A Tale of Two Indices

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A Tale of Two Indices: An Overview

- In 1993, the CBOE introduced the volatility index (VIX)
  - Widely followed benchmark for stock market volatility.
- On September 22, 2003, the CBOE revamped its definition of the VIX and back calculated the new VIX up to 1990.
  - The ticker for the old VIX is switched to VXO.
- The CBOE plans to launch a new Exchange, the CBOE Futures Exchange (CFE) on March 26, 2004 to trade futures and options on the new VIX.
- Questions addressed in this talk:
  - *What are the differences between the two indices?*
  - *Why switch? Why not issue futures and options on the VXO?*
  - *How do VIX and VXO behave historically?*
  - *How do we price futures and options on VIX?*

Difference in Definitions and Calculations

- **VXO**: Average over 8 near-the-money Black-Scholes implied volatilities at the two nearest maturities on S&P 100 index.
  - Essentially an estimate of the one-month at-the-money implied volatility.
- **VIX**: An average of out-of-money option *prices* (Q) across *all available strikes* on S&P 500 index:

$$\sigma^2 = \frac{2}{T} \sum_i \frac{\Delta K}{K_i^2} e^{rT} Q(K_i, T) - \frac{1}{T} \left[\frac{F}{K_0} - 1 \right]^2, \tag{1}$$

  - Linearly interpolate (extrapolate) over the nearest two maturities to obtain a 30-day estimate, in volatility percentages.

$$VIX = 100 \sqrt{\frac{365}{30} \left[T_1 \sigma_1^2 \frac{N_{T_2} - 30}{N_{T_2} - N_{T_1}} + T_2 \sigma_2^2 \frac{30 - N_{T_1}}{N_{T_2} - N_{T_1}} \right]}, \tag{2}$$

Theoretical Underpinnings of the Old VXO

- Black-Scholes implied volatility.
- Under more general settings, an accurate approximation of the *volatility swap rate*,

$$VolS \equiv \mathbb{E}_0^{\mathbb{Q}} \sqrt{RV} = ATMV + O(T^{\frac{3}{2}}). \tag{3}$$

- The payoff of a volatility swap contract:

Vol Swap Rate ($VolS$)	$\rightleftharpoons$	**Realized Volatility** ($\sqrt{RV}$)

- The difference is converted into dollar amount based on a notional figure.
- The contract has zero market value at inception, so that

$$VolS_T = \mathbb{E}_t^{\mathbb{Q}}[\sqrt{RV}_T].$$

Theoretical Underpinnings of the New VIX

- Under very general settings, an approximation of the *variance swap rate*,

$$VarS \equiv \mathbb{E}_0^{\mathbb{Q}} RV_T = \frac{2}{T} \int_0^\infty \frac{Q_0(K,T)}{K^2} e^{rT} dK + \varepsilon, \quad (4)$$

- The approximation error ε is zero when the futures dynamics is continuous; otherwise the error is third order $(dF/F)^3$.

- The payoff of a variance swap contract:

