

**COURANT INSTITUTE OF MATHEMATICAL SCIENCES**

**Department of Mathematics**

## **General Oral Examination Outlines**

**Suggested by the Mathematics Faculty**

**Please note: general oral examinations will not be limited in content to the outlines presented in this document. The outlines are intended only as guidelines for students preparing for the oral preliminary examinations.**

## **Complex Variables**

1. Definitions of analytic functions, Cauchy-Riemann equation, Cauchy theorem, Morera and Goursat theorems.
2. Cauchy's formula, power series, Laurent series, domains of convergence, Liouville theorem, fundamental theorem of algebra, partial differential equations.
3. Branch cuts, elementary Riemann surfaces, logarithm.
4. Isolated singularities, local structure of analytic function.
5. Principle of the argument, Rouché's theorem.
6. Conformal mapping, Riemann map, Schwarz reflection, Schwarz-Christoffel map.
7. Analytic continuation, monodromy theorem, Lesser Picard theorem.
8. Harmonic functions, relations to analytic functions, Poisson formulas, Poisson-Jensen, etc.
9. Special functions: Gamma, Zeta, infinite products.

## **Real Variables**

1. A few items from classical analysis:
  - Weierstrass theorem, Ascoli-Arzelà theorem.
  - Monotone and BV functions (Helly's principle), convex functions, polynomials and Fourier series (a few basic facts).
2. Functions (Function spaces) and measures:
  - Measurable functions (Lusin's Theorem).
  - $L_p$  spaces (basic concepts of Banach and Hilbert spaces).
  - Riesz representation theorem.
  - Basic concepts about measures (countable additivity, Lebesgue measure, Fundamental theorem of calculus).
3. Convergence:
  - Almost everywhere convergence and Egoroff theorem.
  - Weak convergence and convergence in measure (various relations between them).
  - Riesz-Fischer theorem, approximation of identity and density of smooth functions in  $L_p$  (or Sobolev spaces). Weak compactness.
4. Integration:
  - Lebesgue's dominated convergence theorem, Fatou's lemma, monotone convergence theorem.

- Fubini's theorem, integration by parts.
- Absolute continuity of functions, integrals and measures, Radon-Nikadim theorem.

The recommended book is Bartle's *The Elements of Integration and Lebesgue Measures*.

## **Ordinary Differential Equations**

1. Existence, uniqueness, regularity:
  - Short time existence and uniqueness via Picard iteration.
  - Examples of finite time blow up, conditions for global existence, Gronwall's lemma.
  - Global existence via conservation laws or Liapunov functions.
  - Differentiability with respect to parameters and initial data.
  - Ordinary differential equations as vector fields, solution as maps parametrized by  $t$ .
2. Linear ordinary differential equations:
  - Constant coefficients. Solution via eigenvalues/eigenvectors, effect of Jordan blocks, matrix exponential, fundamental solution, Duhamel's principle (also known as variation of constants formula).
  - Periodic coefficients, Floquet exponents, orbital stability.
  - Use of Wronskian and trace.
3. Local behavior near a fixed point for nonlinear ordinary differential equations:
  - Notions of stability, asymptotic stability.
  - The relation (or lack thereof) between linearized and local stability and asymptotic stability.
  - Local invariant manifolds (stable, unstable, center).
4. Autonomous equations in the plane:
  - Hamiltonian systems, phase space diagrams.
  - Connections between fixed points, role of separatrix.
  - Poincare-Bendixson theorem.

## **Partial Differential Equations**

1. First-order Equations: characteristics, quasilinear equations, shocks, weak solutions, first order hyperbolic systems.
2. Cauchy-Kowalevsky theorem: proof by power series.
3. Classification of second order partial differential equations: canonical forms,

propagation of discontinuities, characteristic and noncharacteristic curves.

