

Determining Volatility Surfaces and Option Values From an Implied Volatility Smile

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Abstract

Using only the implied volatility smile of a single maturity T and an assumption of path-independence, we analytically determine the risk-neutral stock price process and the local volatility surface up to an arbitrary horizon $T' \geq T$. Our path-independence assumption requires that each positive future stock price S_t is a function of only time t and the level W_t of the driving standard Brownian motion (SBM) for all $t \in (0, T')$. Using the T -maturity option prices, we identify this stock pricing function and thereby analytically determine the risk-neutral process for stock prices. Our path-independence assumption also implies that local volatility is a function of the stock price and time which can be explicitly represented in terms of the known stock pricing function. Finally, we derive analytic valuation formulae for standard and exotic options which are consistent with the observed T -maturity smile.

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I Introduction

The Black-Scholes model can be described as an ambitious attempt to explain the prices of all standard and exotic options using only a single parameter for each underlying. It is well-known that this volatility parameter can either be estimated from historical spot prices or implied from the price of a single option. In general, the latter approach is preferred in practice since it is “forward-looking” and at least partially captures model error, albeit crudely. When an option price is used to imply volatility, all standard and exotic options on the same underlying are valued using a risk-neutral probability measure consistent with the price of this option, as well as with the prices of the primitive assets.

The persistence of a volatility smile in options markets has lead to several proposed alternatives to the standard Black-Scholes model. For example, modeling the price of the underlying as a CEV process[3], a displaced diffusion[16], or as a call price itself[9], have all been proposed as parsimonious generalizations of the Black-Scholes model which retain the model’s analytic flavor. All of these approaches introduce an additional parameter and thus require at least two options for calibration.

Continuing the trend of expanding the set of basis options, Rubinstein[17] introduced the implied tree approach which values claims relative to standard options with strikes at every terminal node in a binomial tree. In his framework, standard options of any nearer maturity and exotic options are valued using a risk-neutral measure consistent with the prices of the primitive assets and of all options of the terminal maturity. Further continuing the trend of expanding the set of basis options, Derman and Kani[6] use standard options with strikes and maturities corresponding to every node in their implied tree. Similarly, working in a continuous setting, Dupire[8] requires a continuum of strikes and maturities. Since all standard option prices are used in the specification of the risk-neutral process, these last two approaches can only be used to determine the values of exotic options.

The expansion of the set of assets used to determine the risk-neutral measure has the important theoretical advantage of generating consistency with a wider set of assets and allowing weaker assumptions on the stochastic process governing the underlying spot prices. When all strikes and maturities are used as in Dupire, the diffusion coefficient can be any reasonable function of the contemporaneous spot price and time. Furthermore, this diffusion coefficient, which is termed the local volatility, need not be specified

ex ante, since it can be analytically determined from the double continuum of European option prices.

One obvious drawback of these more recent developments is the requirement of an option strike and maturity for every price and time at which dynamic trading is possible. For listed stock options, the minimum distance between strikes and maturities is \$5 and one month respectively. Consequently, interpolation is required to implement these approaches in the equity case. Furthermore, extrapolation is always required both from the last maturity out to perpetuity and from the nearest maturity in to the initial time. When implemented naïvely, these interpolations or extrapolations can engender option price quotes which are not arbitrage-free.

A second drawback associated with the expansion of the set of basis options is the comparative silence of these models regarding mis-pricings of standard options. Although the models require that the input option prices be arbitrage-free, these models blithely accept option prices inconsistent with any reasonable equilibrium framework. Since options markets are often illiquid and options' prices have wide bid ask spreads, the expansion of the set of basis options to all strikes and maturities can often induce model mis-specification. To illustrate this point, note that all of the above models embed the Black-Scholes model as a special case. Thus, in the unlikely event that the spot price actually does follow geometric Brownian motion, the incorporation of noisy option prices into the determination of the risk-neutral measure will lead to incorrect valuations of any remaining options. In fact, empirical evidence supporting this observation in the case of Dupire's local volatility model has been recently promulgated by Dumas, Fleming, and Whaley[7]. Even if the spot price follows a more complicated stochastic process than geometric Brownian motion, certain models may not permit independent specification of initial option prices at every strike and maturity. For example, if Rubinstein's implied tree describes the actual underlying spot price, then intermediate maturity option values are determined by the model. If the market prices of these options differ from the implied tree value for any reason, then incorporating these option prices into the determination of the risk-neutral process will result in mis-pricing, both for these options and exotic options.

In general, the quantity and quality of the options data should determine the appropriate tension between ex-ante process specification on the one hand and calibration to market prices on the other. The purpose of this paper is to propose an analytic and arbitrage-free approach to option valuation when only

one maturity is liquid. Working in a continuous setting, we assume as in Rubinstein that one can observe option prices of all strikes maturing at some common expiration T .

In order to determine the entire local volatility surface out to an arbitrarily distant date, we further assume that each positive future spot price S_t depends only on the contemporaneous level W_t of the driving standard Brownian motion (SBM) and not on its path. This path-independence property was also assumed by Black and Scholes, who further assumed that S_t is an exponential function of W_t ¹. In contrast to Black-Scholes, we infer this spot pricing function as a consequence of our three primary assumptions, namely path-independence, no arbitrage, and the liquidity of the T -maturity options.

In contrast to the primary assumption of the Black-Scholes model, stock prices sometimes hit zero in reality. To allow for this contingency, we allow for the possibility of an absorbing lower barrier on the standard Brownian motion (SBM) and assume that the stock pricing function evaluates to zero along this boundary². For analytic tractability, we restrict attention to boundaries for the SBM which are constant over time³. Given our path-independence assumption, both the stock pricing function and the constant absorbing boundary can be inferred from the option prices. For underlyings such as commodities and currencies, future spot prices of zero are generally considered impossible and in this case, the option prices will imply that the absorbing barrier for the SBM is negative infinity. Whether or not bankruptcy can occur, we characterize the *entire* class of risk-neutral path-independent stock price processes, which vanish when the driving SBM hits a constant lower boundary (possibly negative infinity) and which can be supported over an arbitrarily long time horizon. We also identify the unique member of this class consistent with the option prices observed at maturity T . Once the stock price becomes a known function of the driving SBM, the stock price process inherits the analytical tractability of this SBM. For example, the probability density function (PDF) for the first passage time (if any) of the stock price to the origin is just the known PDF for the first passage time of SBM to its absorbing barrier.

Our path-independence assumption also implies that local volatility is a function of stock price and

¹In a second model, they instead assumed that the spot is a call on the assets of the firm, and thus a different function of the SBM.

²Our model can be generalized to allow for the possibility that option prices imply a positive liquidation value in bankruptcy.

³It would be straightforward to consider linear boundaries.

time which can be explicitly represented in terms of the known stock pricing function. Consequently, we can generate the entire local volatility surface from the observation time out to an arbitrary date occurring after the last maturity. We also show that our path-independence assumption implies that local volatility satisfies a new nonlinear partial differential equation (PDE). Conversely, we show that if the local volatility function satisfies this PDE, then spot prices are path-independent and we provide the map transforming spot prices into the SBM.

Given that spot prices are path-independent, or equivalently that local volatility has the required form, we derive analytic formulas relating the values of standard and exotic options to the contemporaneous spot price and time. These analytic formulas exist for a wide range of volatility surfaces and as in Rubinstein, the values are fully consistent with the liquid option prices at maturity T . In fact, our results may be characterized as analytically extending Rubinstein's results to continuous-time diffusion processes, and to the valuation of options maturing both after and before T .

The outline for this paper is as follows. The next section shows how the twin assumptions of path-independence (PI) and no arbitrage (NA) lead to a simple linear PDE governing the spot pricing function and how the T -maturity option prices determine the spot pricing function at time T . We present the general solution to this problem and illustrate it with a simple example. The next section shows how PI and NA also determine the volatility function in terms of the known stock pricing function. We show that our volatility function satisfies a fully non-linear PDE. We also present a converse result showing that the nonlinear PDE is both a necessary and sufficient condition for prices to be path-independent. Assuming that spot prices are path-independent, or equivalently that local volatility solves the nonlinear PDE, the penultimate section shows how to analytically determine the pricing functions for path-independent standard and exotic options of any maturity. A final section summarizes and discusses extensions. Tedious derivations are banished to the appendices.

II The Stock Pricing Function

II-A Assumptions and PDE

We assume that European options of a fixed maturity $T > 0$ and their underlying asset (termed the stock) trade in a frictionless market with no arbitrage (NA). We assume that the initial stock price is positive and also assume that the limited liability of the stock induces absorption of the stock price at the origin. We let τ denote the first passage time of the stock price to the origin, if any⁴. To determine volatility surfaces and option values out to some arbitrary fixed horizon $T' \in [T, \infty)$, we assume that the stock price process is *path-independent* (PI). More specifically, we assume that over the horizon $[0, T' \wedge \tau]$, each future spot price S_t is related to the contemporaneous level B_t of the driving standard Brownian motion (SBM) and time t by a non-negative⁵ $C^{2,1}$ function $s(x, t)$, $x \in \mathfrak{R}$, $t \in [0, T']$:

$$S_t = s(B_t, t), \quad t \in [0, T' \wedge \tau]. \quad (1)$$

Our assumption of a positive initial stock price implies $s(0, 0) = S_0 > 0$. We will later show that NA and PI imply that $s(x, t)$ is increasing in $x \in \mathfrak{R}$ for each $t \in [0, T']$. Thus, for each t , there exists at most one root $L_t \in (-\infty, 0)$ of the stock pricing function:

$$s(L_t, t) = 0, \quad t \in [0, T'].$$

For analytic tractability, we restrict attention to the class of stock pricing functions such that the lower liquidation boundary⁶ L_t is constant at L over time.