Variance Swap Rate ($VarS$)	$\rightleftharpoons$	**Realized Variance (RV)**

- The contract has zero market value at inception, so that

$$VarS_T = \mathbb{E}_t^{\mathbb{Q}}[RV_T].$$

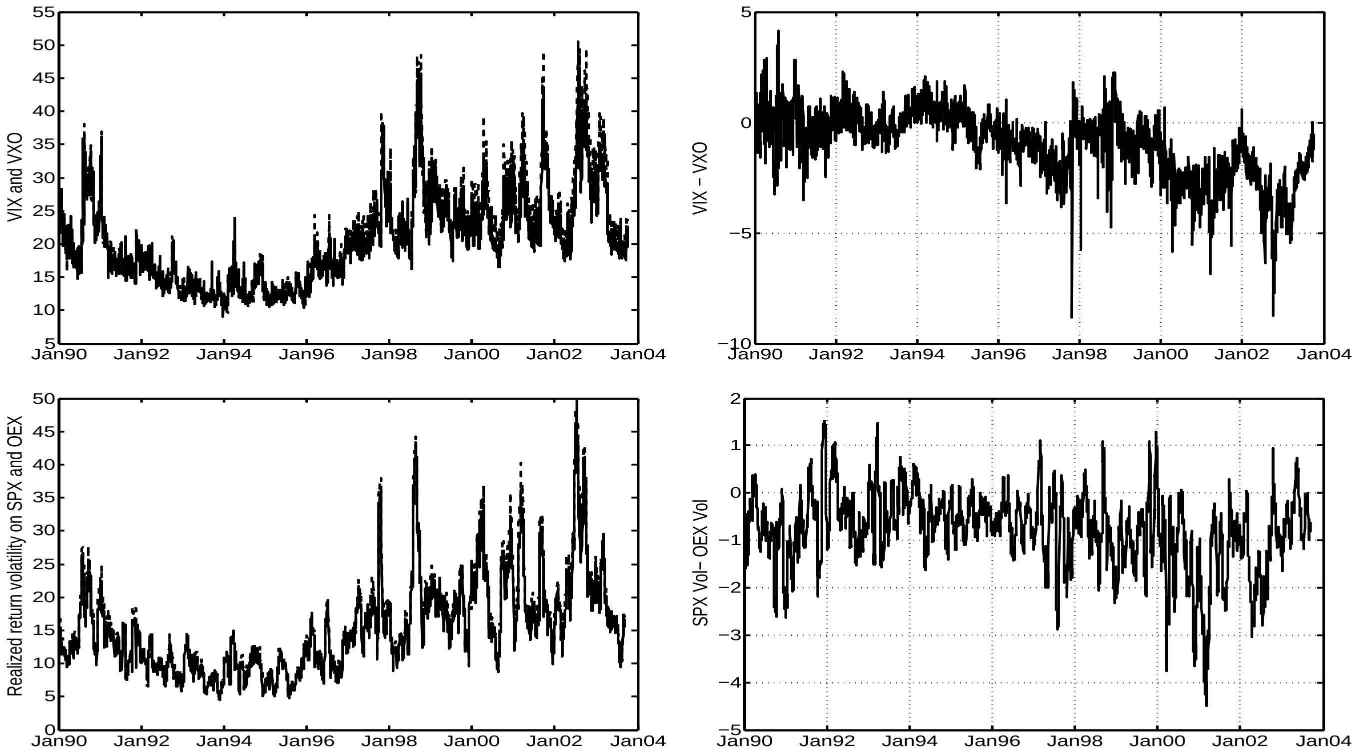
Numerical Errors Are Very Small

$\ln(v_t/\theta)$	Variance Swap				Volatility Swap		
	$\mathbb{E}^{\mathbb{Q}}[RV]$	$\widehat{SW}$	Error	ε	$\left[\mathbb{E}^{\mathbb{Q}}[\sqrt{RV}]\right]^2$	$ATMV^2$	Error
A. Black-Scholes Model							
0.0	0.1369	0.1369	0.0000	0.0000	0.1369	0.1369	0.0000
B. Merton Jump-Diffusion (MJD) Model							
0.0	0.1387	0.1367	0.0020	0.0021	0.1318	0.1319	-0.0001
C. MJD-Stochastic Volatility Model							
-3.0	0.0272	0.0262	0.0010	0.0021	0.0111	0.0114	-0.0002
0.0	0.1387	0.1362	0.0025	0.0021	0.1279	0.1279	-0.0000
3.0	2.3782	2.3865	-0.0083	0.0021	2.4056	2.4016	0.0039

Why Switch from VXO to VIX?

- The general meaning of VXO is not known until very recently.
- The volatility swap contract underlying VXO is much more difficult to hedge than the variance swap contract underlying VIX.
- To hedge a variance swap contract, we need to take
  - A static position in a portfolio options
  - A dynamic position in futures trading
- VIX^2 can also simply be regarded as the value of a portfolio of options.

