \leq y \leq \pi$ and $x \geq 0\}$ onto a disk.
4. Map $\mathbf{C} - \{|z| \leq 1\}$ onto outer region of disk.
5. Map lines onto circle, semi-circle $\rightarrow$ quarter plane, square $\rightarrow$ circle (Simply Connected)
6. Find most general mapping of circle onto itself (has 3 parameters). Prove most general $\overline{s}$ using Riemann Mapping thm. (Reflection principle or Schwarz lemma).
7. Prove Montel's thm.
8. State & sketch Riemann Mapping thm. Montel
9. Can you use the solution of Laplace with boundary condition to make a Riemann mapping function?
10. Can you solve $1 = \int_{-1}^{1} f(t)e^{ixt}dt$, $-1 \leq x \leq 1, \int |f| < \infty$? A: $F(z) = \int_{-1}^{1} f(t)e^{izt}dt$ and $F(z) = 1$ on segment $\Rightarrow$ const. But Riemann-Lebesgue lemma, $F(z) \rightarrow 0$ as $|z| \rightarrow \infty \Rightarrow$ Contradiction.
11. Under what conditions can you map a domain onto unit circle? (Riemann mapping thm) How many free paras? 3 real When is function continuous up to boundary? Jordan arc and every point access to interior. When differ?
12. What can you say if region is double connected - map onto annulus only specifying one radius arbitrary; the other will be determined. Prove.
13. Can any two doubly connected regions be conformally mapped onto each other? No, the corresponding annuli may differ.
14. Why don't the 2 functions $f(z) = \frac{z-a}{z\overline{a}-1}$ and $g(z) = (\frac{z-a}{z\overline{a}-1})^2$ violate uniqueness of Riemann mapping thm: not Schlicht.
15. Why doesn't Riemann Mapping thm hold if region being considered is whole plane minus one point. A: Liouville's thm.
16. Where does proof of Riemann Mapping thm break down if the region is multiply connected?
17. Given circle and 2 points inside, find circle passing through them. Then mapping domain onto disk, for how many points can you specify the images?

10.3 Other

April 2007

1. What is the Open mapping theorem? How do you prove it?

October 2005

1. What is the canonical product representation for $\sin(z)$?

April 2005

1. What is analytic continuation along a curve? If you do it with 2 curves that have the same initial and end points, when is the result the same?
2. Describe $\sqrt{x}$ in detail.
3. What is the Weierstrauss factorization theorem? What if you have a sequence of zeros converging to some point in the finite plane? What if your sequence of zeros accumulates in a finite number of points in the plane?
4. What does an entire function with no zeros look like?
5. Can you build an analytic function on the open ball with zeros accumulating on the boundary? Why can't zeros of an analytic function accumulate inside the domain?

February 2005

1. What do you know about representing an entire function as an infinite product? What would happen if there were infinitely many zeros of an entire function inside a bounded region? Explain?
2. What is the Riemann Sphere? Why do we think about it?
3. What does it mean to have a pole at infinity?
4. If a meramorphic function has finitely many poles of finite order on the Riemann sphere, what can you say about it (Ans: It's a rational function)? Prove it.

October 2004

1. Use the complex potential function to solve for 2D irrotational flow past a cylinder. The same for flow given by a point source. The same for flow past a cylinder under the influence of a point source external to a cylinder. (The student also studied fluid dynamics)
2. Can you define $z^{1/2}$ analytically in the whole plane (no, you have to cut the negative real axis). Can you cut along any other line? (yes, as long as you can define the argument analytically). What about $(z - z^2)^{1/2}$? (You have to cut the interval $[0, 1]$)

3. Derive the product expansion for $\cos(z)$. If I give you a set A of zeros, when is it possible to find an entire function that vanishes precisely on A? Let B = complex plane - $\{3 \text{ points}\}$. Suppose A, a subset of B, has no accumulation points in B. Can you find an f which is analytic on B and vanishes precisely on A?

April 2004

1. What can you say about the range of values taken on by a holomorphic function on $\mathbb{C}$? Start with polynomials. (Ans: fundamental theorem of algebra.) Prove it.

2. How do you prove Liouville's theorem?

3. What can you say about the range of values of a general entire function? (Ans: the lesser Picard theorem) Do you know how to prove that? (Ans: no.) How about a weaker version? (Ans: prove range is dense near essential singularity.)

April 2003

1. What are the Cauchy-Riemann equations?

October 2002

1. State and prove your favorite theorem from complex variables. [I said Cauchy on the rectangle but was stopped before I started the proof.] Can you state the most general version (or a more general version) of this theorem? Is there some "inverse" theorem to Cauchy's theorem? [Morera] What is required of the function f in Morera's theorem? [continuity]

2. Describe analytic functions from the plane to the unit disk, and those from the unit disk to the plane.

3. What can you say about analytic functions on an annulus including the unit circle? Specifically, give the relationship between Fourier series and power series in this case.

4. Consider the L^2 analytic functions. What can you say about them? Do they form a Hilbert space? (student's outline also included functional analysis)

April 2002

1. Write down the Cauchy-Riemann equations, and show that a function with real and imaginary parts satisfying them is analytic.

October 2001

1. What's your favorite theorem in Complex? Second favorite?

February 2000

1. what is the conection between complex variables and fluid dynamics? i wrote down w=(u,-v) (w is the velocity vector), then the div and curl of this expression. what can you say about the consequence of the result?

2. classify and give examples of the different types of isolated singularities.

3. write down the product representation of cos(z). can you say anything about the form of the undetermined function in the representation?

October 1998

1. State and prove Poisson's formula.

Pre 1996

1. Elliptic functions, uses of them e.g. S. Chris.

2. Weierstrass p function. Weierstrass product thm: convergence – in exponent to converge. Write down its differential equation.