By Itô's lemma, the spot price dynamics can be written as:

$$dS_t = \left[\frac{\partial s}{\partial t}(B_t, t) + \frac{1}{2} \frac{\partial^2 s}{\partial x^2}(B_t, t) \right] dt + \frac{\partial s}{\partial x}(B_t, t) dB_t, \quad t \in [0, \tau \wedge T']. \quad (2)$$

Since we can invert (1) for B_t , (2) implies that both the drift and diffusion coefficients are functions $b(S, t)$ and $\sigma(S, t)$ of only the spot price S and the calendar time t . Thus, over the time interval $(0, \tau \wedge T')$, the stock price process is Markov in the pair (B_t, t) or in the pair (S_t, t) .

⁴If the stock price never hits the origin as in the Black-Scholes model, then we set $\tau = \infty$.

⁵For options on underlyings which can go negative (eg. spread options), this assumption can be dropped.

⁶If there is no root, we set the liquidation boundary $L = -\infty$.

For simplicity, we assume a constant riskless rate $r \geq 0$ and a constant dividend yield $q \geq 0$ from the stock. Under these assumptions, it is well-known that there exists a unique “risk-neutral” probability measure Q , which is defined by the following spot price dynamics:

$$dS_t = (r - q)S_t dt + \sigma(S_t, t)S_t dW_t, \quad t \in [0, \tau \wedge T'], \quad (3)$$

where W is a Q -SBM. Letting W_t^a denote SBM absorbing at a fixed $L < 0$, (1) and (2) may be re-written as:

$$S_t = s(W_t^a, t), \quad t \in [0, T'], \quad (4)$$

and:

$$dS_t = \left[\frac{\partial s}{\partial t}(W_t^a, t) + \frac{1}{2} \frac{\partial^2 s}{\partial x^2}(W_t^a, t) \right] dt + \frac{\partial s}{\partial x}(W_t^a, t) dW_t^a, \quad t \in [0, T']. \quad (5)$$

Equating coefficients on dt in (3) and (5) and using (4) gives a partial differential equation (PDE) for $s(x, t)$:

$$\frac{\partial s}{\partial t}(x, t) + \frac{1}{2} \frac{\partial^2 s}{\partial x^2}(x, t) = (r - q)s(x, t), \quad t \in [0, T'], x > L, \quad (6)$$

subject to the boundary conditions:

$$\lim_{x \downarrow L} s(x, t) = 0, \quad t \in [0, T'], \quad (7)$$

and:

$$\lim_{x \uparrow \infty} s(x, t) = o(e^x), \quad t \in [0, T']. \quad (8)$$

The last boundary condition ensures that the payoff is integrable.

II-B Implied Stock Payoff Function

This subsection show how our assumption of path-independence can be combined with the implied volatility smile at maturity T to determine the stock pricing function at T , $s(x, T) \equiv f(x)$. Using the Black-Scholes formula, the implied volatility smile can be converted to the strike structure of T -maturity option prices. Ross[15] and Breeden and Litzenberger[2] have pointed out that differentiating this strike structure twice and future-valuing yields the “risk-neutral” probability density for the future stock price. Interpreting this

density as arising from a change of variables of the known density describing future levels of an absorbing SBM will yield the desired stock payoff function $f(x)$.

To operationalize these observations in our continuous context, we assume that a continuum of T -maturity put prices $\{P_0(K), K > 0\}$, are observable. We further assume that these prices are given by a C^2 function, which by the absence of arbitrage must be positive, upward sloping, and convex. Let $P_k(K) \equiv \lim_{\Delta K \downarrow 0} \frac{P_0(K+\Delta K) - P_0(K)}{\Delta K}$ denote the observed slope in strike at K , which is also the value of an infinitesimal vertical spread struck at K . Given our path-independence assumption, the following relationship exists between this vertical spread and the inverse of the implied spot payoff, $f^{-1}(K)$:

$$e^{rT} P_k(K) = \int_L^{f^{-1}(K)} \frac{1}{\sqrt{2\pi T}} \left\{ \exp \left[-\frac{1}{2} \left(\frac{z}{\sqrt{T}} \right)^2 \right] - \exp \left[-\frac{1}{2} \left(\frac{z-2L}{\sqrt{T}} \right)^2 \right] \right\} dz, \quad K > 0, \quad (9)$$

since the integrand is the risk-neutral probability density for SBM absorbing at L . Solving the LHS for K gives the implied payoff function:

$$\begin{aligned} f(x) &= P_k^{-1} \left\{ e^{-rT} \int_L^x \frac{1}{\sqrt{2\pi T}} \left\{ \exp \left[-\frac{1}{2} \left(\frac{z}{\sqrt{T}} \right)^2 \right] - \exp \left[-\frac{1}{2} \left(\frac{z-2L}{\sqrt{T}} \right)^2 \right] \right\} dz \right\} \\ &= P_k^{-1} \left\{ e^{-rT} \left[N \left(\frac{x}{\sqrt{T}} \right) - N \left(\frac{L}{\sqrt{T}} \right) - N \left(\frac{x-2L}{\sqrt{T}} \right) + N \left(\frac{-L}{\sqrt{T}} \right) \right] \right\}, \quad x > L, \end{aligned} \quad (10)$$

where $N(d) \equiv \int_{-\infty}^d \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$ is the standard normal distribution function.

Since P_k is a positive increasing function of K , P_k^{-1} is a positive increasing function of its argument. Since the argument of P_k^{-1} is increasing in x for $x > L$, it follows that $f(x)$ is a positive increasing function on this domain. If $f(x) = 0$ for x less than some point, then we identify this point as the lower liquidation level L .

II-C Solving for the Stock Pricing Function

If one is interested in obtaining the stock pricing function on the time interval $[0, T]$, then the Feynman-Kac theorem can be used to find the continuous solution to the BVP consisting of the PDE (6) with $t \in [0, T]$, the boundary conditions (7) and (8), and the terminal condition $f(\cdot)$ determined in (10):

$$s(x, t) = e^{-(r-q)(T-t)} E_{x,t}^Q [f(W_T) 1(\tau > T)], \quad x > L, t \in [0, T \wedge \tau], \quad (11)$$

where $\{W_u; u \in [t, T]\}$ is a Q-SBM starting at x . Since $f(\cdot)$ is a positive increasing function of x , so is $s(x, t)$ for each t .

We now consider the determination of the stock pricing function on the time interval $[0, T')$, where $T' > T$ can be arbitrarily large. If we can obtain the horizon payoff at T' from $f(\cdot)$, then we can again use the Feynman-Kac theorem to represent the solution on $[0, T')$. It is tempting to write:

$$f(x) = e^{-(r-q)(T'-T)} \int_L^\infty \frac{1}{\sqrt{2\pi(T'-T)}} \left\{ \exp \left[-\frac{1}{2} \left(\frac{z-x}{\sqrt{T'-T}} \right)^2 \right] - \exp \left[-\frac{1}{2} \left(\frac{z-x-2L}{\sqrt{T'-T}} \right)^2 \right] \right\} g(z) dz, \quad x > L \quad (12)$$

as an implicit definition for the horizon payoff $g(\cdot)$ arising at T' . Since $f(x)e^{(r-q)(T'-T)}$ is a convolution of the transition density of absorbing SBM with $g(z)1(z > L)$, Fourier transforms can be used to explicitly express $g(\cdot)$ in terms of $f(\cdot)$. However, this formulation may be ill-posed in that there is no result on the existence of $g(\cdot)$. To appreciate that this is not merely a technical concern, suppose that $f(x)$ described the value at T of a call on the SBM paying $(W_M - K)^+$ at some date M fixed strictly between T and T' . Then it will be impossible to obtain the stock pricing function for any time after M . On the other hand, in the Black Scholes model, an implied payoff $f(x) = S_0 e^{(r-q-\sigma^2)T + \sigma W_T}$ would generate a horizon payoff $g(z) = S_0 e^{(r-q-\sigma^2)T' + \sigma W_{T'}}$ for any $T' \geq T$ without difficulty.

Thus, the question arises as to whether there exist other processes besides geometric Brownian motion which are sustainable over an arbitrarily long horizon. Fortunately, an extension of a result of Robbins and Siegmund[14] (see Karatzas and Shreve[12], pg. 262) proves that the *entire* set of nonnegative functions $\phi(z, t)$ which satisfy the backward diffusion equation $\frac{\partial \phi}{\partial t}(z, t) + \frac{1}{2} \frac{\partial^2 \phi}{\partial z^2}(z, t) = 0$ on the half plane $(0, \infty) \times \mathfrak{R}$ subject to:

$$\begin{aligned} \lim_{t \downarrow 0} \phi(0, t) = 1, \quad \lim_{t \uparrow \infty} \phi(0, t) = 0, \quad \text{and} \\ \lim_{z \downarrow -\infty} \phi(z, t) = 0, \quad \lim_{z \uparrow \infty} \phi(z, t) = \infty, \quad 0 < t < \infty, \end{aligned}$$

are given by:

$$\phi(z, t) = \int_0^\infty e^{\theta z - \frac{\theta^2}{2} t} dG(\theta), \quad (13)$$

where $G(\theta)$ is a distribution function with $G(\infty) = 1$ and $G(0+) = 0$. Surprisingly, the process $\phi(W_t, t), t > 0$ is just the familiar geometric Brownian motion for the futures price relative $\frac{F_t}{F_0}$, where the volatility

parameter has been randomized⁷ by a probability measure G defined on \mathfrak{R}^+ .