Time Series Behaviors

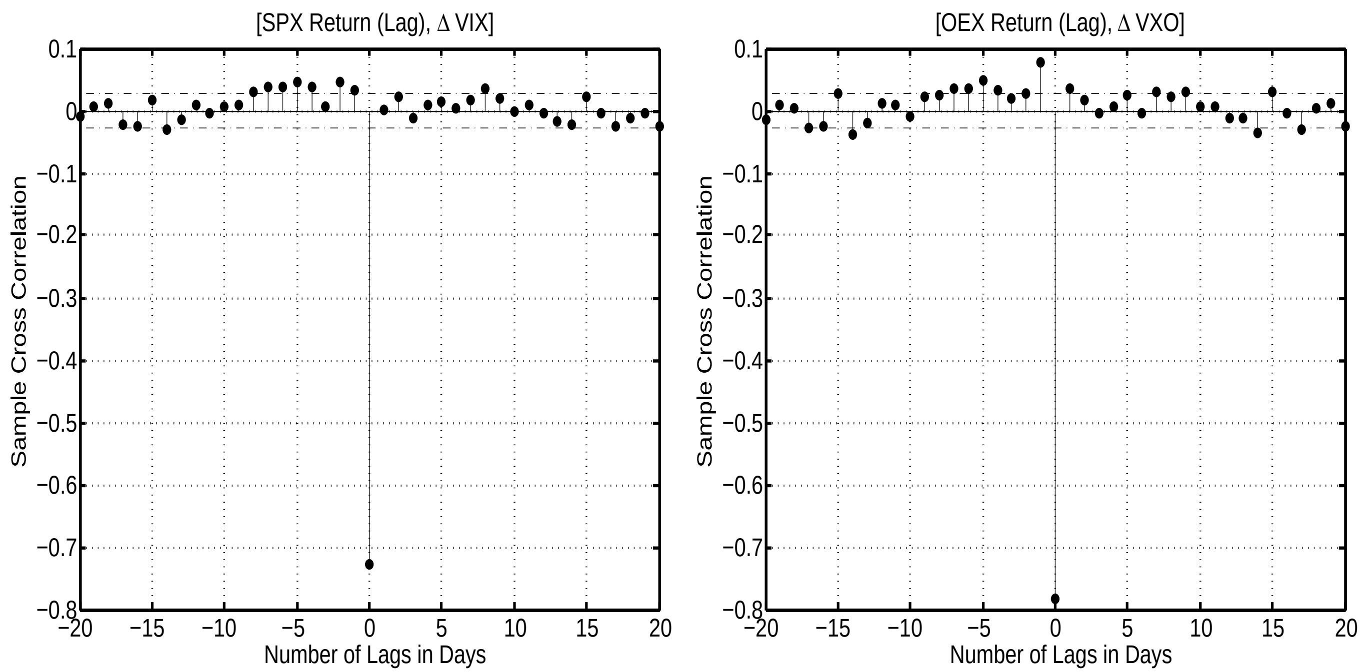


VIX and VXO follow each other closely; level difference reflects mainly difference in the underlying.

Summary Statistics

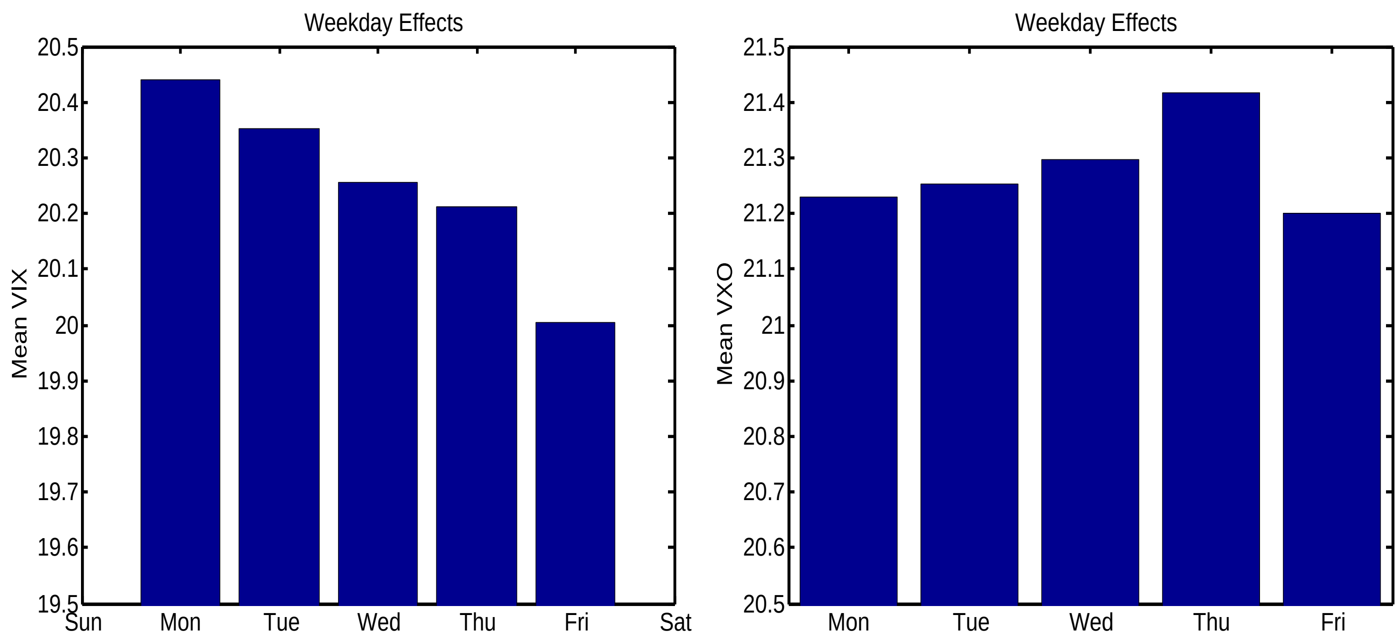
	VIX	SPX Vol	VXO	OEX Vol	VIX	SPX Vol	VXO	OEX Vol
	Levels				Daily Differences			
Mean	20.180	15.241	21.254	16.030	0.000	-0.000	0.001	-0.000
Stdev	6.486	7.087	7.391	7.527	1.060	0.863	1.216	0.907
Skew	0.807	1.282	0.811	1.257	0.668	0.728	0.676	0.539
Kurt	0.519	2.061	0.538	1.865	9.638	30.288	12.916	27.775
	Log Levels				Daily Log Differences			
Mean	2.955	2.625	2.998	2.674	0.000	-0.000	0.000	-0.000
Stdev	0.314	0.442	0.342	0.446	0.047	0.055	0.049	0.055
Skew	0.104	0.103	0.060	0.118	0.736	0.485	0.600	0.387
Kurt	-0.651	-0.383	-0.697	-0.424	6.729	17.840	7.157	18.549