3. Prove Picard's thm. Prove again $\overline{s}$ elliptic modular function. Find map, by construct (elliptic integral verify props) or function except that takes over role of elliptic modular function which coves plane with except of 0,1 and maps onto upper half plane. Send upper half plane to Δ by elliptic integral reflect about [0,1].

4. What about elliptic functions of 1^{st} order? None.

5. Construct elliptic modular function.

6. Open mapping theorem: non-constant analytic function is open.

7. Calculate inverse Laplace transformation using Contour integral. Example: $\omega/(\omega^2 + s^2)$

8. Can $e^z + z = 0$? State the great Picard thm.

9. Casoratti-Weierstrass Proof. A: If z_0 is an essential singular point than $\forall w \in C, \exists z_i \to z_0$ such that $f(z_i) = e^{g(z)} z^k \Pi E(z/a_n, n) \to w$, where $E(z,h) = exp(z + z^2/2 + \ldots + z^h/h)$.

11 Ordinary differential equation

11.1 Existence of ODE

October 2006

1. Continuous dependence on parameters.

$$\varphi'\varphi'' = \epsilon^2 \varphi''' \text{ on } [-1, 1]$$

$\varphi(-1) = \varphi(1) = 0$, $\varphi'(-1) = 1$. Discuss the solutions. Start with $\epsilon = 0$.

2. Local existence and uniqueness of ODE.

3. Global existence $\dot{x} = f(x)$. (I said solution exists globally if $f \cdot x < 0$). Then the question: global forward or backward? Answer: only backward.

4. Under what conditions does an ODE have short time existence and and uniqueness? Can you give an example of an ODE with more than one solution?

5. Describe the pendulum. ($x'' = -\sin x$). What does x correspond to? (Linearize look at eigenvalues).

6. How do you find the flow lines everywhere else? (Find the Hamiltonian, draw potential diagram.)

October 2005

1. Conditions for existence and uniqueness. Example of non-uniqueness.

2. Give a basic existence result (Student did Cauchy-Peano).

3. What if you have ODE $\dot{x} = f(t, x)$ with a uniform Lipschitz constant on f, i.e. $|f(t, x) - f(t, y)| \leq L|x - y|$, what can you say about existence and uniqueness? What if the Lipschitz constant is local in space?

4. Give and example of an ODE for which the solution exists for all time but is not unique. Give an example of an ODE that blows up in finite time.

April 2005

1. Local existence and uniqueness for $\dot{x} = f(x)$. Can you say anything about global existence? (Hint: Try to consider $\dot{x} = -x/ \parallel x \parallel$ and to derive something from this example)

2. What's an ODE? Do you know conditions for existence of a solution? Give conditions for uniqueness and an example of non-uniqueness. Give example of finite-time blow-up. Do you know conditions for global uniqueness?

February 2005

1. Discuss conditions that give existence and uniqueness to solutions of $\dot{x} = f(x)$. Global existence? Show an example when global existence fails. What are conditions that provide existence (but not necessarily uniqueness)? Show an example when uniqueness fails.

April 2004

1. State an existence and uniqueness theorem. How would you prove it? (Ans: contraction mapping argument.)

2. If $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ and f is C^k, how smooth is x? How do you know that? (Ans: use integral equation and bootstrap.) What if f is real analytic? (Ans: Cauchy-Kowalewski)

October 2003

1. What do you know about existence of solutions to an ODE? (Ans: Lipschitz condition implies existence and uniqueness.) Prove it. What about dependence on initial conditions, or differentiating with respect to a parameter?

October 2002

1. Example of nonuniqueness? How do you know it's not unique?

April 2002

1. Given the Lipschitz condition, sketch the proof of local existence and uniqueness. (Ans: Picard iteration, with contracting mapping principle or Schauder fixed-point theorem.)

2. Short time existence; what happens near the boundary of the maximal interval of existence? Give examples of finite-time blow up and non-uniqueless.

February 2002

1. Can an ODE have exactly two solutions?

April 2000

1. existence and uniqueness (conditions on f in y'=f and idea of proof)

April 1999

1. Given $\frac{dy}{dt} = f(t, y), y(0) = 1$, find a $f(t, y)$ st. the inital value problem has many solutions.

Pre 1996

1. How can you prove existence of solution when you use Picard iteration method. A: Banach contraction mapping theorem or Ascoli-Arzela for equicontinuous functions.

2. Existence and uniqueness of 1^{st} order systems. A: Lipschitz condition.

3. Existence and uniqueness of $y' = x^2 + y^2$ A: continuous so exists in small.

4. Can you say existence of $w'(z) = f(z)$ where $f(z)$ is analytic on complex plane?

5. $y'' = f(x, y, y')$. State problem. What happens without Lipschitz condition? Continuation in the large.

6. $dx/dt = P(x)$ polynomial. A: Not necessarily be continuable since $P(x)$ not necessary bounded. Ex: $y' = 1 + y^2 \Rightarrow y = \tan x$ can't be continuous beyond $\pi/2$. Continuable if polynomial of degree – linear.

7. $y(0) = 1$. $dy/dx = \sin(x + y)e^{-(x^2+y^2)}$. What can you say? A: Unique solution exists in entire plane since y' bounded. A: Analytic equation so exists a solution. Majorizing problem use explicitly here Cauchy-Kowaleski.

8. Do you have any condition to guarantee global existence of solution of $x' = f(x)$.

9. Show that there is a critical point in closed unit ball K which is positively invariant under the flow.

10. Solve a linear first order ordinary differential equation.

11. Discuss solution of linear equation $dy/dx = A(x)y$. Solution analytic how to prove it. What about single valuedness? A: iff A is analytic in simply connected domain.

12. Does boundary value problem always exist? Give example where no solution exists.

13. Boundary value problems for ordinary differential equation. When does solution exist? A: for linear with certain boundary condition (Sturm-Liouville self adjoint) use alternative thm.

11.2 Periodic Solution

October 2006

1. State Poincare-Bendixson. Give an example for all cases that can appear. If you know that a solution has a unique limit cycle, can you conclude that the solutions converge to it? Up to time reversal?