Adapting these results to a stock price process with absorption at zero, Appendix 1 proves that the entire set of stock pricing functions $s(x, t)$ which vanish at $x = L$, are positive, increasing, and unbounded in x on $x > L$, and which satisfy the PDE (6) for $x > L$ and $t \in (0, \infty)$ are given by:

$$s(x, t) = S_0 e^{(r-q)t} \int_0^\infty \frac{\sinh[\theta(x-L)]}{\sinh[-\theta L]} e^{-\frac{\theta^2}{2}t} dG(\theta), \quad x > L, t \in (0, T'), \quad (14)$$

where $\sinh(x) \equiv \frac{e^x - e^{-x}}{2}$. Assuming that $G(\cdot)$ is the distribution function of a continuous random variable with a probability density function $G'(\theta)$, evaluation of (14) at $t = T$ relates the implied stock payoff $f(x)$ to the density function $G'(\theta)$:

$$s(x, T) \equiv f(x) = S_0 e^{(r-q)T} \int_0^\infty \frac{\sinh[\theta(x-L)]}{\sinh[-\theta L]} e^{-\frac{\theta^2}{2}T} G'(\theta) d\theta, \quad x > L. \quad (15)$$

Appendix 1 proves that for any function $f(x)$ such that $f(x)e^{-x^2}$ is L^1 , a unique density function $G'(\theta)$ exists which satisfies (15). Appendix 1 also shows how to invert (15) to obtain this density function $G'(\theta)$ from the known implied payoff $f(x)$:

$$G'(\theta) = \begin{cases} \frac{\sqrt{T}}{S_0 e^{(r-q)T}} \frac{\sinh(-\theta L)}{\pi\sqrt{2\pi}} \int_{-\infty}^\infty e^{-i\omega\theta\sqrt{T} + \frac{\omega^2}{2}} \int_{-\infty}^\infty e^{i\omega z - \frac{z^2}{2}} f(L + z\sqrt{T}) dz d\omega & \text{if } \theta \geq 0; \\ 0 & \text{if } \theta < 0. \end{cases} \quad (16)$$

Thus, the stock pricing function $s(x, t)$ is given by substituting (16) in (14):

$$\begin{aligned} s(x, t) &= \frac{e^{-(r-q)(T-t)}}{\pi\sqrt{2\pi}/T} \int_0^\infty \sinh[\theta(x-L)] e^{-\frac{\theta^2}{2}t} \int_{-\infty}^\infty e^{-i\omega\theta\sqrt{T} + \frac{\omega^2}{2}} \int_{-\infty}^\infty e^{i\omega z - \frac{z^2}{2}} f(L + z\sqrt{T}) dz d\omega d\theta \\ &= \frac{e^{-(r-q)(T-t)}\sqrt{T}}{2\pi} \int_{-\infty}^\infty e^{\frac{\omega^2 + Ty_t^2}{2}} \int_{-\infty}^\infty e^{i\omega z - \frac{z^2}{2}} f(L + z\sqrt{T}) [N(y_t) - N(-y_t)] dz d\omega, \end{aligned} \quad (17)$$

for $x > L, t \in (0, T')$, where $y_t \equiv \frac{x-L-i\omega\sqrt{T}}{\sqrt{t}}$ and $T' \geq T$ is finite but can be arbitrarily large. For certain specifications of the implied payoff function $f(\cdot)$, both integrals in (17) can be done analytically, as the next subsection demonstrates. It can be shown that $s(x, t)$ is a positive increasing function of x on $x > L$ and $t \in (0, T')$. The horizon payoff is obtained by evaluating $s(x, t)$ at $t = T'$:

$$g(x) \equiv s(x, T') = \frac{e^{(r-q)(T'-T)}\sqrt{T}}{2\pi} \int_{-\infty}^\infty e^{\frac{\omega^2 + Ty_{T'}^2}{2}} \int_{-\infty}^\infty e^{i\omega z - \frac{z^2}{2}} f(L + z\sqrt{T}) [N(y_{T'}) - N(-y_{T'})] dz d\omega, \quad (18)$$

Furthermore, the risk-neutral process is given by $S_t = s(W_t^a, t), t > (0, T')$, where recall that W_t^a is SBM absorbing at a fixed $L < 0$.

⁷This randomized process would describe the futures price relative in a filtering context if the futures price process were assumed to be geometric Brownian motion with an unknown positive volatility.

II-D Example

We now illustrate our result with a simple example. Suppose that the initial T -maturity option prices are such that vertical spreads have the form:

$$e^{rT} P_k(K) = \int_L^{\frac{\sinh^{-1}(\frac{K}{\beta})+L}{\alpha}} \frac{1}{\sqrt{2\pi T}} \left\{ \exp \left[-\frac{1}{2} \left(\frac{z}{\sqrt{T}} \right)^2 \right] - \exp \left[-\frac{1}{2} \left(\frac{z-2L}{\sqrt{T}} \right)^2 \right] \right\} dz, \quad K > 0. \quad (19)$$

Then from (9):

$$f^{-1}(S) = \begin{cases} \frac{\sinh^{-1}}{\alpha} \left(\frac{S}{\beta} \right) + L & \text{if } S > 0; \\ 0 & \text{if } S < 0. \end{cases}$$

Inverting this function:

$$f(x) = \begin{cases} \beta \sinh[\alpha(x - L)] & \text{if } x > L; \\ 0 & \text{if } x < L. \end{cases} \quad (20)$$

Figure 1 graphs the implied payoff function and its inverse. Using (16), we obtain that:

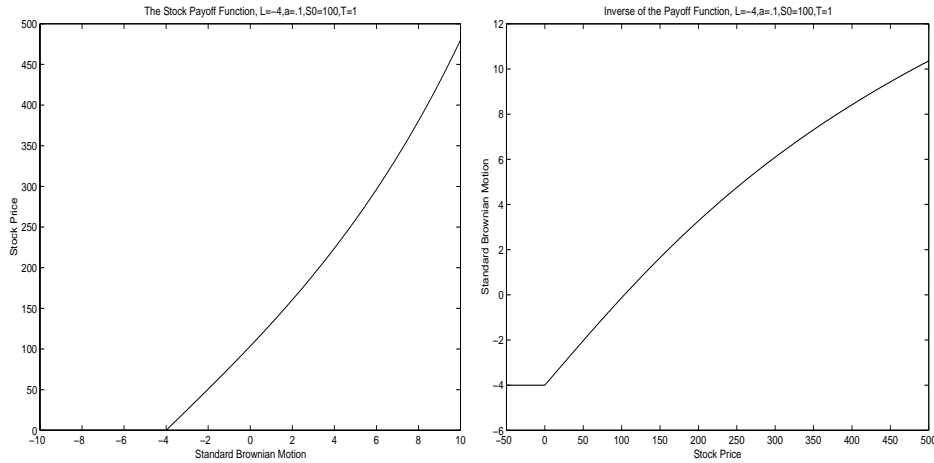


Figure 1: The Implied Stock Payoff Function and Its Inverse

$$G'(\theta) = \delta(\theta - \alpha), \quad (21)$$

where $\delta(\cdot)$ is Dirac's delta function. Substituting (21) into (18) implies that:

$$s(x, T') = g(x) = \beta' \sinh[\alpha(x - L)]. \quad (22)$$

The solution of (6) subject to (7), (8), and (22) is:

$$s(x, t) = \beta' e^{\mu(T'-t)} \sinh[\alpha(x - L)], \quad t \in [0, T'], x > L, \quad (23)$$

where $\mu \equiv r - q - \alpha^2/2$. Setting $s(0, 0) = S_0$ implies:

$$\beta' = S_0 e^{-\mu T'} \operatorname{csch}(-\alpha L), \quad (24)$$

where $\operatorname{csch}(x) \equiv \frac{1}{\sinh(x)}$. Hence from (23):

$$s(x, t) = S_0 e^{\mu t} \operatorname{csch}(-\alpha L) \sinh[\alpha(x - L)], \quad t \in [0, T']. \quad (25)$$

Figure 2 graphs this stock pricing function against the driving SBM and time. For each time, the stock price is an increasing convex function of the SBM, although the slope and convexity are *larger* than in the Black Scholes model. The volatility surface corresponding to the risk-neutral process $S_t = S_0 e^{\mu t} \operatorname{csch}(-\alpha L) \sinh[\alpha(W_t^a - L)], t \in [0, T']$ is determined in the next section.

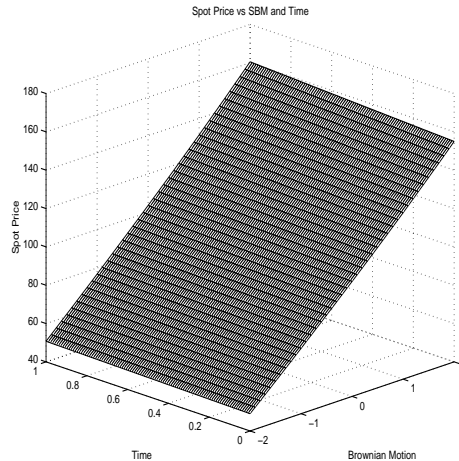


Figure 2: The Stock Pricing Function

III Local Volatility

The next subsection shows how the local volatility can be represented in terms of the known stock pricing function derived in the last section from the given implied volatility smile. The following subsection illustrates our results with the same example developed in the last section. The last subsection shows that our volatility function $\sigma(S, t)$ satisfies a certain nonlinear PDE. Conversely, it shows that if the local volatility function satisfies the nonlinear PDE, then the stock price process is path-independent and the stock pricing function can be determined.

