Topics for Algebra Orals

Items marked (*) are suggested as optional topics. At the end of each block we occasionally offer a few questions whose answers might stimulate a review of these topics.

1. Linear Algebra.


Linear operators and matrices: rank, invertibility, determinant, similarity, and matrix representations relative to bases.

3. Groups of matrices, the classical matrix groups, groups that preserve quadratic, hermitian, skew symmetric, and other bilinear forms.
   - Orthogonal groups over $\mathbb{R}$ and $\mathbb{C}$: $SO(n, \mathbb{R})$, $SO(n, \mathbb{C})$.
   - Unitary groups $U(n)$
   - Symplectic group $Sp(2n)$

4. Quadratic forms, signature, type classification over $\mathbb{R}$ and $\mathbb{C}$.

5. Minimal and characteristic polynomials of a linear operator

6. Jordan canonical form
   - structure of nilpotent operators
   - Generalized eigenspaces and the decomposition $A = N + D$ with $D$ diagonal, $N$ nilpotent, and $[N, D] = 0$
   - The Jordan form as a cross section for similarity classes of matrices.

7. Inner product spaces
   - eigenvalues and eigenvectors
   - Spectral theorem
   - Diagonalization of special operators: self-adjoint, unitary, normal.


Sample questions.

1. If $A$ is an $n \times n$ complex matrix that is symmetric ($A_{ij} = A_{ji}$) does that mean you can diagonalize $A$? If so, over what kind of basis; if not, what can you say about $A$?

2. Explain why $\text{Tr } A = \sum \lambda_i$ and $\det A = \prod \lambda_i$, where $\{\lambda_i\}$ are the complex eigenvalues counted according to their multiplicities.

3. Do you know any criteria for diagonalizability of a linear operator on a complex vector space that is not assumed to be an inner product space?

2. Group theory

1. Homomorphisms, normal subgroups, and the isomorphism theorems for quotient groups.
   - Lagrange theorem and its variants
   - Coset spaces and permutation actions on them
2. Transformation groups \( G \times X \to X \)
   - Orbits and stabilizers
   - Transitive actions (homogeneous spaces) and their realizations as left actions on coset spaces \( G/H \).
   - Geometric examples – e.g. the action of \( \text{SO}(3) \) on the 2-sphere \( S^2 \approx \text{SO}(3)/\text{SO}(2) \).

3. Direct products and semidirect products
   - Criteria for \( G \) to be a direct or semidirect product of two subgroups
   - Construction of semidirect products from actions of \( H \) on \( N \) by automorphisms.
   - Basic examples

4. Dihedral groups, their geometric meaning, and their semidirect product structure

5. The symmetric group \( S_n \)
   - Cycles and the commuting cycle decomposition
   - Parity and the signature map \( \text{sgn}(\sigma) = \pm 1 \)
   - Conjugacy classes in \( S_n \)
   - The alternating groups \( A_n \)

6. Conjugacy classes and the class equation

7. Sylow’s theorems and their applications.
   - Determination of structure of various groups of low orders.

8. Cyclic groups and the structure of finitely generated abelian groups.

*9. Nilpotent, solvable groups.

*10. Matrix groups over finite fields: \( \text{GL}(n, \mathbb{F}) \), \( \text{SL}(n, \mathbb{F}) \)

*11. Free groups, generators, and relations in groups.

*12. Basic facts about the classical matrix groups and their Lie algebras.

Sample questions

1. If \( G \) is a cyclic group (finite or infinite), what can you say about arbitrary subgroups \( H \subseteq G \)? Are they all cyclic?

2. Classify all the groups \( G \) with (a) \( |G| = 7 \), (b) \( |G| = 14 \), (c) \( |G| = 21 \), where \( |G| \) is the order of the group.

3. Classify all the groups \( G \) with (a) \( |G| = 9 \), (b) \( |G| = 18 \).

4. Classify all groups with (a) \( |G| = 8 \), (b) \( |G| = 12 \).

5. For various \( n \) and primes \( p > 1 \) can you compute the orders of the groups \( \text{GL}(n, \mathbb{Z}_p) \) and \( \text{SL}(n, \mathbb{Z}_p) \)?

6. Explain why the circle group \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \) is isomorphic to the quotient group \( \mathbb{R}/\mathbb{Z} \).

7. Compute the centers and conjugacy classes for the dihedral groups \( D_n \) (with \( |D_n| = 2n \)).

8. Explain why \( \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{15} \) but \( \mathbb{Z}_3 \times \mathbb{Z}_6 \) is not isomorphic to \( \mathbb{Z}_{18} \). What is the group theoretic basis for the Chinese Remainder theorem?
3. Ring theory.

1. Homomorphisms and ideals
   - Basic theorems about maximal ideals in commutative rings
   - Prime ideals in commutative commutative rings
   - Nilradical.