April 2002

1. Discuss 2-dimensional autonomous dynamical systems (Poincaré-Bendixson).

April 2000

1. autonomous system. Characterization of ass. st. solutions: enough to show knowledge of solution of y'=Ay for 2x2 constant matrix A.

2. Poincare-Bendixon Thm.

Pre 1996

1. Discuss stabilities of singular points of $x'' + sin(x) = 0$.

2. Define autonomous system. What can you say about solutions Poincare-Bendixson thm. What if there are singular points in bounded domain. Solution approaches closed limit set? What if singular point is in limit set? A: Solution approaches singular point.

3. 2-dimension autonomous system. (a) flow points inwards $\to \exists$ periodic solution. (b) If $\exists$ singular point, consider linear part.

4. Floquet theory. When do you get periodic solution?

5. What can you say about solution of $\Phi_t = A\Phi$ where $A(x+T) = A(x)$. What's necessary condition for periodic solution?

6. Make a necessary and sufficient condition for $x' = f(x,t)$ with $f(t) = f(t+T)$ to have a periodic solution. A: $\lambda(A) \neq 2\pi ik/T$, where $A = \partial f_0/\partial x(0)$.

7. Equation of linear oscillation with damping. If force is impressed, which terms dominate in solution?

8. Periodic solution: ... integral of system of ODE's.

11.3 ODE else

October 2005

1. Give me an example of an ODE for which the linearized picture doesn't say anything about the stability of the fixed point.

2. What does the phase plane look like for the pendulum?

February 2005

1. Discuss stability for the autonomous system $\dot{x} = Ax$. What do solutions look like for different A? What kind of growth behavior do you get when A is a two-by-two Jordan block with eigenvalue 1 and a "1" above the diagonal?

2. What is a stiff ODE? (student was also asked about numerical methods)

3. Consider second-order, self-adjoint boundary value problems. What kinds of boundary conditions are appropriate? (besides periodic)

4. Suppose I have an autonomous ODE in the plane with a critical point at zero. What can I say about trajectories that originate near zero? (Ans: Apply Hartman-Grobman theorem to linearized dynamics)

5. What if I start far from zero, at x_0. Can you characterize the dependence of the solution on the initial conditions? Can you compute $\frac{\partial x}{\partial x_0}$?

6. What is the Liouville equation? How would you prove it?

October 2004

1. Given linear, homogeneous ODE, what conditions on the matrix imply that all solutions go to zero? What conditions imply that all solutions are bounded for positive time?

October 2001

1. What is stability? Suppose you have a fixed point where all of the eigenvalues lie on the left side of the complex plane, what can you say about stability? What if some of the eigenvalues are on the imaginary axis?

April 2000

1. what do you like in ODE?

2. Solution of scalar eqn. y'=f(f) (t-t0=int(1/f(y)) and invert whenever possible)

3. Behavior of a nonlinear autonomous equation near a singular point of the vector field.

February 2000

1. What is *Phase Plane*?

2. What is the definition of stability? Consider an autonomous system. How would you find out if a critical point is stable? Discuss stability of a linear autonomous system. [Real part of eigenvalues.] Draw the orbits for the following cases in $\mathbf{R}^2$:
   (a) $\lambda_1 = 1,\ \lambda_2 = 1;$
   (b) $\lambda_1 = -1,\ \lambda_2 = -1;$
   (c) $\lambda_1 = 1,\ \lambda_2 = -1;$
   (d) $\lambda_1 = i,\ \lambda_2 = -i;$
   (e) $\lambda_1 = i + c,\ \lambda_2 = -i + c,\ c \in (0, \infty);$
   (f) $\lambda_1 = i - c,\ \lambda_2 = -i - c,\ c \in (0, \infty).$

3. for $\frac{dx}{dt} = Ax$ (A a matrix) what conditions on A gaurantee that the solutions go to 0? what conditions gaurantee that the solution is bounded?

October 1999

1. What is a linear first order ODE?.

2. Discuss the solutions of a linear autonomous first order system of ODE's. Asymptotic behaviour of the solutions.

April 1999

1. Sturm Liouville problem. What can you say about the eigenvalues? Where do they cluster, plus or minus infinity? Existance of evalues?

Pre 1996

1. Write down Sturm-Liouville problem and state a theorem on eigenvalues and eigenspace.

2. Write most general solution for linear equation in complex plane. regular point, regular singular point, irregular singular point.

3. Write Bessel's equation. Is it Fuchsian? A: No. Which term makes it non-Fuchsian.

4. Fuchsian theory. Singularities of 1^{st} and 2^{nd} kind, regular and irregular. Representation of solutions as $z^{\alpha}P(z)$. Single valuedness.

5. Fuchsian theory and example of its equations. n^{th} order equation. What does solution look like?

6. What are properties of solutions of Bessel's equation near the origin? What can you say about solution in neighborhood of ∞ ?

12 Partial differential equation

12.1 PDE General

October 2005

1. Do you know anything about the method of characteristics? (Student went over Burger's equation)

February 2005

1. How would you use Fourier series to solve a PDE? Sometimes you can use series of other functions - how would these arise?

2. How would you solve Burger's equation? How about a general non-linear first-order PDE? (Student was also asked about numerical methods)

October 2004

1. Characterize the viscous Burger's equation. Write down a nonlinear hyperbolic equation. What is a shock?

April 2004

1. To which equations does the maximum principle apply? (Ans: some parabolic and elliptic)

October 2003

1. How do Fourier series help you solve the heat equation? How would you solve the heat equation by separation of variables? Can you relate the two methods?

April 2003

1. How would you solve $u_t + a(x,t)u_x = 0$ with initial data at t=0? Can you always solve $b(x,t)u_t + a(x,t)u_x = 0$? Ans: No, the data might be characteristic. Example used: $b = -x$, $a = t$. What fails in this case, existence or uniqueness?