III-A The Local Volatility Surface

Since the stock pricing function $s(x, t)$ is increasing in x on $x > L$, it can be inverted for $x = s^{-1}(S, t)$. Equating coefficients on dW_t^a in (2) and (3) determines local volatility $\sigma(S, t)$ in terms of this inverse and $\frac{\partial s}{\partial x}(x, t)$:

$$\sigma(S, t) = \frac{1}{S} \frac{\partial s}{\partial x}(s^{-1}(S, t), t), \quad t \in [0, T'], x \in \mathfrak{R}. \quad (26)$$

Since $s(\cdot, t)$ is increasing in x for each t , $\sigma(S, t)$ is positive. Local volatility is an explicit function of S and t if $s_x(x, t)$, and $s^{-1}(S, t)$ can both be written explicitly in terms of their arguments. We have thus shown that the path-independence assumption combined with the T -maturity implied smile fully determines the local volatility surface. By evaluating the function $\sigma(S, t)$ at historical spot prices and times, the theory can be tested using historical local volatilities.

III-B Example

To illustrate our results, again suppose that T -maturity option prices satisfy (19), so that the stock pricing function is given in (23). Differentiating (23) w.r.t. x implies that:

$$s_x(x, t) = \alpha \beta' e^{\mu(T'-t)} \cosh[\alpha(x - L)], \quad (27)$$

where $\cosh(x) \equiv \frac{e^x + e^{-x}}{2}$. Multiplying and dividing by $\sinh[\alpha(x - L)]$ gives:

$$s_x(x, t) = \alpha S \coth[\alpha(x - L)], \quad t \in (0, T'), \quad (28)$$

from (23), where $\coth(x) \equiv \frac{\cosh(x)}{\sinh(x)}$. Inverting (23) w.r.t. x implies that:

$$s^{-1}(S, t) = L + \frac{\sinh^{-1}}{\alpha} \left(\frac{S}{\beta'} e^{\mu(T'-t)} \right), \quad t \in (0, T'). \quad (29)$$

Substituting (28) and (29) in (26) determines the local volatility surface:

$$\sigma(S, t) = \alpha \coth \left[\sinh^{-1} \left(\frac{S}{\beta'} e^{\mu(T'-t)} \right) \right], \quad t \in (0, T'), \quad (30)$$

where β' is given in (24). Figure 3 graphs this local volatility surface.

To understand the behavior of this local volatility as a function of the stock price, note that the stock pricing function is proportional to $\sinh[\alpha(x - L)]$, which behaves linearly in x for x near L and exponentially

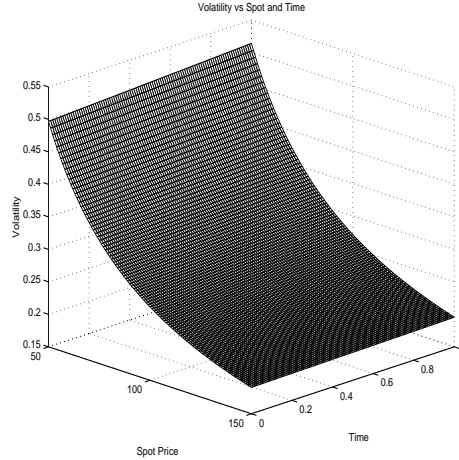


Figure 3: The Local Volatility Surface

in αx for x large. Thus at each future date, the volatility smile is approximately hyperbolic in S (“normal volatility”) for S near zero, while it is asymptoting to the constant α (“lognormal volatility”) for S high. Mathematically, as $S \downarrow 0$ in (30), $\sinh^{-1}(\cdot) \downarrow 0$, $\coth(\cdot) \uparrow \infty$, as does volatility. Conversely, as $S \uparrow \infty$ in (30), $\sinh^{-1}(\cdot) \uparrow \infty$, $\coth(\cdot) \downarrow 1$, and volatility asymptotes to α . As S increases from 0 to ∞ , the volatility smile slopes downward in a convex fashion.

To understand the behavior of this local volatility as a function of the future time t , substitute β' given in (24) into (30):

$$\sigma(S, t) = \alpha \coth \left[\sinh^{-1} \left(\frac{S}{S_0 e^{\mu(T'-t)}} \sinh(-\alpha L) \right) \right], \quad t \in [0, T']. \quad (31)$$

Thus, along the path $S = S_0 e^{\mu(T'-t)}$, volatility is constant at $\alpha \coth(-\alpha L)$. Thus, just as α controls the asymptotic height of the volatility smile, the parameter L controls this “at-the-money” volatility. For fixed S_0 , as $L \downarrow -\infty$, $\coth(\cdot) \uparrow 1$ and at-the-money volatility approaches α . Conversely, as $L \uparrow 0$, $\coth(\cdot) \uparrow \infty$ as does volatility. Thus, in our model, the more likely is bankruptcy, the greater is the relative volatility perhaps due to shareholders’ greater willingness to gamble.

As the terminal time T' approaches infinity, then for any fixed level of S , the terminal volatility smile $\sigma(S, T')$ asymptotes down to α if $\mu < 0$ and becomes unbounded above if $\mu > 0$. Since $\mu = r - q - \alpha^2/2$ is increasing in the risk-neutral drift $r - q$, it appears counter-intuitive that raising μ from a negative value to a positive one would raise asymptotic volatility from α to infinity. The explanation is that along a fixed

stock price path, shifting μ from a negative value to a positive value lowers the corresponding path of the driving SBM, making bankruptcy more likely.

III-C PDE and Converse

Using (26) as a change of variables in (6), Appendix 2 proves that $\sigma(S, t)$ obeys the following nonlinear PDE⁸:

$$\frac{\sigma^2(S, t)S^2}{2} \frac{\partial^2 \sigma(S, t)}{\partial S^2} + [r - q + \sigma^2(S, t)]S \frac{\partial \sigma}{\partial S}(S, t) + \frac{\partial \sigma}{\partial t}(S, t) = 0, \quad S > 0, t \in (0, T'). \quad (32)$$

Setting $t = T'$ in (26), the T' -maturity (local) volatility smile is

$$\sigma(S, T') = \frac{g'(g^{-1}(S))}{S} \equiv v(S), \quad S > 0, \quad (33)$$

where $g'(x)$ and $g^{-1}(S)$ are determined by respectively differentiating and inverting $g(x)$, determined in (18). Since $g(x)$ is an increasing function for $x > L$, $g^{-1}(S)$ is well-defined for $S > 0$ and $\sigma(S, T')$ is positive. Equation (26) represents a class of solutions of the nonlinear PDE (32) subject to (33).

We have seen that the assumptions of path-independence and no arbitrage imply that the volatility function satisfies a fully non-linear PDE with a known solution class. Conversely, Appendix 3 proves that if a given local volatility function $\sigma(S, t)$ satisfies the nonlinear PDE (32), then the stock price is path-independent. The stock pricing function $s(x, t)$ is given by (12), where the horizon payoff $g(S)$ is still given by (18). Alternatively, Appendix 4 proves that the implied payoff can be determined from the T' -maturity local volatility smile $v(S) \equiv \sigma(S, T')$:

$$f(x) = I_v^{-1}(x + c_0), \quad x \in \mathfrak{R},$$

where:

$$I_v(f) \equiv \int_{c_1}^f \frac{1}{v(S)S} dS, \quad f \geq 0,$$

and c_0 and c_1 are constants. Finally, Appendix 3 shows that the inverse of the stock pricing function is given by:

$$s^{-1}(S, t) = \int_{S_0}^S \frac{1}{\sigma(Z, t)Z} dZ + \int_0^t \left[\frac{1}{2} \frac{\partial}{\partial S} [\sigma(S, u)S] \Big|_{S=S_0} - \frac{(r - q)}{\sigma(S_0, u)} \right] du, \quad S > 0, t \in [0, T'].$$

⁸If the local volatility of the spot price S is modelled as a function of the T' -maturity forward price F instead of S , then PDE (32) is altered by replacing F with S and setting $r = q = 0$.

IV Pricing Path-Independent Options

The next subsection derives explicit valuation formulas for standard options and risk-neutral densities of any maturity under a wide class of volatility functions consistent with the given smile and path-independence. The following subsection illustrates our results with our running example. The final section shows how the driving Brownian motion and the local volatility can each be interpreted as a path-independent exotic option written on the stock price.

IV-A Standard Options

To value European calls of some intermediate maturity $M \in [0, T]$ in terms of the contemporaneous spot price, we first determine the function $\gamma(x, t)$ relating the call value to the SBM W_t and time t . By Itô's lemma, this pricing function solves:

$$\frac{1}{2} \frac{\partial^2 \gamma}{\partial x^2}(x, t) + \frac{\partial \gamma}{\partial t}(x, t) = r\gamma(x, t), \quad x > L, t \in (0, M), \quad (34)$$

subject to the boundary conditions:

$$\lim_{x \downarrow L} \gamma(x, t) = 0, \quad \lim_{x \uparrow \infty} \gamma(x, t) = s(x, t)e^{-q(M-t)} - Ke^{-r(M-t)}, \quad t \in [0, T], \quad (35)$$

where $K > 0$ is the call strike, and subject to the terminal condition:

$$\gamma(x, M) = [s(x, M) - K]^+, \quad x > L. \quad (36)$$

By the Feynman-Kac theorem, the continuous solution to this BVP is:

$$\gamma(x, t) = e^{-r(M-t)} E_{x,t}^Q [s(W_M, M) - K]^+, \quad x > L, t \in [0, M \wedge \tau), \quad (37)$$

where $\{W_u, u \in [t, M]\}$ is a Q-SBM starting at x at time t . Equation (37) relates the call value to the SBM W_t and time t . To instead relate the call value to the stock price and time, let $c(S, t) = \gamma(x, t)$ in (37), where $x = s^{-1}(S, t)$:

$$c(S, t) = e^{-r(M-t)} \int_L^\infty \frac{[s(z, M) - K]^+}{\sqrt{2\pi(M-t)}} \left\{ \exp \left[-\frac{1}{2} \left(\frac{z - s^{-1}(S, t)}{\sqrt{M-t}} \right)^2 \right] - \exp \left[-\frac{1}{2} \left(\frac{z + s^{-1}(S, t) - 2L}{\sqrt{M-t}} \right)^2 \right] \right\} dz, \quad (38)$$

for $S \geq 0, t \in [0, M \wedge \tau)$. This solution will be an explicit function of S and t if $s^{-1}(S, t)$ can be written explicitly in terms of its arguments.