2. Euclidean domains and principal ideal domains
   - Greatest common divisors and the g.c.d. algorithm

3. Ring \( \mathbb{Z}_n \) of integers (mod \( n \)) and its group of multiplicative group of units \( U_n \).
   - \( \text{Aut}(\mathbb{Z}_n, +) \cong (U_n, \cdot) \).

4. Polynomial rings \( R[x] \) and \( F[x] \)
   - Roots and multiplicities
   - Criteria for multiple roots
   - Irreducibility of polynomials and factorization
   - Eisenstein’s criterion for irreducibility in \( \mathbb{Q}[x] \)
   - Discriminants

5. Unique factorization domains
   - Unique factorization in \( F[x_1, \ldots, x_n] \) and \( \mathbb{Z}[x_1, \ldots, x_n] \).
   - Theorem of Gauss


Sample questions

1. Prove that \( \text{Aut}(\mathbb{Z}_n, +) \cong (U_n, \cdot) \).

2. Does unique factorization hold in the rings (a) Gaussian integers \( \mathbb{Z}[i] \), (b) \( \mathbb{R}[x_1, \ldots, x_n] \), (c) \( \mathbb{Z}[\sqrt{5}] \) ?

3. If \( p > 1 \) is a prime, is it a prime in the ring of Gaussian integers \( \mathbb{Z}[i] \)? What are the primes in this ring?

4. Give an example in which the ring of polynomials \( F[x] \) over a field is not isomorphic to the ring \( F \) of functions obtained by regarding \( f \in F[x] \) as a mapping \( f : F \to F \).

5. Find the greatest common divisor of the polynomials \( f(x) = \ldots \) and \( g(x) = \ldots \) in \( \mathbb{Q}[x] \).

5. Field theory and Galois theory.

1. Construction of fraction fields
2. Finite fields and their structure.
3. Field extensions
   - Algebraic vs transcendental extensions
   - Minimal polynomial
   - Splitting field of a polynomial over \( \mathbb{Q} \)
   - Separable vs inseparable extensions
   - Cyclotomic extensions
4. The Galois group
   - Definition of the Galois group; Galois extensions
   - Abelian Galois extensions
   - Angle trisection
   - Solvability in radicals
   - Main theorem of Galois.

*5. p-adic fields and valuations on fields; real fields.

6. Representations of finite groups.
   1. Irreducibility, direct sum decompositions, Maschke’s theorem
   2. The group algebra \( \mathbb{F}[G] \) and convolution product
   3. Conjugacy classes and their relation to irreducible representations
   4. Decomposition of the regular representation into irreducibles
      - Spectrum and multiplicities
      - Orthogonality relations
   5. Character tables and examples
   6. Divisibility of \(|G|\) by the dimension of any irreducible representation
   7. Induced representations.
      - Frobenius reciprocity theorem,
      - Artin’s theorem (rational decomposition of representations induced from cyclic subgroups).

Sample questions
1. Construct the character tables for the groups (a) \( \mathbb{Z}_n \), (b) the symmetric group \( S_3 \), (c) the dihedral group \( D_4 \) (of order 8). How many irreducible representations in each case, and what are their dimensions?
2. What are the orthogonality relations between trace characters of representations? How can you tell if a representation \( \rho \) is irreducible by looking at its trace character \( \chi_\rho = \text{Tr}_\rho \)?
3. What is the trace character of the left regular representations \( \rho \), with \( \rho_g f(x) = f(g^{-1}x) \)? Explain why \( \sum \dim(\pi) \text{Tr}_\rho_\pi = \delta_\epsilon \) where \( \delta_\epsilon(g) = 1 \) if \( g = e \) and is zero otherwise (sum taken over the irreducible representations).
4. The symmetric group \( S_3 \) acts on \( X = \{1, 2, 3\} \) in the obvious way. Consider the representation \( \rho \) we get by letting \( G \) act on functions \( f : X \to \mathbb{C} \) via \( \rho_g f(x) = f(g^{-1} \cdot x) \). Find the trace character \( \chi_\rho = \text{Tr}_\rho \) and the decomposition of \( \rho \) into irreducible representations of \( S_3 \).

*7. Tensor products and their universal mapping properties.
   1. Multilinear algebra, multilinear forms, exterior algebra
   2. General construction of tensor product for vector spaces and for R-modules.
      - Universal mapping property
      - Change of base field \( V \otimes_\mathbb{F} \mathbb{K} \) for vector spaces
3. The tensor algebra $T = \bigoplus_{k=0}^{\infty} \otimes^k V$, the symmetric algebra $S = \bigoplus_{k=0}^{\infty} S^k(V)$, and the exterior algebra $\bigoplus_{k=0}^{n} \wedge^k(V)$.


1. Examples
2. Hilbert Nullstellensatz
3. Elements of algebraic geometry