

# Robust Replication of Volatility Derivatives

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# Introduction

- For a continuous positive price process:

$$dF_t = \sigma_{t-} F_t dW_t,$$

a contract that pays the realized variance:

$$\int_0^T \sigma_{t-}^2 dt$$

can be created by combining a static position in  $T$ -expiry Europeans (at all strikes) with dynamic trading in the underlying futures.

- The replication requires no assumptions on volatility dynamics.
- See Neuberger (1990), Dupire (1992), Carr-Madan (1998), Demeterfi et. al. (1999), and Carr-Lewis (2002) for details.

## Introduction (Con'd)

- But how does one create a contract that pays realized volatility = the square root of realized variance?
- Previous research has done this by specifying a stochastic process for volatility.
- We show that by also allowing dynamic trading in options, we can synthesize general functions of realized variance.
- We make a correlation assumption, but avoid specifying a process for volatility. In this sense, the replication is robust.

# Outline

- Review static hedging of path-independent payoffs
- Review variance swaps
- Robust replication of functions of realized variance
- Extensions

## Static Hedging of P-I Payoffs

- For general  $f$  and for any expansion point  $\kappa \geq 0$ :

$$f(S) = f(\kappa) + f'(\kappa)(S - \kappa) + \int_{\kappa}^{\infty} f''(K)(S - K)^+ dK + \int_0^{\kappa} f''(K)(K - S)^+ dK.$$

- Breeden-Litzenberger (1978), Carr Madan (2000).
- In terms of the bond price  $B_0$ , and call and put prices  $C_0(K)$  and  $P_0(K)$  at all strikes, therefore, a claim on the payoff  $f$  has initial value:

$$V_0[f(S_T)] = f(\kappa)B_0 + f'(\kappa)[C_0(K) - P_0(K)] + \int_{\kappa}^{\infty} f''(K)C_0(K)dK + \int_0^{\kappa} f''(K)P_0(K)dK.$$

- No restrictions on the underlying price process.

## Example: Log

- Suppose the payoff to be replicated is  $X_T = \ln(S_T/F_0)$ .
- Then expand  $f(S) \equiv \ln(S/F_0)$  about  $\kappa = F_0$ .
- The value of a claim on  $X_T$  is:

$$V_0[X_T] = - \int_0^{F_0} \frac{1}{K^2} P_0(K) dK - \int_{F_0}^{\infty} \frac{1}{K^2} C_0(K) dK,$$

i.e., the initial value of a static position in OTM options maturing at  $T$ .

- All assets have the same maturity.
- Useful for synthesizing a variance swap.

## Example: Powers

- Again  $X_T \equiv \ln(S_T/F_0)$ .
- Then for  $p$  real, the power payoff:

$$e^{pX_T} = (S_T/F_0)^p$$

decomposes into the payoffs from calls and puts on  $S_T$ .

- For  $p = a + bi$  complex, the power “payoff”:

$$e^{pX_T} = e^a[\cos(bX_T) + i \sin(bX_T)]$$

has real part  $e^a \cos(b \ln(S_T/F_0))$  and imaginary part  $e^a \sin(b \ln(S_T/F_0))$ , each of which decomposes into the payoffs from calls and puts on  $S_T$ .

- Useful for synthesizing volatility derivatives.

# Example of a Variance Swap

**Bank of America Securities LLC**

**Indicative Terms**

(For Discussion Only)

October 8, 1999

## S&P 500 Index Realized Variance Swap

**Equity Payer:** Bank of America, N.A. ("BofA")  
**Equity Receiver:** Merrill Lynch International  
**Trade Date:** October 8, 1999  
**Maturity Date:** May 7, 2003  
**Underlying Index:** The Standard & Poor's 500 Composite Stock Price Index  
**Equity Calculations:**

- (a) "Initial Price" means 0.305  
 (b) "Final Price" means the actual realized index Variance defined in accordance with the following formula and definition:

$$\sqrt{\frac{\sum_{i=1}^{n-1} \left[ \ln \left( \frac{P_{i+1}}{P_i} \right) \right]^2}{n-2}} \times \sqrt{52}$$

- (c) "Natural Logarithm" means for any Daily Quotient, as determined by the Calculation Agent, the exponential number which equates 2.718281828 to such Daily Quotient;  
 (d) "n" means the total number of Valuation Dates;  
 (e) "P<sub>i</sub>" means the closing level of the index on the ith valuation date (i.e.: P<sub>1</sub> is the closing level of the index on October 6, 1999, P<sub>2</sub> is the closing level of the index on the first Wednesday that is an Exchange Business Day following the Trade Date and P<sub>n</sub> is the closing level of the index on the Final Valuation Date.  
 (f) "Valuation Dates" means, commencing on October 6, 1999, and each Wednesday thereafter up to and including the Final Valuation Date and if any such date is not an Exchange Business Day, the next following day that is an Exchange Business Day, subject to the Market Disruption Events as set forth in the 1996 ISDA Equity Derivatives Definitions.  
 (g) " $\sum_{t=1}^n$ " means the summation from t=1 to t=n.