Differentiating (38) twice w.r.t. strike gives the implied risk neutral probability that the stock price is at K at time M , given that the stock is at S at time t . Alternatively, the change of variables $S = s(x, t)$ for the absorbing SBM transition density expresses this probability as:

$$q(Z, M; S, t) = \frac{1}{\sqrt{2\pi(M-t)}} \left\{ \exp \left\{ -\frac{1}{2} \left[\frac{s^{-1}(Z, M) - s^{-1}(S, t)}{\sqrt{M-t}} \right]^2 \right\} - \exp \left\{ -\frac{1}{2} \left[\frac{s^{-1}(Z, M) + s^{-1}(S, t)}{\sqrt{M-t}} \right]^2 \right\} \right\} \frac{\partial s^{-1}}{\partial S}(Z, M), \quad S > 0, t \in [0, M). \quad (39)$$

The density will be positive only if $s^{-1}(S, M)$ is increasing in S and it will be explicit in Z only if $s^{-1}(Z, M)$ is explicit in Z .

An alternative derivation of the call pricing function $C(S, t), S \geq 0, t \in [0, M \wedge \tau)$ is obtained by integrating its payoff $(S - K)^+$ against this density, and discounting at the riskfree rate:

$$C(S, t; K, M) = e^{-r(M-t)} \int_K^\infty \frac{Z-K}{\sqrt{2\pi(M-t)}} \left\{ \exp \left\{ -\frac{1}{2} \left[\frac{s^{-1}(Z, M) - s^{-1}(S, t)}{\sqrt{M-t}} \right]^2 \right\} - \exp \left\{ -\frac{1}{2} \left[\frac{s^{-1}(Z, M) + s^{-1}(S, t)}{\sqrt{M-t}} \right]^2 \right\} \right\} \frac{\partial s^{-1}}{\partial S}(Z, M) dZ. \quad (40)$$

To the extent that markets for intermediate maturity calls exist, our theory can be tested in the usual manner by comparing market prices with model values, obtained from either (38) or (40). To obtain an initial *implied* volatility surface, one must numerically solve $c_{BS}(S_0, 0, \sigma_i(K, M); K, M) = c(S_0, 0; K, M)$ for $\sigma_i(K, M)$ at each $K > 0$ and $M \in (0, T')$, where $c_{BS}(S, t, \sigma_i; K, M)$ is the Black-Scholes call pricing formula.

IV-B Example

To illustrate our results, suppose again that $T-$ maturity option prices satisfy (19), so that the stock pricing function is given by (23) and the inverse of this function is given in (29). Evaluating this inverse at $(S, t) = (Z, M)$ yields:

$$s^{-1}(Z, M) = L + \frac{\sinh^{-1} \left(\frac{Z}{\beta'} e^{\mu(T'-M)} \right)}{\alpha}. \quad (41)$$

Differentiating w.r.t the first argument yields:

$$\frac{\partial_{S^{-1}}}{\partial S}(Z, M) = \frac{1}{\alpha\sqrt{Z^2 + (\beta')^2 e^{-2\mu(T'-M)}}}. \quad (42)$$

Substituting (41), (29), and (42) in (39) implies that the risk-neutral stock pricing density is:

$$q(Z, M; S, t) = \frac{1}{\sqrt{2\pi(M-t)}} \left\{ \exp \left\{ -\frac{1}{2} \left[\frac{\sinh^{-1} \left(\frac{Z}{\beta'} e^{\mu(T'-M)} \right) - \sinh^{-1} \left(\frac{S}{\beta'} e^{\mu(T'-t)} \right)}{\alpha\sqrt{M-t}} \right]^2 \right\} \right. \\ \left. - \exp \left\{ -\frac{1}{2} \left[\frac{\sinh^{-1} \left(\frac{Z}{\beta'} e^{\mu(T'-M)} \right) + \sinh^{-1} \left(\frac{S}{\beta'} e^{\mu(T'-t)} \right)}{\alpha\sqrt{M-t}} \right]^2 \right\} \right\} \frac{1}{\alpha\sqrt{Z^2 + (\beta')^2 e^{-2\mu(T'-M)}}}, \quad (43)$$

for $S > 0, t \in [0, M \wedge \tau)$. Figure 4 graphs this density (termed the arcsinhnormal) against the future spot price and time. The downward sloping volatility surface graphed in Figure 3 cancels much of the positive skewness of the lognormal density leading to a close approximation of a Gaussian density.

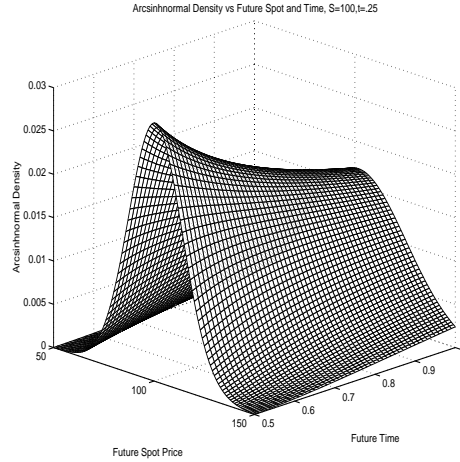


Figure 4: The Arcsinhnormal Probability Density Function

Integrating the call's payoff against this density yields the following pricing formula:

$$C(S, t) = \frac{e^{-q(M-t)}}{2} \left(S + \sqrt{S^2 + \beta^2 e^{-2\mu(T-t)}} \right) [N(d_+ + \alpha\sqrt{M-t}) + N(d_- - \alpha\sqrt{M-t})] \\ - \frac{\beta^2 e^{-q(M-t)}}{2e^{2\mu(T-t)}} \frac{1}{S + \sqrt{S^2 + \beta^2 e^{-2\mu(T-t)}}} [N(d_+ - \alpha\sqrt{M-t}) + N(d_- + \alpha\sqrt{M-t})] \\ - K e^{-r(M-t)} [N(d_+) - N(d_-)],$$

where:

$$d_{\pm} \equiv \frac{\pm \sinh^{-1} \left(\frac{S}{\beta} e^{\mu(T-t)} \right) - \sinh^{-1} \left(\frac{K}{\beta} e^{\mu(T-M)} \right)}{\alpha \sqrt{M-t}}.$$

Figure 5 graphs the call value and time values of this model against the corresponding values in the Black-Scholes model with the same at-the-money implied volatility. The negative skewness apparent in the volatility surface and arcsinhnormal density function is manifested in higher out-of-the-money put prices and lower out-of-the-money call prices. Figure 6 plots the arcsinhnormal call value against the current

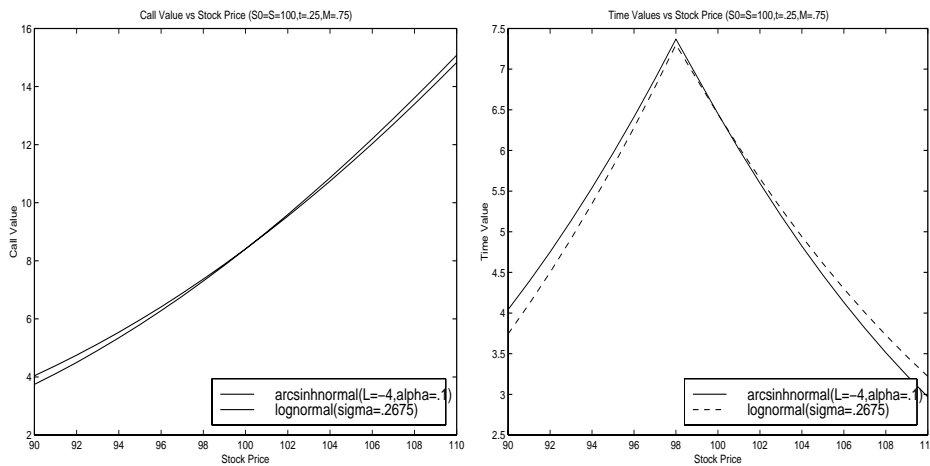


Figure 5: The Arcsinhnormal Call Value and Time Value vs. Black-Scholes

stock price and time.

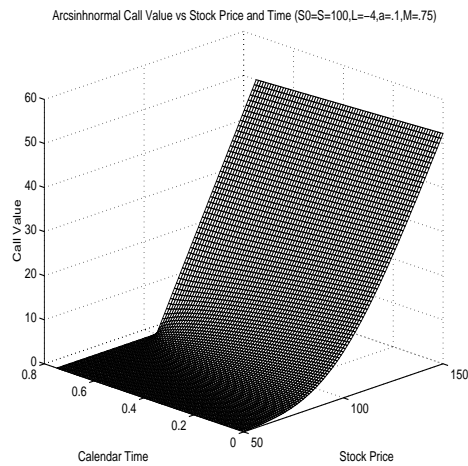


Figure 6: The Arcsinhnormal Call Valuation Function

IV-C Path-Independent Exotic Options

The SBM W_t is the forward price of a path-independent exotic option with the payoff $g^{-1}(S_{T'})$ at T' , where $g^{-1}(\cdot)$ is the inverse of $g(x)$ determined in Appendix 1 from $f(\cdot)$. This follows from the observation that deferring the payment of this exotic's premium to T' induces zero drift under Q and the payoff induces unit volatility. Thus, $s(x, t)$ is properly called the spot pricing function, since it relates the spot price of the underlying to the (forward) value of an asset. Similarly, $s^{-1}(S_t, t)$ is a standard pricing function relating the time t forward price of the Brownian exotic paying $g^{-1}(S_{T'})$ at T' to the time t spot price and time.