Cross-Correlation between Return and VIX(VXO)



Instantaneous correlation only; no lead-lag effects.

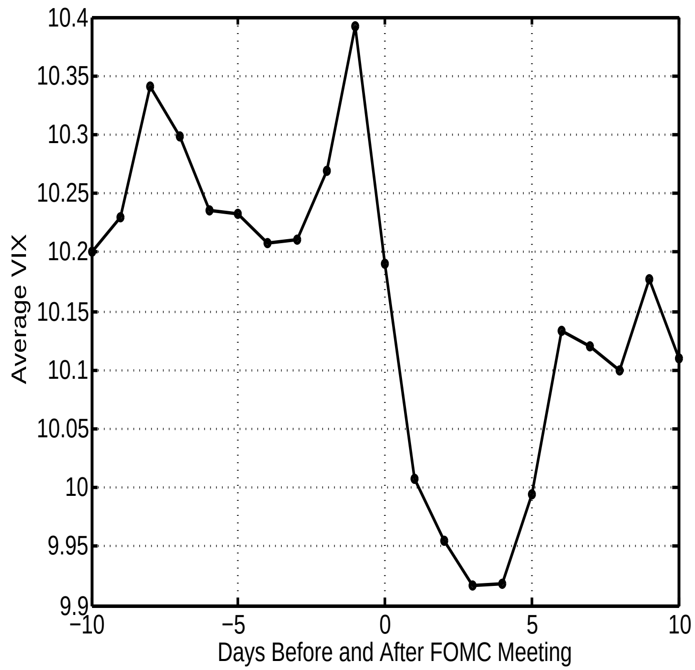
Weekday Effects

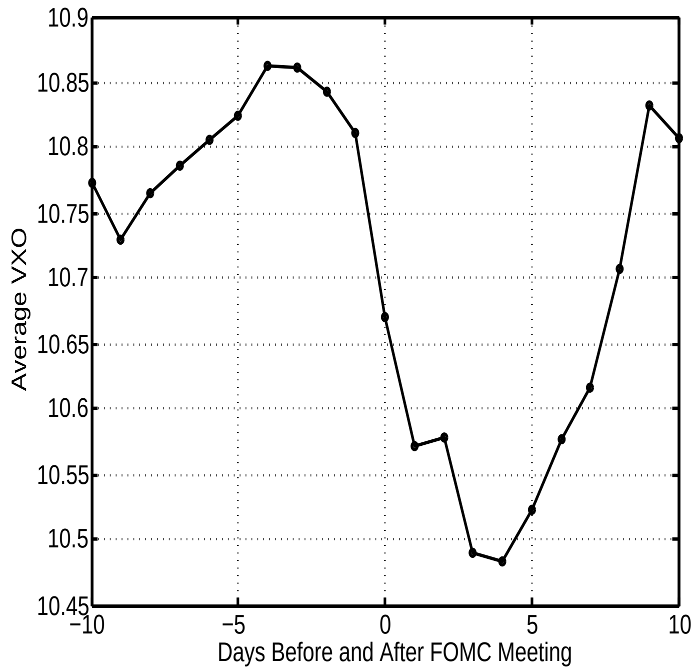


Friday average VIX is lower than averages on other weekdays.
VXO does not have this effect.