**Notional:** 111,230,666  
**Equity Payment:** Notional \* [ Final Price<sup>2</sup> - Initial Price<sup>2</sup> ]  
 If the Equity Payment is a positive value, then the Equity Payer pays the Equity Receiver this value.  
 If the Equity Payment is a negative value, then the Equity Receiver pays the Equity Payer the absolute value of this number.  
**Credit Terms:** n/a

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**NO BANK GUARANTEE**

# Variance Swaps

- Assume a continuous positive price process  $F_t$  for  $t \in [0, T]$ , where:

$$dF_t = \sigma_{t-} F_t dW_t,$$

and  $W$  is Brownian motion under a risk-neutral probability measure.

- Let  $X_t \equiv \ln(F_t/F_0)$ . By Itô's lemma:

$$dX_t = \frac{1}{F_t} dF_t + \frac{1}{2} \left( \frac{-1}{F_t^2} \right) \sigma_{t-}^2 F_t^2 dt$$

so:

$$X_T = \int_0^T \frac{1}{F_t} dF_t - \frac{1}{2} \int_0^T \sigma_{t-}^2 dt.$$

# Hedging a Variance Swap

- Re-arranging:

$$\langle X \rangle_T = \int_0^T \sigma_{t-}^2 dt = -2X_T + \int_0^T \frac{2}{F_t} dF_t.$$

- So a static options position with initial value:

$$\int_0^{F_0} \frac{2}{K^2} P_0(K) dK + \int_{F_0}^{\infty} \frac{2}{K^2} C_0(K) dK,$$

together with dynamic trading in futures replicates the variance payoff.

- But what about a nonlinear function of variance?

# Example of a Vol Swap

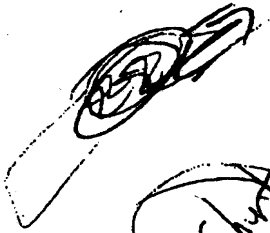

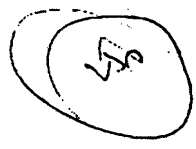
 **Merrill Lynch**  
S&P 500 INDEX VOLATILITY SWAP

**Indicative Terms and Conditions**

As of November 6 1997

<b>Swap Party A:</b>	Merrill Lynch International (MLI)
<b>Swap Party B:</b>	Investor
<b>Underlying Index:</b>	Standard & Poor's 500 Index (SPX)
<b>Notional Amount:</b>	\$50 million
<b>Maturity Date:</b>	1 year
<b>Valuation:</b>	Closing prices
<b>Party A Final Payment:</b>	Notional * Max [ 0, (Vol - ActVol)]
<b>Party B Final Payment:</b>	Notional * Max [ 0, (ActVol - Vol)]
<b>Vol:</b>	22 %
<b>ActVol:</b>	Actually realized index volatility defined according to the following formula:


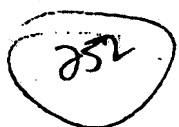
22 2.65 712 53.1

$$\sqrt{\frac{\sum_{i=1}^{n-1} \left[ \ln\left(\frac{P_{i+1}}{P_i}\right) - Avg \right]^2}{n-2}} \times \sqrt{252}$$

where  
 n = number of business days from the Trade Date up to and including the Maturity Date  
 P<sub>i</sub> = daily closing price of S&P 500 Index on the i<sup>th</sup> business day starting on Trade Date (i-1)  
 P<sub>n</sub> = closing price of the index on the Maturity Date

$$Avg = \frac{1}{n-1} \ln\left(\frac{P_n}{P_1}\right)$$

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# Literature on Volatility Derivatives

- All of the following rely on a specification of a volatility process:
  - Grünbuchler-Longstaff (1993)
  - Brockhaus-Long (1999)
  - Detemple-Osakwe (1999)
  - Heston-Nandi (2000)
  - Matytsin (2000)
  - Brenner-Ou-Zhang (2001)
  - Howison-Rafailidis-Rasmussen (2002)
  - Javaheri-Wilmott-Haug (2002).
- The majority are diffusion models, in which one can hedge with dynamic trading in one option and the underlying.