October 2002

1. State Cauchy Kowalewski.
2. Find a solution to Burger's equation for small time. Explain the behavior of solutions at points where the characteristics intersect.
3. Define a characteristic surface in general.
4. Consider the equation $u_{tt} - a(x)u_{xx} = 0$. Is it hyperbolic? Describe the range of influence. Can we solve it in backward time? Is there a PDE for which solving in backward time is a problem?
5. What's Burger's equation? Where does it come from? What are the characteristics? Discuss that they can cross, and what happens when they do, etc...
6. Write down a hyperbolic equation. How about in more than one spatial dimension? Now take the multi-dimensional wave equation and put an $a(x)$ in front of the u_{tt} term. What condition do we need on $a(x)$ to guarantee existence? (Don't mention Cauchy-Kowalewski; "we hate Cauchy-Kowalewski!")
7. What are characteristics? What's the relationship to the principle symbol?
8. Say the initial data is $u = 0$ everywhere and u_t is the characteristic function on a ball. Where is the solution 0? Prove it. Will the solution be continuous? Is the discontinuity in the function or in its derivative?

February 2002

1. Give the general solution to $u_t + u_x = 0$.

April 2000

1. Burgers equation: Inviscid vs. Viscous; Shocks, discontinuities; Rarefaction shocks.

February 2000

1. write down examples of the 3 types of PDE with boundary or initial data.

April 1999

1. $u_{xx} + 2u_{xy} + u_{yy} = 0$. Type? Do change of coordinates to standard form.
2. Can you solve $u_t = u_x$ in the first quarter of the x-t plane with $u(x, 0) = f(x)$ and $u(0, t) = g(t)$. How about $u_t = -u_x$ for the same data?

October 1998

1. On an open $D \subset R^n$, consider: $\Delta u(x) = f(x)u(x)$. What can you say about uniqueness of solutions? (since this is linear in u, the question is actually: is $u \equiv 0$, when boundary conditions are 0?) Hint. Consider $n = 1$, $D = (0, 1)$, and f constant! Then obviously we don't have uniqueness when $f < 0$: $u = \sin(2\pi x/\sqrt{|f|})$ does the job. So in general: we have uniqueness when the function f is non-negative.. to prove it multiply by u, and integrate by parts!

Pre 1996

1. Characteristics?
2. Well posed problem.
3. linear 1^{st} and 2^{nd} order equation.
4. Existence thm for quasi-linear hyperbolic systems?
5. Define of hyperbolicity. A: Cauchy problem well-posed.
6. Why Cauchy problem isn't well posed in Elliptic equation?
7. Linear equation of 2^{nd} order in 2 variables. Criterion for hyperbolic, elliptic, and parabolic?
8. Characteristics of wave equation and heat equation.
9. Domain of dependency for wave equation in 3-D, 2-D, 1-D? Domain of dependency for heat equation in 3-D, 2-D, 1-D?
10. Prove that characteristic lines of a linear first order hyperbolic equation don't intersect.
11. In which case the characteristic lines intersect? What do you call it? A: Shock for quasi-linear case.
12. What is shock evolution equation for conservation equation?

12.2 Hyperbolic PDE

October 2004

1. Consider the solution of the wave equation on $\mathbb{R}^3$. It looks like an integral of Cauchy data. What kind of integral is it? (Ans: surface integral)
2. The wave equation on $\mathbb{R}$: what initial conditions do you need? Write the formula for the solution.

April 2004

1. What is the wave equation on $\mathbb{R}$? What form does the solution take? (Ans: $F(x + ct) + G(x - ct)$. Change variables first to turn equation into $u_{st} = 0$.

2. Do you know where the wave equation comes from? (Ans: string tension, $F = ma$)

3. How would you prove uniqueness for the wave equation on a line? What about on a bounded domain? (Ans: use an energy argument)

4. What's the energy for a solution of the wave equation, and why do we care? (Ans: write down $E = \int (u_t)^2 + (u_x)^2 \, dx$; it's constant in time, which gives uniqueness of solution)

October 2003

1. Why do you have uniqueness for the wave equation? How can you solve it in higher dimensions? Domain of dependence?

October 2002

1. Write down a hyperbolic PDE. What are its characteristics? In general what is the meaning of characteristic (or non-characteristic)?

2. What are the characteristics of the 1-dimensional wave equation? How does this change of velocity depends on position? Consider first the transport equation with this kind of velocity: what are the characteristics? Do you have existence for all times?

April 2002

1. Write down the wave equation. Define its energy and prove that it is conserved in the case $u = 0$ at the boundary (Dirichlet) or normal derivative $= 0$ (Neumann). What happens if one has mixed boundary conditions? (Ans: modify the energy by adding the integral of $\frac{1}{2}\left(\frac{\partial u}{\partial n}\right)^2$ over the boundary—physically this is like a spring.

2. Solve the 1-dimensional wave equation. How do you do it in higher dimensions? (Ans: outline 3-dimensional method and Hadamard descent.)

February 2000

1. what's the difference between solutions to the wave equation in 3-D and 2-D? how can you solve the 2-D equation using the solution of the 3-D equation?

October 1999

1. Wave. Basics in 1-d. Some connections with Fluid Mechanics, group velocity, phase velocity, etc. (person asked this was also doing fluids)

Pre 1996

1. Solve $u_{xx} - u_{tt} = 0$ with $u(0,t) = u(t), u'(0) = 0$ and $u'(x,0) = 0$.

2. Prove existence for semi-linear wave equation, $u_{tt} - \Delta u = f(x,u)$.

3. Solution of wave equation in 3-D.
4. Define Huygens principle.
5. Huygens Principle, Kirchhoff's principle?
6. Wave equation in 5-dimension.
7. Space-like and time-like surface.
8. What is space-like surface? Why define such a thing?
9. What is type of Maxwell's equation? A: symmetric hyperbolic.
10. Given symmetric hyperbolic system. Find characteristics. See Pertrovsky. What does characteristic mean? Cauchy-Kowalewski thm. Necessary conditions A: analytic coefficient and initial data.