The local volatility $\sigma(S_t, t)$ can also be interpreted as the price process for an exotic equity derivative. To determine the payoff, note from (33) that the T' -maturity *absolute volatility* $a(S, T') \equiv \sigma(S, T')S$ is the following function of the time T' spot price $S_{T'}$:

$$a(S_{T'}, T') = g'(W_{T'}^a) = \frac{1}{\frac{\partial g^{-1}}{\partial S}(S_{T'})} = \lim_{h \downarrow 0} \frac{h}{g^{-1}(S_{T'} + h) - g^{-1}(S_{T'})}.$$

The T' -maturity (relative) volatility smile $v(S)$ arises from quantoing this dollar payoff into shares i.e. the payoff in shares is:

$$\sigma(S_{T'}, T') = v(S_{T'}) = \frac{g'(W_{T'}^a)}{S_{T'}} = \frac{1}{S_{T'}} \lim_{h \downarrow 0} \frac{h}{g^{-1}(S_{T'} + h) - g^{-1}(S_{T'})}.$$

If we also assume that the premium on this exotic is specified in shares and paid at T' , then this share-denominated forward price matches the local volatility $\sigma(S_t, t)$ at all times up to T' . The appearance of $\sigma^2(S, t)$ in the middle term of the PDE (32) is now easily interpreted as a quanto correction. The absence of the usual discounting term in (32) is due to the deferral of the premium payment to maturity.

V Summary and Future Research

By interpreting the stock as essentially a path-independent derivative on the driving Brownian motion (or conversely), we were able to develop a PDE for the stock pricing function. Adding the information from a complete strike structure of option prices and the economic restrictions implied by no arbitrage and limited liability allowed us to uniquely determine this stock pricing function. Similarly, by interpreting local volatility as an exotic derivative on the stock, we were able to develop a nonlinear PDE for the

volatility function, which we solved analytically using the known stock pricing function. Conversely, if local volatilities are assumed to satisfy the nonlinear PDE, then stock prices are path-independent and the function linking the SBM to the stock price can be determined analytically. Finally, we derived closed form formulas for option prices and risk-neutral densities consistent with a wide class of local volatility functions.

In principle, this work can be extended to valuing American and other path-dependent options. For standard American and standard barrier options, the PDE's developed in this paper apply on restricted domains. Similarly, for exotic options with mild path-dependence such as compound, chooser and discretely sampled barrier options, the PDE's developed in this paper would be solved recursively. For options with stronger path-dependence such as Asian, lookback and passport options, additional state variables would be required. Since numerical solutions may be required in any case, the question of the optimal discretization of the PDE's should be considered.

If we relax the assumption of this work that the T -maturity implied volatility smile is given, then a second extension would involve calibrating the smile to a matrix of initial option prices. Evaluating (38) at $(S, t) = (S_0, 0)$ determines initial call values of any maturity M and strike K in terms of the functions $s(S, M)$ and $s^{-1}(S_0, 0)$, which can both be expressed in terms of $f(S)$ via (17). Thus, the inverse problem of finding $f(\cdot)$ from a matrix of initial option prices can be posed once we select an objective function such as minimizing squared distance from a prior or maximizing smoothness. Under the assumptions of this paper, the identification of $f(\cdot)$ is tantamount to the determination of the risk-neutral process and volatility surface for all times. Given that $f(\cdot)$ relates the stock price at T to normally distributed SBM, a natural basis representation of $f(\cdot)$ arises in the form of Hermite polynomials. As these techniques are fairly involved, this extension is best left for future research.

A minor extension of these results which would generate a wider class of analytic volatility surfaces is to assume path-independence after a deterministic time change. To illustrate, suppose that the stock price is an Ornstein-Uhlenbeck process absorbing at the origin as in Cox and Ross[5]. Then the stock price depends on the path of the driving SBM, and as a consequence does not fit the path-independence property described in this paper. However, it is well known (see eg. Cox and Miller[4] pg. 229) that such

a stock price process is proportional to a standard Brownian motion run on a deterministically different clock.

A major extension of these results which would generate a much wider class of analytic volatility surfaces would involve replacing the one dimensional SBM with an alternative process (eg. Bessel squared) or even a d -dimensional diffusion process whose solution is known. One can also add jumps in order to generate the casual empirical observation that implied volatilities at low spot prices tend to be higher at short maturities than long ones. One way to generate jumps is to do a stochastic time change⁹ on the standard Brownian motion driving all assets. For example if a gamma time change was used, then the fundamental driver becomes a symmetric VG process (see Madan, Carr, and Chang[13]), while the spot price process would be a generalized VG process. As in the continuous case, a generalized VG Levy measure when written as a function of jumps in the spot price would revert to the standard symmetric VG Levy measure when written in terms of jumps of the driving variable. In the interest of brevity, such extensions are best left for future research.

⁹In fact, a stochastic time change on a GBM is an alternative to the scale change performed in this paper. Both approaches can generate a volatility smile.

Appendix 1: Characterizing Stock Pricing Functions

In this appendix, we characterize the entire set of stock pricing functions $s(x, t)$ which vanish at $x = L$, are positive, increasing, and unbounded in x for $x > L$, and are consistent with no arbitrage on the time interval $(0, T')$, where T' can be arbitrarily large. Robbins and Siegmund[14] have shown that the entire class of positive martingale functions $m(y, u)$ which satisfy the backward diffusion equation $\frac{\partial m}{\partial u} + \frac{1}{2} \frac{\partial^2 m}{\partial y^2} = 0$ on the region $\mathfrak{R}^+ \times (0, U)$ are given by:

$$m(y, u) = \int_0^\infty \frac{e^{-\frac{1}{2}\left(\frac{\theta-y}{\sqrt{U-u}}\right)^2} - e^{-\frac{1}{2}\left(\frac{\theta+y}{\sqrt{U-u}}\right)^2}}{\sqrt{2\pi(U-u)}} dG(\theta) + \int_u^{U^+} \frac{ye^{-\frac{1}{2}\left(\frac{y}{\sqrt{v-u}}\right)^2}}{\sqrt{2\pi(v-u)^3}} dH(v), \quad (44)$$

for some measures G on $(0, \infty)$ and H on $(0, U)$. The kernel of the first integral is recognized as the transition density for SBM absorbing at the origin. The kernel of the second integral is recognized as the density for the first passage time of SBM to the origin. If we set $H = 0$, then (44) describes the set of all nonnegative martingale functions satisfying the backward diffusion equation on $\mathfrak{R}^+ \times (0, U)$ and vanishing at the origin.

To extend this class of functions to the entire time axis $(0, \infty)$, let:

$$v(z, t) \equiv \frac{m(y, u)}{n(y, u)}, \quad (45)$$

where $z \equiv \frac{y}{U-u}$, $t = \frac{1}{U-u}$, and:

$$n(y, u) \equiv \frac{e^{-\frac{1}{2}\left(\frac{y}{\sqrt{U-u}}\right)^2}}{\sqrt{2\pi(U-u)}}$$

is a normal density function. Then for any solution m to the backward diffusion equation, it is straightforward to verify that $v(z, t)$ also solves the backward diffusion equation $\frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial^2 v}{\partial z^2} = 0$ on the region $\mathfrak{R}^+ \times \left(\frac{1}{U}, \infty\right)$. Letting U approach infinity in the original problem implies that $v(z, t)$ is defined on the open quarter plane $\mathfrak{R}^+ \times \mathfrak{R}^+$. When m is given by (44) with $H(\cdot) = 0$, then (45) implies that:

$$v(z, t) = 2 \int_0^\infty \sinh(\theta z) e^{-\frac{1}{2}\theta^2 t} dG(\theta),$$

which vanishes at $z = 0$.

To obtain (46), we let $x = z + L$ so that the vanishing arises at $x = L < 0$ and we also account for the stock's initial value S_0 and risk-neutral drift $r - q$:

$$s(x, t) = S_0 e^{(r-q)t} \int_0^\infty \frac{\sinh[\theta(x-L)]}{\sinh[-\theta L]} e^{-\frac{\theta^2}{2}t} dG(\theta), \quad x > L, t \in (0, T). \quad (46)$$

We next show how to determine the distribution function $G(\theta)$ from the known implied payoff $f(x)$. We assume that the distribution function is that of a continuous random variable, so that $dG(\theta) = G'(\theta)d\theta$. Evaluating (46) at $t = T$ and re-arranging implies:

$$\frac{f(x)}{F_0} = \int_0^\infty \sinh[\theta(x-L)] e^{-\frac{\theta^2}{2}T} \frac{G'(\theta)}{\sinh(-\theta L)} d\theta,$$

where $F_0 = S_0 e^{(r-q)T}$ is the T -maturity forward price. Letting $z = \frac{x-L}{\sqrt{T}}$ and $y = \theta\sqrt{T}$ yields:

$$\sqrt{T} \frac{f(L + z\sqrt{T})}{F_0} = \int_0^\infty \sinh(yz) e^{-\frac{y^2}{2}} \frac{G'(y/\sqrt{T})}{\sinh(-Ly/\sqrt{T})} dy.$$

Substituting in the definition of $\sinh(\cdot)$:

$$2\sqrt{T} \frac{f(L + z\sqrt{T})}{F_0} = \int_0^\infty e^{zy - \frac{y^2}{2}} \frac{G'(y/\sqrt{T})}{\sinh(-Ly/\sqrt{T})} dy - \int_0^\infty e^{-zy - \frac{y^2}{2}} \frac{G'(y/\sqrt{T})}{\sinh(-Ly/\sqrt{T})} dy.$$