## Difficulties with SV Diffusion Models

- Simple SV models may be mis-specified:
  - Trouble simultaneously fitting long- and short-dated option prices.
  - Out-of-sample pricing errors.
  - Vol vol implied from short-dated option prices is unrealistically high.
  - Hedges fail when there are jumps.
- Even if an SV model is correctly specified, it depends on parameters which are not directly observable.

## The Need for Robust Replication

- According to a managing director at a major Wall Street dealer (2002):

“[T]here is no replicating portfolio for a volatility swap and the magnitude of the convexity adjustment is highly model-dependent.”

“As a consequence, market makers’ prices for volatility swaps are both wide (in terms of bid-offer) and widely dispersed.”

“[P]rice takers such as hedge funds may occasionally have the luxury of being able to cross the bid-offer – that is, buy on one dealer’s offer and sell on the other dealer’s bid.”

# Dynamic Option Trading

- We show how to dynamically trade European vanilla options to replicate volatility derivatives, without specifying a stochastic process for volatility.
- Such strategies may be expensive at the individual contract level, but are more practical at the aggregate portfolio level.
- In particular, let us replicate the payoff  $e^{\lambda\langle X \rangle_T}$ , for  $\lambda \in \mathbb{C}$ .
- Laplace inversion yields general functions of  $\langle X \rangle_T$ .
- Assume  $X$  has continuous sample paths, and assume frictionless continuous trading in  $F$  and in Europeans on  $F$  expiring at  $T$ .
- For simplicity, assume zero interest rates.

## Relating the Distributions of $X$ and $\langle X \rangle$

- Suppose that  $dF_t = \sigma_t F_t dW_t$  where  $\sigma$  and  $W$  are independent.
- Conditional on  $\mathcal{F}_T^\sigma$ :

$$X_T = \int_0^T \sigma_t dW_t - \frac{1}{2} \langle X \rangle_T \sim \text{Normal} \left( -\frac{1}{2} \langle X \rangle_T, \langle X \rangle_T \right).$$

- Hence, for each  $p$ :

$$\begin{aligned} E e^{pX_T} &= E \left[ E(e^{pX_T} | \mathcal{F}_T^\sigma) \right] \\ &= E \left[ e^{E(pX_T | \mathcal{F}_T^\sigma) + \text{Var}(pX_T | \mathcal{F}_T^\sigma)/2} \right] \\ &= E \left[ e^{(p^2/2 - p/2) \langle X \rangle_T} \right] \\ &= E e^{\lambda \langle X \rangle_T}, \end{aligned}$$

where  $\lambda \equiv p^2/2 - p/2$ . Or equivalently,  $p(\lambda) \equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}$ .

## European and QV Claims

- The same argument holds at time  $t$  instead of time 0, so:

$$E_t e^{p(\lambda)(X_T - X_t)} = E_t e^{\lambda(\langle X \rangle_T - \langle X \rangle_t)},$$

where recall  $p(\lambda) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}$ .

- Then the QV Laplace claim and the European power claim:

$$L_t \equiv E_t e^{\lambda \langle X \rangle_T}$$

$$P_t \equiv E_t e^{p(\lambda) X_T}$$

are related by:

$$L_t = N_t P_t,$$

where:

$$N_t \equiv e^{\lambda \langle X \rangle_t - p(\lambda) X_t}.$$

# Replicating the QV Laplace claim

- We have  $L_t = N_t P_t = e^{\lambda \langle X \rangle_t - p(\lambda) X_t} P_t$ , so:

$$\begin{aligned} dL_t &= N_t dP_t + P_t dN_t + dP_t dN_t \\ &= N_t dP_t + P_t N_t \left( -p(\lambda) \frac{dF_t}{F_t} \right), \end{aligned}$$

where all drift terms vanish because  $L_t \equiv E_t e^{\lambda \langle X \rangle_T}$  is a martingale.

- Integrating over time:

$$L_T = P_0 + \int_0^T N_t dP_t - \int_0^T \frac{p(\lambda) P_t N_t}{F_t} dF_t,$$

since  $N_0 = 1$ .

- Hence, when  $\lambda$  and  $p$  are both real, we replicate  $L_T = e^{\lambda \langle X \rangle_T}$  via the self-evidently self-financing strategy:

$$\begin{array}{ll} N_t & \text{claims on } e^{p(\lambda) X_T} \\ -p(\lambda) P_t N_t / F_t & \text{futures.} \end{array}$$

- Even if  $\lambda$  is real,  $p \equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}$  can be complex, so we need to deal with this case.