4. Laplace's Equation: fundamental solution, Green's identities, maximum principle, mean value property, Dirichlet problem, Hilbert space/energy methods.
5. Variational methods: Sobolev spaces, weak solutions, Dirichlet principle.
6. Heat Equation: physical derivation, fundamental solution, Cauchy problem, maximum principle, uniqueness, regularity, Fourier analysis (finite and infinite domains), general second-order linear parabolic equations (finite difference method and maximum principle).
7. Wave Equation: physical derivation, Cauchy problem in one dimension, domain of dependence, solution of boundary value problem (1D) by Method of Characteristics, fundamental solution, spherical means.

## **Differential Geometry**

1. Calculus on smooth manifolds:
  - Smooth manifolds, tangent and cotangent vectors, differentials of maps.
  - Vector fields, flows, 1-parameter groups of diffeomorphisms, Lie bracket and Lie derivative, Frobenius' Theorem.
  - Tensors, differential forms, exterior derivative.
  - Stokes theorem. Darboux's theorem.
2. Riemannian geometry:
  - Riemannian metric, Riemannian connection, geodesics and exponential map.
  - Curvature tensor, geometric interpretation of curvature.
  - Submanifolds, second fundamental form, Gauss and Codazzi equations, Gauss map, Gauss-Bonnet in 2-dimensions.

## **Fluid dynamics**

1. Eulerian versus Lagrangian description of fluid continua. Equations of mass and momentum conservation. The Navier-Stokes equations for incompressible viscous flow. The equations for compressible flow of a homogeneous perfect gas, including energy and state equations.
2. The inviscid incompressible fluid. Bernoulli theorems. Kinematics of vorticity and Kelvin's theorem. Potential flow and complex variable methods in two dimensions. Airfoil theory and the use of conformal mapping in flow problems.
3. Viscous flow and the Reynolds number. The stress tensor. The approximation of Stokes flow. Simple exact solutions such as Poiseuille flow.
4. Gas dynamics of a perfect gas in one dimension: characteristics and waves, simple waves, expansion fans. Conservation laws, shocks, Rankine-Hugoniot conditions and the entropy condition.

5. Specific problem areas such as the approximations of acoustics, or waves on the surface of water, are occasionally touched upon.

## **Numerical Analysis**

1. Floating point arithmetic.
2. Conditioning and stability.
3. Numerical solution of linear systems of algebraic equations by LU and Cholesky's methods.
4. QR and SVD factorizations and least squares;
5. Eigenvalue algorithms.
6. Conjugate gradient and Lanzos algorithms.
7. Numerical quadrature including Gaussian quadrature.
8. Newton's method.
9. Numerical solution of initial value problems for ordinary differential equations by Runge-Kutta and linear multistep schemes and error control.
10. Finite difference approximations of elliptic partial differential equations.
11. Classical iterative methods.
12. The FFT and applications to numerical solution of partial differential equations.
13. Polynomial and spline approximation.
14. Convergence and stability of difference schemes for hyperbolic and parabolic partial differential equations.

## **Functional Analysis/Hilbert Spaces**

1. Completeness:
  - Stone Weierstrass.
  - Baire category.
  - Banach Steinhaus.
  - Open Mapping Theorem.
2. Convexity:
  - Hahn Banach.
  - Weak topologies.
  - Banach algebra.
3. Duality in Banach spaces:

- The normed dual.
  - Adjoint.
  - Compact operators.
4. Operators in Hilbert spaces:
- Bounded operators (self-adjoint, normal, projections).
  - Riesz representation theorem.
  - Square roots and functional calculus.
  - Spectral theorem for bounded self-adjoint operators.
5. Unbounded operators:
- Graphs and symmetric operators.
  - Cayley transform.
  - Spectral theorem for unbounded self-adjoint operators.
6. Applications:
- Integral equations.
  - Fourier transform and definition of L<sup>2</sup>-based Sobolev spaces.
  - Some basic semigroup theory with application to heat equation and Schrodinger equation.

## **Probability**

1. Weak convergence of probability measures on the real line and characteristic functions.
2. Sums of independent random variables and the weak and strong laws of large numbers.
3. The central limit theorem.
4. The law of the iterated logarithm.
5. Markov chains.
6. Martingales and martingale convergence theorems.