12.3 Laplace and Elliptic PDE

April 2007

1. Laplace equation for a bound smooth domain. What kind of boundary condition do you need? Can you tell anything about the solution?

October 2004

1. How do you prove the maximum principle for Laplace's equation? (Ans: mean value property). Write down the solution to the Dirichlet problem in the disk. How do you know it takes the values that you want on the boundary? Is it unique?

April 2004

1. Can you write down the solution to Laplace's equation on the upper half plane: $\Delta u = 0$, $u(0, x) = g(x)$? (Ans: Poisson's formula)
2. Is Laplace's equation on $\Omega \subset \mathbb{R}^2$ a well posed problem? (Ans: yes, if Ω is bounded) What if it's not bounded? (Ans: e.g. on upper half plane, $u = e^x \sin y$ and $u \equiv 0$) Can you give a simpler example? (Ans: $u = y$)
3. What does the maximum principle state? How would you prove it?

October 2003

1. What is Laplace's equation? When do you have a solution? (Ans: bounded domain, Dirichlet and Neumann problems.) Why do you have uniqueness? How can you solve it for irregular domains? (Ans: conformal mapping.) How does the Laplacian transform under a conformal map?

April 2003

1. What is Poisson's equation? What kinds of boundary conditions can you impose?

2. What can you say about uniqueness of the Dirichlet problem for the Poisson equation? How can you prove this? (Ans, for what it's worth: I started off using an energy method type approach, but it turned out they really wanted me to use the maximum principle.)

3. What is the eigenvalue problem for the Laplacian? What can you say about the eigenvalues?

October 2002

1. Mean value theorem for harmonic functions.

2. Discuss the eigenvalues of a symmetric elliptic operator. How do you find them for a general domain in $\mathbb{R}^n$? (see Courant and Hilbert)

April 2002

1. How many eigenvalues of the Laplace operator are there, given zero boundary conditions on the unit sphere? (Ans: countably many.)

2. Maximum principle.

3. How would you solve $\Delta u = f > 0$? (Ans: Green's function.) Physical interpretation of Green's function? (Potential.)

February 2002

1. Consider the PDE $\Delta u = 0$ in $(0,1) \times (0,1)$. What are some conditions you can put on this to get uniqueness/existence? What if you wanted to prescribe $u(x,0) = f(x)$ and $u_y(x,0) = g(x)$ on $[0,1] \times \{0\}$? Existence/uniqueness?

2. Poisson's formula.

October 2001

1. State the maximum principle for the Laplace equation. How could you prove it? (Ans: from mean value property.) How could you prove it for a general elliptic operator?

2. What is a well-posed problem for the Laplace equation? And for an unbounded domain?

3. Given the Laplace equation in the upper half-plane, how would you solve it? (Hint: use Fourier transform in x.) How are the constants in the solution determined?

April 2000

1. What is the value at the origin of an harmonic function inside a rectangle around the origin with zero boundary data in all sides of the rectangle except at one side where it has one as boundary data. What is the value for the same problem with a cube in R^3.

2. With what kind of boundary conditions can you expect to solve Laplace eqn, say on disk?

3. Use RMT to prove existence of solution of lap(u)=0 on a general simp. conn. domain.

4. Solution of lap(u)=0 on disk. Then on Sphere (Green function)

5. Uniqueness of solution with Dirichlet/Neumann conditions (e.g using energy principle)

6. (still lapl.) if Neumann conditions, du/dn=g, what conditions on g? (mean value zero, by one of Green's identities)

7. Physical intuition of solution of lapl. and of this mean-value-free du/dn.

8. Uniqueness of Laplace.

9. Existence of Laplace: Fundamental soluuntions; Poisson for ball; Greens functions; Methods for existence.

October 1999

1. Solve $\Delta u = 1$ with $\|u\| < C + r^3$.

April 1999

1. Given a square in the plane, find the value of a harmonic fn at the center of the square which takes 1 on the bottom side and 0 on the other 3 sides.

Pre 1996

1. Liouville's thm for harmonic functions?

2. Prove maximum thm from maximum modulus thm for analytic function.

3. Dirichlet problem, Conditions on data and boundary?

4. What's application of Riemann Mapping thm to Laplace equation? A: Enables to find fundamental solution. Prove existence of solution to Laplace equation?

5. Apply Riemann mapping thm to find fundamental solution of Laplace equation.

6. $u = 0, \partial u/\partial n = g, \Delta u = 0$ Is this well posed? What must be true for g? By Schwartz reflection thm, u analytic on boundary $\Rightarrow g$ analytic.

7. $u_{xx} + u_{yy} = 0$ on $\{(x,y)|\ y \geq 0\}$ with $u(x, y = 0) = f(x), u_y(x, y = 0) = g(x)$. Is this well-posed? A: No. $(1/n^2)e^{n\varepsilon y} sin(nx), n \to \infty$.

8. What is the condition of $f(x)$ at ∞ for the following equation to be well posed? $u_{xx} + u_{yy} = 0$ on $\{(x,y)|\ y \geq 0\}$ with $u(x, y = 0) = f(x)$ A: Uniqueness through Liouville's them.

9. Find Green's function for operator d^2/dt^2 with boundary condition $u(0) = u(1) = 0$.

10. Green's formula for $u(P)$ satisfying $u_{xx} + u_{yy} = 0$ in R in terms of u on boundary (if $u \in C$ on boundary)

11. Prove Perron method of existence for $\Delta u = 0$.

12. Discuss general elliptic eigenvalue problem for $\frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i}) + c_i u_{x_i} + \lambda u = 0$. A: elliptic if $a_{ij}\partial u/\partial x_i$ is positive definite.