The FOMC Meeting Day Effect

Vol index levels 10 days before and after FOMC meeting days.





Volatility drops by about half a percentage point after the meeting.

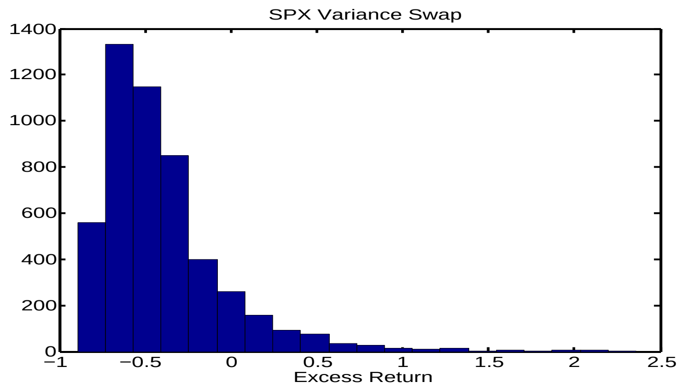
The Information Content

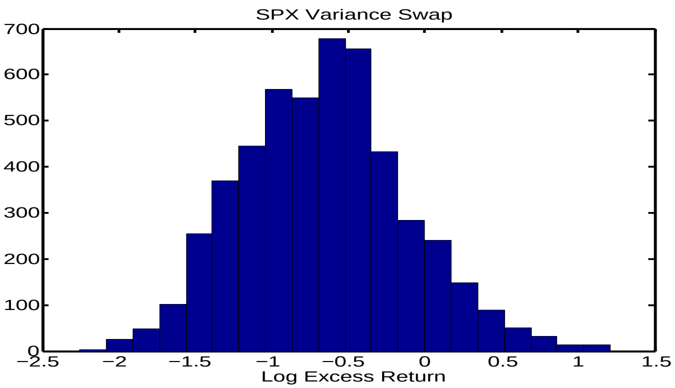
Series	Intercept		VIX/VXO		GARCH		R-square
			A. Forecasting $\ln \sqrt{RV}_{SPX}$				
$\ln VIX$	-0.685	(0.197)	1.120	(0.066)	—	—	0.631
GARCH	0.180	(0.166)	—	—	0.907	(0.061)	0.568
Joint	-0.580	(0.201)	0.846	(0.147)	0.262	(0.125)	0.641
			B. Forecasting $\ln \sqrt{RV}_{OEX}$				
$\ln VXO$	-0.519	(0.177)	1.065	(0.059)	—	—	0.663
GARCH	0.173	(0.169)	—	—	0.911	(0.061)	0.572
Joint	-0.500	(0.179)	0.969	(0.143)	0.098	(0.132)	0.666

The vol indices are efficient forecasts of realized volatility;
GARCH vol is not needed once the index is included.

The Excess Returns of Longing Variance Swaps

$(RV - VIX^2)/VIX^2$ $\ln RV/VIX^2$

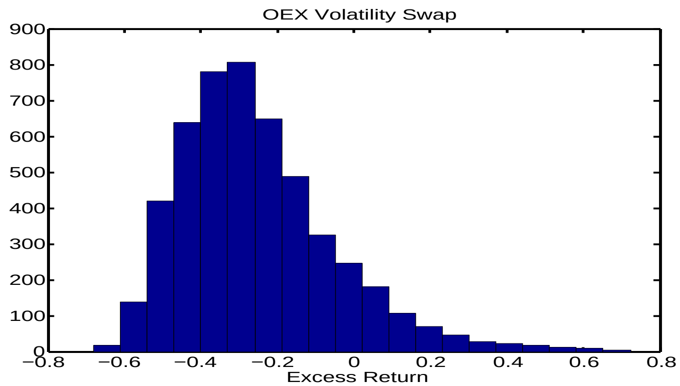


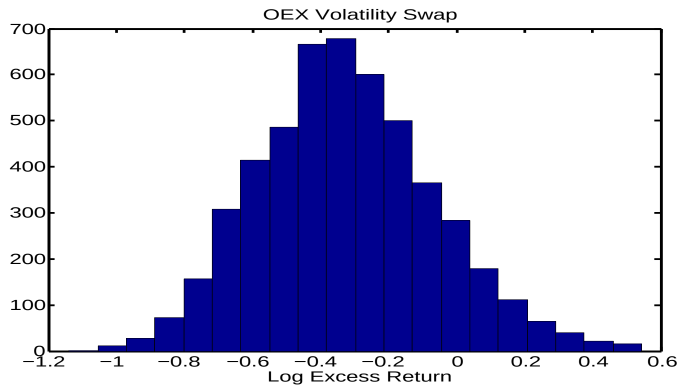


Highly negative on average, but positively skewed.