# Review of Laplace Inversion

- If a function  $g$  has Laplace transform  $G$ , then:

$$g(\langle X \rangle_T) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} G(\lambda) e^{\lambda \langle X \rangle_T} d\lambda,$$

for appropriately chosen  $\sigma$ .

- So in general, a function  $g$  is essentially a weighted sum of its complex-valued Laplace transform  $G$ , where the weights are also complex-valued.
- However, if  $g$  is real, then we can take the real part of both sides to conclude that the desired real function of variance is just the real part of some mixture of Laplace claims.
- Other approaches:
  - Alternative inversion algorithms, e.g. Post-Widder uses  $\lambda$  negative real.
  - Deformation of the contour.

# Replication in the Complex Case

- For complex  $\lambda$  and  $p(\lambda)$ , and complex  $\alpha(\lambda)$ :

$$\begin{aligned} \operatorname{Re}[\alpha(\lambda)L_T] = & \operatorname{Re}[\alpha(\lambda)P_0] + \int_0^T \operatorname{Re}[\alpha(\lambda)N_t]d\operatorname{Re}(P_t) - \\ & \int_0^T \operatorname{Im}[\alpha(\lambda)N_t]d\operatorname{Im}(P_t) - \int_0^T \frac{\operatorname{Re}[p(\lambda)\alpha(\lambda)P_tN_t]}{F_t}dF_t. \end{aligned}$$

- Hence, we replicate  $\operatorname{Re}[\alpha(\lambda)L_T]$  by trading cosine and sine claims:

$\operatorname{Re}[\alpha(\lambda)N_t]$	claims on $\operatorname{Re}[e^{p(\lambda)X_T}]$
$-\operatorname{Im}[\alpha(\lambda)N_t]$	claims on $\operatorname{Im}[e^{p(\lambda)X_T}]$
$-\operatorname{Re}[p(\lambda)\alpha(\lambda)P_tN_t/F_t]$	futures.

## Example: Volatility Swap

- For  $q \geq 0$ :

$$\sqrt{q} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-sq}}{s^{3/2}} ds.$$

- Therefore, the fixed rate to charge on a vol swap is:

$$\begin{aligned} E_0 \sqrt{\langle X \rangle_T} &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - E_0 e^{-s\langle X \rangle_T}}{s^{3/2}} ds \\ &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - E_0 e^{(1/2 + \sqrt{1/4 - 2s})X_T}}{s^{3/2}} ds \\ &= \frac{1}{2\sqrt{\pi}} \left[ \int_0^{1/8} \frac{1 - E_0 e^{(1/2 + \sqrt{1/4 - 2s})X_T}}{s^{3/2}} ds \right. \\ &\quad \left. + \int_{1/8}^\infty \frac{1 - E_0 e^{X_T/2} \cos(\sqrt{2s - 1/4} X_T)}{s^{3/2}} ds \right], \end{aligned}$$

which can be expressed in terms of the initial prices of European options.

## Example: Call on Variance

- Laplace transform of  $q \mapsto q^+$  is  $s \mapsto 1/s^2$ .
- So for  $\sigma > 0$ :

$$q^+ = \frac{1}{\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{sq}}{s^2} ds$$

- Therefore:

$$(\langle X \rangle_T - K)^+ = \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[ \frac{e^{-(\sigma+iw)K}}{(\sigma+iw)^2} e^{(\sigma+iw)\langle X \rangle_T} \right] dw,$$

which we can replicate.

## Conclusion and Extensions

- Without specifying a stochastic process for instantaneous or implied volatility, we have synthesized claims paying functions of realized variance, including realized volatility.
- The inputs are the same as for a variance swap, i.e. an implied volatility smile for the same underlying and maturity.
- Extensions we have done:
  - $\langle \ln L, X \rangle = \beta \langle X \rangle$
  - Letting above  $\beta$  depend on  $X$
  - Generalizing  $X = \ln(F/F_0)$  to  $X = h(F)$ .
  - Replicating joint claims on  $X_T$  and  $\langle X \rangle_T$ .
- Extensions we need/want to do:
  - Letting above  $\beta$  depend on  $t$ .
  - Determine numerical accuracy of the inversion.
  - Theoretical and numerical investigation of the effects of discretizing the path monitoring, strike sampling, and hedge frequency.