13. Discuss $\nabla^2 u + \lambda u = 0$. Can every function be expanded in series of eigenfunction? A: sufficient, Lipschitz; necessary, undetermined.

14. Show not every continuous function can be expanded in a Fourier series which converges uniformly (even point-wise).

15. What additional conditions must you impose on biharmonic equation $\Delta\Delta u = u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$ in order to obtain well posed problem?

12.4 Parabolic PDE

October 2005

1. Solve the 1D heat equation on the real line (Ans: use Fourier transform). For what initial conditions does the solution make sense? Solve the heat equation in $[0, 1]$ with periodic initial and boundary conditions. How would you do this if the initial/boundary conditions were not periodic?

2. How do you solve the heat equation on a line? What about a general domain? (Ans: Use eigenfunctions of the Laplacian) How do you know that eigenfunctions corresponding to different eigenvalues are orthogonal? What if an eigenvalue has multiplicity greater than one, can you make the functions orthogonal?

February 2005

1. Can you solve the initial value problem for the heat equation on the line?

October 2004

1. What can't you solve the heat equation backwards?

October 2003

1. What is the solution of the heat equation on S^1? (Ans: use Fourier series)
2. What does the maximum principle say for the heat equation on S^1?
3. How would you solve the heat equation on $\mathbb{R}$? (Ans: use Fourier transform)

April 2003

1. Tell us about the backwards heat equation.
2. How can you use the eigenvalues and eigenfunctions of the Laplacian to solve the heat equation?
3. How do you solve the heat equation with periodic initial data?

October 2002

1. Define $g(x)$ as a non-negative function on $[0, 1]$. Consider the solution to the heat equation on $[0, 1]$ with boundary conditions $g(0)$ and $g(1)$ given, and then the solution on the whole real line, with the same g but extended by 0 outside of $[0, 1]$. How do the solutions in $[0, 1]$ compare in forward time? (maximum principle)

April 2002

1. Solve the heat equation on the line given initial values. Why do we consider only positive time? What if the Fourier transform of the initial values is compactly supported?
2. Uniqueness of solution to the heat equation. Do you have uniqueness with zero initial values if you require the solution to go uniformly to 0 as $t \to 0$?
3. Fundamental solution of the heat equation and behavior of derivatives of the heat kernel.
4. Consider the backwards heat equation $u_t + u_{xx} = 0$ with g as initial value. Is this a well-posed problem? Switch to Fourier domain and find the Fourier transform of the solution, assuming it exists. What happens? Now show an example of a sequence of g's converging to 0 in, say, L^2, but such that $\hat{u}$ blows up. (Ans: I used $\hat{g_n} = n^{-1}$; characteristic function of $[n, n + 1]$).
5. How do you solve the heat equation in 1 dimension? (Ans: heat kernel.) What is necessary for uniqueness? Smoothness?

October 2001

1. Write down the heat kernel (can derive it by Fourier transform). Differentiate it with respect to x. Now we have a nonzero function that solves the heat equation but goes to 0 as $t \to 0$. Why is that allowed? (Ans: it depends on *how* the solution goes to 0; pointwise convergence not enough.)

April 2000

1. Solve Heat equation, discuss regularity.
2. Dirichlet problem. Can you solve the dirichlet problem in a Disk in R^2. (answ: poisson formula.) What if the domain is not a disk but a rectangle (answ: conformal mapping). What if the problem is in R^3? Can you do the same ? (answ:no). Then, how can you solve it. (answ: finding a Green's function).
3. Solve the Dirichlet problem in a cube in R^3 with zero boundary data using Fourier Series Techniques.
4. heat eqn: Write eqn and solution (say on the line). What about on the interval $[0, 1]$? What about counterpart of mean value of du/dn=0 in lapl. here, say on $[0, 1]$? (u(0,t)+u(1,t)=0, since there is no source term and thus heat flux must be conserved)
5. After writing solution in $[0, 1]$ generalize for arbitrary bd. domain in hyperplane t=0. Try similar expansion in eigenfns.
6. Duhamel's principle for Heat Equation.

February 2000

1. What is the heat equation? How do you solve it in the whole space?
2. What about the regularity of the solution of Poisson's equation? [If the source term is L^2, then the solution is H^2.] What if the source term is C^0, is the solution C^2? [No. But if the source term is $C^{0,\gamma}$, then the solution is $C^{2,\gamma}$.]
3. write down the heat kernal. what can you say about the solution of the heat equation? (analytic). prove that the solution is analytic.
4. solve the heat equation for a bounded domain with zero for the boundary values.
5. Solve the 1d heat equation in the domain $t < 1$ with $u(1, x) = sin(x)$ on $t = 1$.

October 1999

1. Heat. The basics in 1-d, well posedness, i.v.p., how you derive the heat kernel?

April 1999

1. Use fourier series to solve $u_t - u_{xx} = 0, u(x, 0) = f(x)$ on bounded domain. What condition on f to get convergence?

October 1998

1. Compute solution for the heat equation using Fourier Transform method.

Pre 1996

1. Prove maximum principle of solution of heat equation.
2. Derive the Poisson integral kernel for heat equation with pure initial value condition through Fourier transformation.
3. Heat equation, no discontinuous derivatives. Conditions at ∞ for uniqueness? Cauchy problem not well posed, need additional conditions since $t = 0$ is characteristic.
4. Discuss on well-poseness of heat equation on $[a, b]$.
5. Why isn't the heat equation with the following boundary data well posed? $u(a, t), u(b, t)$ and $u(x, 0), u(x, T)$ where $a \leq x \leq b, 0 \leq t \leq T$. A: Uniqueness is broken down by $sin(n\pi(x - a)/(b - a))sin(n^2\pi t/T)$
6. How can you solve initial-boundary mixed problem of heat equation.
7. Heat equation with $T(t = 0, x)$ on upper half plane. Is it well posed? A: Yes.