The Excess Returns of Longing Volatility Swaps

$(\sqrt{RV} - VXO)/VXO$ $\ln RV/VXO$





Highly negative on average, but positively skewed.

VIX Futures

- The CBOE plans to launches on the VIX level (not VIX^2)

$$F_0^{vix} = E_0^{\mathbb{Q}} VIX_{T_1}, \tag{5}$$

- Direct valuation is not straightforward, but we obtain some bounds.

- Simplify VIX definition as a continuum of OTM options (Q) at one maturity

$$VIX_{T_1} = \sqrt{\frac{2}{(T_2 - T_1)} e^{r(T_2 - T_1)} \int_0^\infty \frac{Q_{T_1}(K, T_2)}{K^2} dK}. \tag{6}$$

VIX Futures bounds

- Assume continuous futures price dynamics, we have

$$VIX_{T_1} = \sqrt{E_{T_1}^{\mathbb{Q}} RV_{T_1,T_2}}, \tag{7}$$

where RV_{T_1,T_2} is the annualized return quadratic variation between T_1 and T_2.

- The VIX futures can be represented as:

$$F_0^{vix} = E_0^{\mathbb{Q}} \sqrt{E_{T_1}^{\mathbb{Q}} RV_{T_1,T_2}}, \tag{8}$$

- The concavity of the square root and Jensen's inequality implies that:

$$E_0^{\mathbb{Q}} \sqrt{RV_{T_1,T_2}} \leq F_0^{vix} \leq \sqrt{E_0^{\mathbb{Q}} RV_{T_1,T_2}}, \tag{9}$$

by the law of iterated expectations.

The Upper Bound for the VIX Futures

$$E_0^{\mathbb{Q}}\sqrt{RV_{T_1,T_2}} \le F_0^{vix} \le \sqrt{E_0^{\mathbb{Q}} RV_{T_1,T_2}}.$$

- The upper bound is the square root of the forward variance swap rate,

$$U_0 \equiv \sqrt{E_0^{\mathbb{Q}} RV_{T_1,T_2}}, \tag{10}$$

- Can be represented as a function of European option prices on SPX

$$\begin{aligned} U_0^2 &= E_0^{\mathbb{Q}} RV_{T_1,T_2} = E_0^{\mathbb{Q}} RV_{0,T_2} - E_0^{\mathbb{Q}} RV_{0,T_1} \\ &= \frac{1}{T_2 - T_1}\int_0^\infty \left[Q_0(K,T_2)e^{rT_2} - Q_0(K,T_1)e^{rT_1}\right]\frac{2}{K^2}dK. \end{aligned} \tag{11}$$

- Hence observable.

The Lower Bound for the VIX Futures

$$E_0^{\mathbb{Q}} \sqrt{RV_{T_1,T_2}} \leq F_0^{vix} \leq \sqrt{E_0^{\mathbb{Q}} RV_{T_1,T_2}}.$$

- The lower bound is essentially the forward vol swap rate.