Multiplying by $e^{-\frac{z^2}{2}}$ and changing variables as $y = -y$ in the second integral yields:

$$\hat{f}(z) \equiv 2\sqrt{T} \frac{f(L + z\sqrt{T})}{F_0} e^{-\frac{z^2}{2}} = \int_{-\infty}^\infty e^{-\frac{(z-y)^2}{2}} \phi(y) dy, \quad (47)$$

where:

$$\phi(y) = \begin{cases} \frac{G'(y/\sqrt{T})}{\sinh(-Ly/\sqrt{T})} & \text{if } y \geq 0; \\ -\frac{G'(-y/\sqrt{T})}{\sinh(Ly/\sqrt{T})} & \text{if } y < 0. \end{cases} \quad (48)$$

Equation (47) expresses $\hat{f}(\cdot)$ as a linear combination of translates of a standard normal density function. Wiener proved that the span of all such translates is L_1 (see Goldberg[10], pg. 33). Thus, so long as $f(x)e^{-x^2}$ is L_1 , there exists a $\phi(\cdot)$ generating it. To identify this ϕ , note that the right hand side of (47) is a convolution. Assuming that $e^{-\frac{(z-y)^2}{2}} \phi(y)$ is a square-integrable function of z for all $y \in \mathfrak{R}$, taking Fourier transforms of both sides yields:

$$\int_{-\infty}^\infty e^{i\omega z} \hat{f}(z) dz = \int_{-\infty}^\infty e^{i\omega y} \phi(y) dy \sqrt{2\pi} e^{-\omega^2/2},$$

since $e^{-\omega^2/2}$ is the Fourier transform of $\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$. Re-arranging this expression yields:

$$\int_{-\infty}^{\infty} e^{i\omega y} \phi(y) dy = \frac{e^{\omega^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega z} \hat{f}(z) dz.$$

Inverting for $\phi(y)$ yields:

$$\phi(y) = \frac{1}{\sqrt{(2\pi)^3}} \int_{-\infty}^{\infty} e^{-i\omega y + \frac{\omega^2}{2}} \int_{-\infty}^{\infty} e^{i\omega z} \hat{f}(z) dz d\omega.$$

Solving (48) for $G'(\cdot)$ when $y \geq 0$ yields:

$$\begin{aligned} G'(\theta) &= \sinh(-L\theta) \phi(\theta\sqrt{T}) \\ &= \frac{\sqrt{T} \sinh(-\theta L)}{F_0 \pi \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\theta\sqrt{T} + \frac{\omega^2}{2}} \int_{-\infty}^{\infty} e^{i\omega z - \frac{z^2}{2}} f(L + z\sqrt{T}) dz d\omega, \end{aligned} \quad (49)$$

for $\theta \geq 0$ and we set $G'(\theta) = 0$ otherwise. We are thus able to identify the risk-neutral stock price process consistent with the T -maturity option prices:

$$S_t = s(W_t, t) = S_0 e^{(r-q)t} \int_0^{\infty} \frac{\sinh[\theta(x-L)]}{\sinh[-\theta L]} e^{-\frac{\theta^2}{2}t} G'(\theta) d\theta, \quad t \in (0, \tau \wedge T'), \quad (50)$$

where $G'(\theta)$ is given in (49).

Appendix 2: The PDE for Local Volatility as a Consequence of Path-Independence

Recall the PDE (6) describing the spot pricing function $s(x, t)$:

$$\frac{\partial s}{\partial t}(x, t) + \frac{1}{2} \frac{\partial^2 s}{\partial x^2}(x, t) = (r - q)s(x, t), \quad t \in [0, T], x \in \mathfrak{R}. \quad (51)$$

Let $s_x(x, t) \equiv \frac{\partial s}{\partial x}(x, t)$. Differentiating (51) w.r.t. x implies:

$$\frac{\partial s_x}{\partial t}(x, t) + \frac{1}{2} \frac{\partial^2 s_x}{\partial x^2}(x, t) = (r - q)s_x(x, t), \quad t \in [0, T], x \in \mathfrak{R}. \quad (52)$$

Consider the change of variables derived in (26):

$$\sigma(S, t) = \frac{1}{S} \frac{\partial s}{\partial x}(x, t) = \frac{s_x(x, t)}{s(x, t)}, \quad \text{where } S = s(x, t), \quad t \in [0, T], x \in \mathfrak{R}. \quad (53)$$

Thus:

$$s_x(x, t) = \sigma(s(x, t), t)s(x, t). \quad (54)$$

$$\frac{\partial s_x}{\partial x}(x, t) = \frac{\partial \sigma}{\partial S}(S, t) \frac{\partial s}{\partial x}(x, t)s(x, t) + \sigma(S, t) \frac{\partial s}{\partial x}(x, t).$$

$$\begin{aligned} \frac{\partial^2 s_x}{\partial x^2}(x, t) &= \frac{\partial^2 \sigma}{\partial S^2}(S, t) \left(\frac{\partial s}{\partial x}(x, t) \right)^2 s(x, t) + \frac{\partial \sigma}{\partial S}(S, t) \frac{\partial^2 s}{\partial x^2}(x, t)s(x, t) \\ &\quad + 2 \frac{\partial \sigma}{\partial S}(S, t) \left(\frac{\partial s}{\partial x}(x, t) \right)^2 + \sigma(S, t) \frac{\partial^2 s}{\partial x^2}(x, t). \end{aligned}$$

$$\frac{\partial s_x}{\partial t}(x, t) = \frac{\partial \sigma}{\partial S}(S, t) \frac{\partial s}{\partial t}(x, t)s(x, t) + \frac{\partial \sigma}{\partial t}(S, t)s(x, t) + \sigma(S, t) \frac{\partial s}{\partial t}(x, t). \quad (55)$$

Substituting (54) to (55) in (52):

$$\begin{aligned} &\frac{\partial \sigma}{\partial S}(S, t) \frac{\partial s}{\partial t}(x, t)s(x, t) + \frac{\partial \sigma}{\partial t}(S, t)s(x, t) + \sigma(S, t) \frac{\partial s}{\partial t}(x, t) + \frac{1}{2} \frac{\partial^2 \sigma}{\partial S^2}(S, t) \left(\frac{\partial s}{\partial x}(x, t) \right)^2 s(x, t) \\ &+ \frac{\partial \sigma}{\partial S}(S, t) \frac{\partial^2 s}{\partial x^2}(x, t)s(x, t) + 2 \frac{\partial \sigma}{\partial S}(S, t) \left(\frac{\partial s}{\partial x}(x, t) \right)^2 + \sigma(S, t) \frac{\partial^2 s}{\partial x^2}(x, t) \\ &= (r - q)\sigma(S, t)s(x, t), \quad t \in [0, T], x \in \mathfrak{R}. \end{aligned} \quad (56)$$

Substituting (51) in (56):

$$\begin{aligned} &\frac{\partial \sigma}{\partial S}(S, t) \frac{\partial s}{\partial t}(x, t)s(x, t) + \frac{\partial \sigma}{\partial t}(S, t)s(x, t) + \frac{1}{2} \frac{\partial^2 \sigma}{\partial S^2}(S, t) \left(\frac{\partial s}{\partial x}(x, t) \right)^2 s(x, t) + \frac{\partial \sigma}{\partial S}(S, t) \frac{\partial^2 s}{\partial x^2}(x, t)s(x, t) \\ &+ 2 \frac{\partial \sigma}{\partial S}(S, t) \left(\frac{\partial s}{\partial x}(x, t) \right)^2 = 0, \quad t \in [0, T], x \in \mathfrak{R}. \end{aligned} \quad (57)$$

Now, multiply (51) by $\sigma(S, t)$:

$$\sigma(S, t) \frac{\partial s}{\partial t}(x, t) + \frac{\sigma(S, t)}{2} \frac{\partial^2 s}{\partial x^2}(x, t) = (r - q)\sigma(S, t)s(x, t), \quad t \in [0, T'], x \in \mathfrak{R}. \quad (58)$$

Also multiply (51) by $\frac{\partial \sigma}{\partial S}(S, t)s(x, t)$:

$$\frac{\partial \sigma}{\partial S}(S, t) \frac{\partial s}{\partial t}(x, t)s(x, t) + \frac{1}{2} \frac{\partial \sigma}{\partial S}(S, t) \frac{\partial^2 s}{\partial x^2}(x, t)s(x, t) = (r - q) \frac{\partial \sigma}{\partial S}(S, t)s^2(x, t), \quad t \in [0, T'], x \in \mathfrak{R}. \quad (59)$$

Substituting (58) and (59) in (57):

$$\frac{1}{2} \frac{\partial^2 \sigma(S, t)}{\partial S^2} \left(\frac{\partial s(x, t)}{\partial x} \right)^2 s(x, t) + (r - q)S \frac{\partial \sigma}{\partial S}(S, t)s^2(x, t) + \frac{\partial \sigma(S, t)}{\partial S} \left(\frac{\partial s(x, t)}{\partial x} \right)^2 + \frac{\partial \sigma}{\partial t}(S, t)s(x, t) = 0, \quad (60)$$

for $S \geq 0, t \in [0, T')$. Dividing by $s(x, t)$:

$$\frac{1}{2} \left(\frac{\partial s(x, t)}{\partial x} \frac{1}{s(x, t)} \right)^2 s^2(x, t) \frac{\partial^2 \sigma(S, t)}{\partial S^2} + \left[r - q + \left(\frac{\partial s(x, t)}{\partial x} \frac{1}{s(x, t)} \right)^2 \right] s(x, t) \frac{\partial \sigma}{\partial S}(S, t) + \frac{\partial \sigma}{\partial t}(S, t) = 0, \quad (61)$$

for $S \geq 0, t \in [0, T')$. Substituting in $\sigma(S, t) = \frac{\partial s(x, t)}{\partial x} \frac{1}{s(x, t)}$ gives the following nonlinear PDE for $\sigma(S, t)$:

$$\frac{\sigma^2(S, t)S^2}{2} \frac{\partial^2 \sigma(S, t)}{\partial S^2} + [r - q + \sigma^2(S, t)]S \frac{\partial \sigma}{\partial S}(S, t) + \frac{\partial \sigma}{\partial t}(S, t) = 0, \quad S \geq 0, t \in [0, T'). \quad (62)$$