13 Fluids

February 2005

1. Can you tell us about the intersection between Complex, PDE and Fluids? (Ans: complex potentials for fluid flows and conformal mapping). Can you give a simple example of where you might use such methods? (Ans: Complex potential of flow around the unit circle)
2. How would you find the flow over a triangular bump sticking out of the real axis? (Ans: Schwarz-Christoffel map).
3. Do you know about Kutta-Jukowski transformations?
4. Write down the vorticity transport equation. How do you go about closing this system (Ans: assuming incompressibility, use Biot-Savart law to get the velocity field). How do you derive the Biot-Savart law?

April 2002

1. Write down Euler's equations for incompressible fluids.
2. Relate the div$(u) = 0$ condition to constant density along streamlines.
3. Bernoulli's theorem: derive conserved quantity along streamlines.
4. Vortex equation: derive, state physical meaning (vortex stretching).

April 2000

1. Wave equation for acoustics.
2. Energy equation.
3. Bernoulli's Theorem–steady, compressible.
4. Can you find internal energy from p, rho, isentropic case?

February 2000

1. Why do we need complex theory for fluid dynamics? (I said we can make complex analytic function for incompressible and irrotational fluid)

October 1999

1. What is a material derivative?
2. Conservation of mass.
3. A little of nonlinear wave equations (shocks, hydarulic jumps, etc.)

14 Numerical Analysis

April 2007

1. How to solve least square problem?

October 2006

1. Hou can you test if a symmetric real matrix is positive definite? (Compute eigenvalues). The computation of eigenvalues explicitly is iterative and cannot be done by a fixed finite number of operations. Do you know a method that would test for positive defniteness with a finite number of operations? (Try Cholesky, see if it fails).
2. What is the most usual method to solve a linear system of equations? (Gaussian elimination). Do you know other methods? QR factorization. Do you know which is faster? Can you use a QR to solve a least squares problem? How?
3. Do you know what is the Householder transformation? Can you draw a picture or write a formula for it? How would you use it for least squares?
4. The error bound of the Euler method, Runge-Kutta with 2 points. How's the stability of Runge-Kutta? What's the difference of the midpoint rule in ODE and the Runge-Kutta method with 2 points? (One is mult-step and the other is one-step).

5. Heat equation in a bounded domain? ($u_t - u_{xx} = 0$) with the boundary conditions: $u(0,x) = \phi(x)$, $u(t,0) = 0$ or periodic. How can you apply the Fourier series? What about the numerical scheme? (Forward Euler or Backward Euler). Which condition does the forward Euler need for stability? (Courant number k/h^2 should be less than $1/2$) with respect to L^∞ norm, can you show? If $k/h^2 > 1/2$, then what happens?

6. Laplace's equation in an unbounded region like $y > 0$ with boundary condition $u = f$ and $\frac{du}{dy} = g$, is it well-posed? (No. Give an example).

October 2005

1. Discuss the Newton-Raphson method: convergence, rate of convergence, etc.

2. Compare multi-step and Runge-Kutta methods

3. Define A-stability. What is the connection to stiff equations?

4. Give a numerical method that you could use to solve the heat equation. Discuss convergence properties.

February 2005

1. How would you find a solution to a non-linear equation? (Ans: Newton's method) When does this method converge, and how fast? In the 1D case, is there a simple criterion that guarentees convergence in some neighborhood of the root?

2. How would you solve a stiff ODE numerically? Do you always care about whether or not you resolve fast timescales? Can you deal with stiffness by setting the initial data to be zero in the stiff directions?

3. Can you solve Burger's equation numerically using spectral or collocation methods?

October 2004

1. Is the work for computing the SVD finite time? (Ans: No, computing eigenvalues requires iteration)

2. Why are normal equations bad for solving least squares problems? (Ans: condition number squared) What would you use instead?

3. Why do we consider error squared, instead of just absolute error?

4. What is the operation count of a convolution? What is a more efficient way to compute a convolution? What is the operation count of the discrete Fourier transform? The FFT?

5. What is your favorite iterative method?

October 2003

1. How can you tell if a matrix is full rank? (Ans: svd) Is there another method? (Ans: look at eigenvalues of $M^T M$)

2. How can you tell if a matrix is positive definite? (Ans: try to take the Cholesky factorization)

3. What is a stiff ODE? Give an example.

4. What is the trapezoidal rule for ODE?

April 2003

1. How would you compute $\int_{-\infty}^{\infty} e^{-x^4}\, dx$ numerically? How do you estimate the error of the trapezoidal integration rule?

2. What is the condition number of a matrix? Give me a 2x2 matrix which has a large condition number.

3. What does the condition number of a matrix tell you? (Ans: I talked about error bounds for solving systems of equations and convergence rates of iterative methods.) What does the error bound tell you in the case that the solution is zero? (It turns out that this is something of a trick question.)

4. What is Newton's method (for finding zeros of a function)? What do you know about the convergence of Newton's method? Can Newton's method fail to converge?

5. What kind of problems can you have with Newton's method? (Ans: problems with using Newton's method in practice include dealing with dense Jacobian matrices, ill conditioning of the Jacobian matrix and very expensive function evaluations required to compute the entries of the Jacobian matrix.) How can you deal with these kinds of problems?

February 2002

1. What's a linear multistep method? When is it convergent? If it's not convergent in general but we pick a right-hand side in such a way that our approximation converges, what about the observed and predicted orders of accuracy?

October 2001

1. How do you know if a matrix is positive definite? How do you determine this numerically? (Cholesky) What's the operations count?

2. How expensive is it to solve a linear system of equations? First, how would you solve it? (Ans: if A in $Ax = b$ is positive definite, I'd do Cholesky factorization.) That only affects the time by a constant factor. Do LU instead. Start with A tridiagonal. What about a full A? What if A is banded?

3. Write down the fluids PDE, tell us how you'd discretize it, and discuss checking the stability of the numerical solution. (This person's specials outline mentioned numerical fluid solvers.) Ans: Use Lax equivalence theorem, which tells us that for a discretized PDE system (of order $p \geq 1$), given consistency, stability is equivalent to convergence. For Navier-Stokes, consistency is $O(\Delta t) + O(\Delta x^2)$.