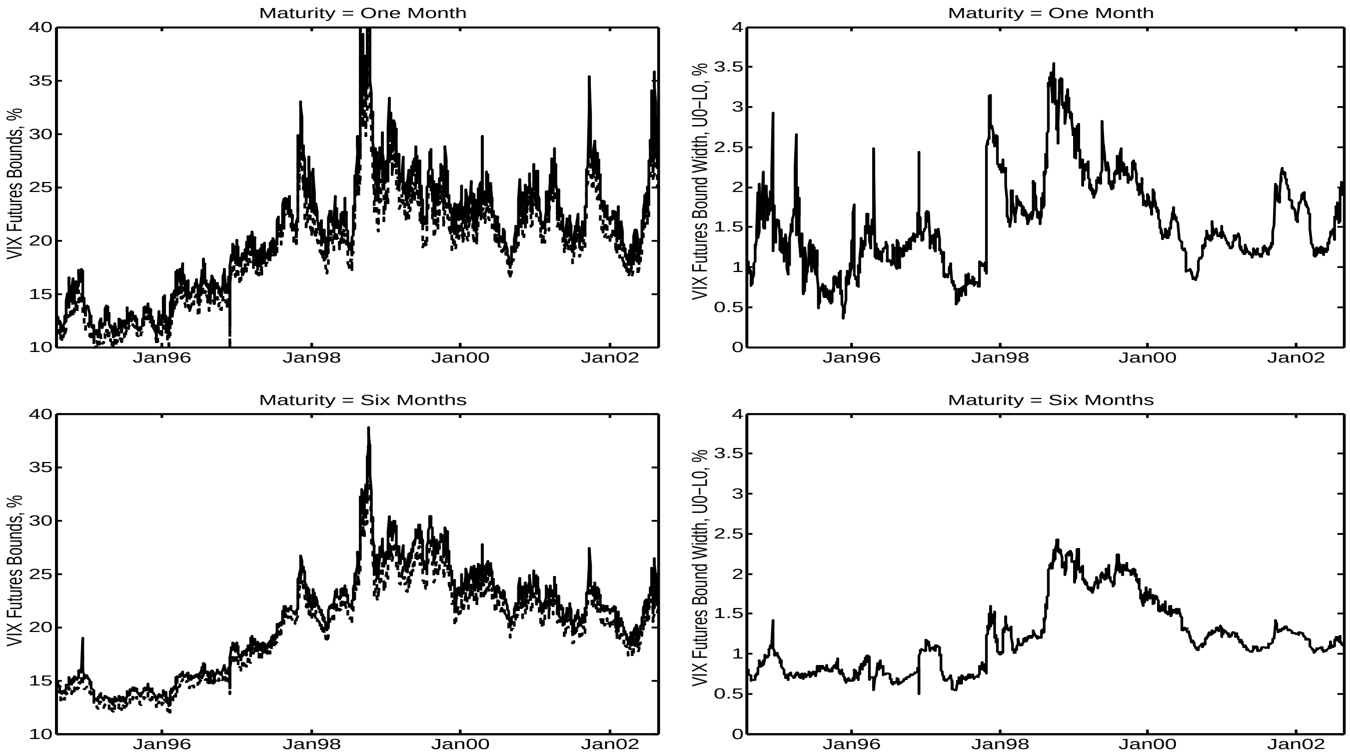
$$L_0 \equiv E_0^{\mathbb{Q}} \sqrt{RV_{T_1,T_2}}. \tag{12}$$

- Under more assumptions, it can be approximated by a forward-start at-the-money forward call option price (FSATMFC),

$$L_0 \approx \sqrt{2\pi} \frac{FSATMFC_0(T_1, T_2)e^{rT_2}}{\sqrt{T_2 - T_1}}, \tag{13}$$

- These forward-start options are active over the counter, hence observable again.

How Tight are the Bounds?



Moves with the vol level; between 0.5-3.5 percentage vol points.

Arbing Upper-Bound Violations

- Recall that no-arbitrage implies

$$E_0^{\mathbb{Q}} \sqrt{RV_{T_1,T_2}} \leq F_0^{vix} \leq \sqrt{E_0^{\mathbb{Q}} RV_{T_1,T_2}} \equiv U_0.$$

- Suppose $F^{vix} \geq U_0$. Do this
  - Buy low implies: Buy one unit of forward variance swap.
  - Sell High implies: Sell $2F_0^{vix}$ units of VIX futures.

- At the VIX futures maturity T_1, VIX futures ($F_{T_1}^{vix}$) converge to VIX_{T_1}. Forward variance swap ($U_{T_1}^2$) converges to $VIX_{T_1}^2$.

P&L from the Arb

- Recall that $F^{vix} \geq U_0$.
- Recall our position: Long one forward variance swap; short $2F_0^{vix}$ VIX futures.
- Our P&L at VIX futures maturity T_1 is

$$\begin{aligned} PL &= VIX_{T_1}^2 - U_0^2 - 2F_0^{vix}\left[VIX_{T_1} - F_0^{vix}\right] \\ &= VIX_{T_1}^2 - 2F_0^{vix}VIX_{T_1} + 2(F_0^{vix})^2 - U_0^2 \\ &\geq VIX_{T_1}^2 - 2F_0^{vix}VIX_{T_1} + (F_0^{vix})^2 \\ &= \left(VIX_{T_1} - F_0^{vix}\right)^2 \geq 0. \end{aligned}$$

Arbing Lower-Bound Violations

- Recall that no-arbitrage implies

$$L_0 \equiv E_0^{\mathbb{Q}} \sqrt{RV_{T_1,T_2}} \leq F_0^{vix} \leq \sqrt{E_0^{\mathbb{Q}} RV_{T_1,T_2}}.$$

- Suppose $F^{vix} \leq L_0$. Do this
  - Buy low implies: Buy one unit of VIX futures.
  - Sell High implies: Sell one unit of forward vol swap.