Appendix 3: The PDE for Local Volatility as a Sufficient Condition for Path-Independence

This appendix proves that if a positive function $\sigma(S, t)$ satisfies the nonlinear PDE (32), then the stock price is path-independent. More specifically, there exists a function $s^{-1}(S, t)$ increasing in S , such that the process:

$$B_t \equiv s^{-1}(S_t, t), \quad t \in [0, \tau \wedge T'], \quad (63)$$

is the driving SBM, i.e. $B_t = W_t$ for all $t \in [0, \tau \wedge T']$. This function is given by:

$$s^{-1}(S, t) = \int_{S_0}^S \frac{1}{\sigma(Z, t)Z} dZ + \int_0^t \left[\frac{1}{2} \frac{\partial}{\partial S} [\sigma(S, u)S] \Big|_{S=S_0} - \frac{(r-q)}{\sigma(S_0, u)} \right] du, \quad S > 0, t \in [0, T']. \quad (64)$$

To prove this result, Itô's lemma applied to (63) implies that for $t \in [0, \tau \wedge T']$:

$$dB_t = \left[\frac{\sigma^2(S_t, t)S_t^2}{2} \frac{\partial^2 s^{-1}}{\partial S^2}(S_t, t) + (r-q)S_t \frac{\partial s^{-1}}{\partial S}(S_t, t) + \frac{\partial s^{-1}}{\partial t}(S_t, t) \right] dt + \frac{\partial s^{-1}}{\partial S}(S_t, t) \sigma(S_t, t) S_t dW_t. \quad (65)$$

For B to be SBM, we need to show that its drift is zero:

$$\frac{\sigma^2(S, t)S^2}{2} \frac{\partial^2 s^{-1}}{\partial S^2}(S, t) + (r-q)S \frac{\partial s^{-1}}{\partial S}(S, t) + \frac{\partial s^{-1}}{\partial t}(S, t) = 0, \quad S > 0, t \in [0, \tau \wedge T'], \quad (66)$$

that its volatility is unity:

$$\frac{\partial s^{-1}}{\partial S}(S, t) \sigma(S, t) S = 1, \quad S > 0, t \in [0, \tau \wedge T'], \quad (67)$$

and that its initial level is zero:

$$B_0 = s(S_0, 0) = 0. \quad (68)$$

Differentiating (64) w.r.t. S :

$$\frac{\partial s^{-1}}{\partial S}(S, t) = \frac{1}{\sigma(S, t)S}, \quad S > 0, t \in [0, \tau \wedge T'], \quad (69)$$

so (67) holds. Furthermore, since $\sigma(S, t)$ is assumed positive, the function s^{-1} is increasing in S . Evaluating (64) at $(S, t) = (S_0, 0)$ implies (68) also holds. To show that (66) also holds, divide (32) by $\sigma^2(S, t)S$:

$$\frac{S}{2} \frac{\partial^2 \sigma(S, t)}{\partial S^2} + \frac{\partial \sigma}{\partial S}(S, t) + \frac{(r-q)S}{\sigma^2(S, t)S} \frac{\partial \sigma}{\partial S}(S, t) + \frac{1}{\sigma^2(S, t)S} \frac{\partial \sigma}{\partial t}(S, t) = 0, \quad S \geq 0, t \in [0, \tau \wedge T']. \quad (70)$$

Noting that:

$$\frac{\partial}{\partial S}[S\sigma(S, t)] = \sigma(S, t) + S \frac{\partial \sigma}{\partial S}(S, t), \quad S > 0, t \in [0, \tau \wedge T'],$$

and:

$$\frac{\partial^2}{\partial S^2}[S\sigma(S, t)] = 2 \frac{\partial \sigma}{\partial S}(S, t) + S \frac{\partial^2 \sigma}{\partial S^2}(S, t), \quad S > 0, t \in [0, \tau \wedge T'],$$

(70) can be re-written as:

$$\frac{1}{2} \frac{\partial^2}{\partial S^2}[S\sigma(S, t)] + \frac{(r-q)S}{\sigma^2(S, t)S^2} \frac{\partial}{\partial S}[S\sigma(S, t)] - \frac{r-q}{\sigma(S, t)} + \frac{1}{\sigma^2(S, t)S} \frac{\partial \sigma}{\partial t}(S, t) = 0, \quad S \geq 0, t \in [0, \tau \wedge T']. \quad (71)$$

Integrating both sides w.r.t. S implies:

$$\frac{1}{2} \frac{\partial}{\partial Z}[\sigma(Z, t)Z] \Big|_{Z=S_0}^{Z=S} - \frac{(r-q)Z}{\sigma(Z, t)Z} \Big|_{Z=S_0}^{Z=S} + \int_{S_0}^S \frac{1}{\sigma^2(Z, t)Z} \frac{\partial \sigma}{\partial t}(Z, t) dZ = 0, \quad S \geq 0, t \in [0, \tau \wedge T'], \quad (72)$$

where the constant of integration has been set to zero. Multiplying through by -1 implies that for $S \geq 0, t \in [0, \tau \wedge T']$:

$$-\frac{\sigma^2(S, t)S^2}{2} \frac{\partial}{\partial S}[\sigma(S, t)S] + (r-q)S \frac{1}{\sigma(S, t)S} + \frac{1}{2} \frac{\partial}{\partial S}[\sigma(S, t)S] \Big|_{S=S_0} - \frac{r-q}{\sigma(S_0, t)} - \int_{S_0}^S \frac{1}{\sigma^2(Z, t)Z} \frac{\partial \sigma}{\partial t}(Z, t) dZ = 0. \quad (73)$$

Differentiating (69) w.r.t. S :

$$\frac{\partial^2 s^{-1}}{\partial S^2}(S, t) = -\frac{\frac{\partial}{\partial S}[\sigma(S, t)S]}{[\sigma(S, t)S]^2}, \quad S > 0, t \in [0, \tau \wedge T'], \quad (74)$$

Differentiating (64) w.r.t. t :

$$\frac{\partial s^{-1}}{\partial t}(S, t) = \frac{1}{2} \frac{\partial}{\partial S}[\sigma(S, t)S] \Big|_{S=S_0} - \frac{(r-q)}{\sigma(S_0, t)} - \int_{S_0}^S \frac{1}{\sigma^2(Z, t)Z} \frac{\partial \sigma}{\partial t}(Z, t) dZ, \quad S > 0, t \in [0, \tau \wedge T']. \quad (75)$$

Substituting (69),(74), and (75) in (73) gives (66).

Appendix 4: Implying the Stock Payoff From the Volatility Smile

When the local volatility surface $\sigma(S, t)$ is given for all $t \in [0, T']$, then the time T' volatility smile $v(S) \equiv \sigma(S, T')$ is also given. Consequently, the implied payoff $g(S)$ must solve the first order non-linear differential equation (33). Under certain conditions on $v(\cdot)$, an implicit solution for $g(\cdot)$ is given by:

$$I_v(g) = x + c_0, \quad x \in \mathfrak{R}, \quad (76)$$

$$\text{where } I_v(g) \equiv \int_{c_1}^g \frac{1}{v(S)S} dS, \quad g \geq 0, \quad (77)$$

and c_0 and c_1 are constants. Since g can be less than c_1 , we define $\int_b^a \frac{1}{v(S)S} dS = -\int_a^b \frac{1}{v(S)S} dS$ for $b > a$.

In order that the implied payoff $g(x)$ be well-defined by the solution (77), the volatility smile $v(S)$ must be restricted such that $I_v(g)$ is invertible. To obtain these restrictions, note that as $x \downarrow -\infty$, the volatility smile should permit the implied stock payoff g to approach zero:

$$I_v(0) \equiv -\int_0^{c_1} \frac{1}{v(S)S} dS = -\infty, \quad \forall c_1 > 0. \quad (78)$$

This condition holds so long as $v(S)$ is bounded as $S \downarrow 0$ or at least does not approach infinity so fast that the integral of $\frac{1}{v(S)S}$ is bounded.

Differentiating the definition of I_v in (77) w.r.t. g implies that $I'_v(g) = \frac{1}{v(g)g}$, which is strictly positive for $g > 0$. In order that $g(x)$ be well-defined by (77) for all real x , we further require that there exist a positive constant c_1 such that $v(S), S > 0$ satisfies:

$$I_v(\infty) \equiv \int_{c_1}^{\infty} \frac{1}{v(S)S} dS = \infty. \quad (79)$$

This condition holds so long as $v(\cdot)$ is bounded above as $S \uparrow \infty$, or at least does not approach infinity so fast that the integral of $\frac{1}{v(S)S}$ over the positive half line is bounded.

To summarize, if the given positive volatility smile $v(S)$ satisfies (78), and (79), then $I_v(0) = -\infty$, $I'_v(g) > 0$ for $g > 0$, and $I_v(\infty) = \infty$. Consequently, the inverse of I_v exists, and from (77), the stock payoff $g(x)$ is given by:

$$s(x, T) \equiv g(x) = I_v^{-1}(x + c_0), \quad x \in \mathfrak{R}. \quad (80)$$

Since I_v is an increasing function defined on \mathfrak{R}^+ with range \mathfrak{R} , I_v^{-1} is an increasing function defined on \mathfrak{R} with range \mathfrak{R}^+ . By (80), g is also an increasing function defined on \mathfrak{R} with range \mathfrak{R}^+ .

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