February 2000

1. Show the convergence of Euler and trapezoidal methods. (I used dahlquist equivalence theorem for trapezoidal case)

April 1999

1. What is a stiff ode? Example. What schemes are good/bad for stiff. What's the highest order multistep A-stable scheme?

2. Conjugate gradient verses LU. The costs, given the condition number and size.

3. $u_t + u_x = 0$. Schemes? Stability of them?

October 1998

1. Explain Crank-Nicholson's method

2. What is Wellposed/Ill-posed

3. How would you solve wave equation numerically

4. Explain how numerical methods can be used to prove "Maximam Principle" for the elliptic equation

15 Probability

April 2007

1. Prove Central Limited Theorem.

2. Suppose $\{Y_i\}$ are iid Binomial random variables.

$$P(Y_i = 1) = P(Y_i = -1) = 1/2.$$

   For what kind of α do we have

$$\sum_{i=0}^{\infty} \frac{Y_i}{i^\alpha}$$

   converges with probability 1? Guess the answer first. ($\alpha > 1/2$) Then prove it (Use Kolomogorov's two series theorem)

October 2006

1. What is a martingale? If X_n, F_n is a martingale sequence is it true that $E[X_{n+1}|X_1, \ldots, X_n] = X_n$? (I said I don't know, but probably not since $\Sigma(X_1, \ldots, X_n)$ may in general be a proper sub-σ algebra of F_n.

2. What is Stirling's formula?

3. What is the entropy of a probability measure?

October 2005

1. What is the Central Limit Theorem? Is it possible for $\frac{S_n}{\sqrt{n}}$ to converge to a random variable almost surely? (Ans: No, by Kolmogorov 0-1 Law)

2. Consider Bernoulli trials. State and prove the String Law of Large Numbers for this case. Central limit theorem for Bernoulli trials. Bernoulli trials are like a random walk in Z; are they recurrent? Prove it.

3. Optional stopping theorem

4. Convergence of L^2 bounded martingales

5. L^2 maximal inequality

April 2005

1. What is the Paul Levy Theorem and can you prove it?

February 2005

1. State the CLT. Can you state Lindberg's condition?

2. Given the appropriate normalizations, what kind of distributions can you get as limits (in the sense of distribution) of sums of random variables? (Ans: stable laws)

3. If x_n is iid and has a finite first moment but no variance, can its partial sums normalized by a power of n converge to some non-degenerate distribution?

4. What is the definition of convergence in distribution? Can you state a theorem about properties equivalent to convergence in distribution?

5. State the Strong Law of Large Numbers and indicate the proof. State the Law of the Iterated Logarithm, and determine the set of limit points.

October 2004

1. What is the Central Limit Theorem? Prove it. What is its relationship to Brownian motion?

2. Explain different definitions of convergence in distribution, and why they are equivalent.

3. State the strong law of large numbers. What if $E(X^+)$ is infinite, but $E(X^-)$ is finite?

October 2003

1. What is the strong law of large numbers? What are strongest sufficient conditions? Can you prove it?

2. Prove the Central Limit Theorem.

3. You know about Brownian motion, right? What do you know about the distribution of the maximum? (Ans: reflection principle.) Can you prove it?

October 2002

1. Zero one laws: what are some?

2. Use Kolm zero one law to prove $S_n/\sqrt{n}$ goes to infinity almost surely. What's the right normalization (not $\sqrt{n}$) to get convergence to something finite? Where does the $\sqrt{2}$ come from?

3. State the Law of Iterated Logs.

4. What is the Law of Large Numbers? If you assume anything you want about the random variables, can you prove it?

5. What is the Central Limit Theorem? What does convergence in distribution mean? Does $S_n/\sqrt{n}$ converge a.s.? Prove that it converges with probability 0.

April 2002

1. What is the Central Limit Theorem? Idea of proof (using characteristic functions).

2. What is standard Brownian motion? What differential equation does $E(f(x_t))$ satisfy if x_t is BM? (Ans: heat equation with $1/2$ coefficient in front of the Laplacian and initial value f. I mentioned also that the space of continuous real-valued functions on the non-negative reals is a complete and separable metric space w.r.t. uniform convergence on compact sets.) Why is this important? (Ans: Prohorov's theorem.)

3. One runs a BM at x from time t up to time 1 and at that time evaluates f at wherever the BM is. What equation does the expected value of this last quantity satisfy? (Ans: the backwards heat equation.)

October 2001

1. What's the characteristic function of the normal distribution? Sketch proof of Central Limit Theorem.

February 2000

1. let's say you have a random walk with equal probability of left and right movement and boundaries that absorb the particle undergoing the random walk. what can you say about the probability of hitting the left boundary before the right boundary? how do things change if we start at different starting points.

April 1999

1. What is central limit theorem? (Use any formulation you want) Idea of the proof? What is convergence in distribution? What is your intuition of why convergence of characteristic functions is the same? (suppose our distribution has density, then can talk about Fourier transforms. No rigorous proof was required)

16 Theory of Computation

April 2002

1. Suppose a software company wanted to try to create a software package that could inspect code written in some language and determine whether the program would run forever (e.g. by getting stuck in an infinite loop), depending on the given input. Is this possible? (Ans: no, this is the Halting Problem, which is undecidable.) Sketch the proof. (diagonal trick.)

2. What is the meaning of NP and NP-completeness? If a polynomial algorithm were found for some NP-complete problem, what would that mean?

3. Give an example of an NP-complete problem. (Ans: Hamiltonian path.) For each graph that has a Hamiltonian path, one has a certificate of that property, i.e. the path itself. Is there a certificate for not having a Hamiltonian path? (Ans: this would be equivalent to NP=coNP, which is only conjectured to be false.)