- At the VIX futures maturity T_1, VIX futures ($F_{T_1}^{vix}$) converge to VIX_{T_1}. Forward vol swap (L_{T_1}) converges to a spot vol swap rate ($VolS_{T_1}$).

- Recall that vol swap rate is always lower than the variance swap rate: $VolS_{T_1} \leq VIX_{T_1}$.

P&L from the Arb

- Recall that $F^{vix} \leq L_0$.
- Recall our position: Long one VIX futures; short one forward vol swap.
- Recall also: $VolS \leq VIX$.
- Our P&L at VIX futures maturity T_1 is

$$\begin{aligned} PL &= VIX_{T_1} - F_0^{vix} - [VolS_{T_1} - L_0] \\ &= [VIX_{T_1} - VolS_{T_1}] + \left[L_0 - F_0^{vix}\right] \geq 0. \end{aligned}$$

The Economic Meanings of the Bounds

Apply the following equality, $var(x) = E[x^2] - (E[x])^2$

- The risk-neutral variance of the future volatility

$$\begin{aligned} \mathrm{Var}_0^{\mathbb{Q}} \sqrt{RV_{T_1,T_2}} &= E_0^{\mathbb{Q}} RV_{T_1,T_2} - \left(E_0^{\mathbb{Q}} \sqrt{RV_{T_1,T_2}} \right)^2 \\ &= U_0^2 - L_0^2. \end{aligned} \tag{14}$$

- The risk-neutral variance of the future *expected* volatility (VIX)

$$\begin{aligned} \mathrm{Var}_0^{\mathbb{Q}} \sqrt{E_{T_1}^{\mathbb{Q}} RV_{T_1,T_2}} &= E_0^{\mathbb{Q}} RV_{T_1,T_2} - \left(E_0^{\mathbb{Q}} \sqrt{E_{T_1}^{\mathbb{Q}} RV_{T_1,T_2}} \right)^2 \\ &= U_0^2 - \left(F_0^{vix} \right)^2. \end{aligned} \tag{15}$$

Pricing VIX Options

- The terminal payoff of a call option on the VIX is $(VIX_{T_1} - K)^+$
- We hence need to know the risk-neutral distribution of VIX for valuation.
- We do not know the distribution, but we (will) **observe** the first two risk-neutral moments of VIX_{T_1},

$$\begin{aligned} Mean: \quad E_0^{\mathbb{Q}}[VIX_{T_1}] &= F_0^{vix} = F_0 \\ Variance: \quad Var_0^{\mathbb{Q}}[VIX_{T_1}] &= U_0^2 - (F_0^{vix})^2 \equiv V_0 \end{aligned}$$

- We can price options with a distributional assumption.

Pricing VIX Options under Black-Scholes

- The Black-Scholes model

$$C_0 = e^{-rT_1}\left[F_0 N\left(d_1\right) - K N\left(d_2\right)\right],$$

where

$$d_1 = \frac{\ln F_0/K + \frac{1}{2}s^2 T_1}{s\sqrt{T_1}}, \quad d_2 = d_1 - s\sqrt{T_1},$$

and s is the annualized standard deviation of $\ln VIX_{T_1}$, which can be represented as a function of the first two moments of VIX_{T_1},

$$s = \sqrt{\frac{1}{T_1}\ln\frac{V_0 + F_0^2}{F_0^2}}. \tag{16}$$

- The key is the observable input for V_0, and hence for s.

Pricing VIX Options under Bachelier

- Under a normal assumption, the call option value is

$$C_0 \;=\; e^{-rT_1}\left[\sqrt{V_0}N'(d) + (F_0 - K)\,N(d)\right],$$

with

$$d = \frac{F_0 - K}{\sqrt{V_0}}.$$

- The key is the observable input for F_0 and for V_0.

Summary

- The VIX approximates the variance swap rate on SPX; the old VXO approximates the volatility swap rate on OEX.

- Payoffs to variance swap contracts are much easier to replicate/hedge than payoffs to vol swaps. Hence, CBOE switched to the new VIX and plans to launch futures and options on the new VIX.

- We provide tight bounds on the VIX futures.

- We can also price VIX options using observables.

- Looking forward to a fascinating market...