Factor Models for Option Pricing

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December 29, 2000

Abstract

Options on stocks are priced using information on index options and viewing stocks in a factor model as indirectly holding index risk. The method is particularly suited to developing quotations on stock options when these markets are relatively illiquid and one has a liquid index options market to judge the index risk. The pricing strategy is illustrated on IBM and Sony options viewed as holding SPX and Nikkei risk respectively.

1 INTRODUCTION

From its early beginnings with the Black-Merton-Scholes [4], [16] model, option pricing has been grounded in the perspectives of a two asset economy. Initially, this was a harmless assumption as the economy studied was complete and pricing was at the cost of replication. Increasingly it is now recognized that with stochastic volatility and jumps constituting an essential (Bates, [2], Bakshi, Cao and Chen [1]) if not indispensible (Carr, Geman, Madan, and Yor [6]) part of the stock price dynamics, pricing must be founded on principles other than the cost of replication. The most general of these principles developed in the finance literature is the absence of arbitrage, due to Ross [17], Harrison and Kreps [11], and Harrison and Pliska [12]. Recently, this principle has been generalized to the absence of acceptable opportunities by Carr, Geman, and Madan [5]. In either case, these theories conclude by demonstrating the existence of a probability, equivalent to the true probabilities and termed risk-neutral probability, with the property that prices of traded assets are expected discounted cash flows under this revised probability. An important special case of these theories is the equilibrium capital asset pricing model (CAPM) of Sharpe, Lintner, and Mossin [18], [13], [15]. The CAPM broadly asserts that the risk-neutral probability may be defined as a function of the return on the market portfolio. Much empirical research has been concerned with testing this theory and the Ross [17] extensions. The basic conclusion has been that although there may be other factors relevant for asset pricing, the return on the market portfolio is certainly one of them.

In this paper, we suppose that the relevant market factor can be proxied by a major financial market index and that we have a liquid market for options on this index. We then price options on the individual stocks by viewing them as indirectly holding the index risk plus an idiosyncratic firm-specific component. The index risk is priced consistent with the index options market, while the idiosyncratic component is priced consistent with the true statistical probability, with no change of probability on this component. We illustrate our methods using the variance gamma model of Madan, Carr, and Chang [14] that is known to calibrate well the index options market and the statistical process to the level of the fourth moment.

Option price computations are done using the Fourier transform of the modified call price as developed in Carr and Madan [7]. This methodology relies on analytical expressions for the characteristic function of the logarithm of the stock price. Since factor price models are essentially linear in log returns, the Fourier transforms are analytically tractable and this makes option pricing feasible using the Carr and Madan [7] techniques. We illustrate factor option pricing by first pricing IBM options as indirect exposure to the S&P 500 index. Here, we may compare the results of factor option pricing with a direct calibration of the liquid options market in IBM options. Next, we price Sony options as indirect exposure to the Nikkei. Since Sony options are relatively illiquid, a comparison of the indirect calibration with the direct calibration is not possible.

An important motivation leading to the development of this research was the need to provide quotes for options on single name stocks in the Tokyo and Hong Kong markets, where such options are relatively thinly traded, while the Nikkei and Hang Seng index options markets are well developed. The factor option pricing model provides a potential solution for such problems.

Section 2 develops the basic theory of factor option pricing. Applications to pricing IBM options off the S&P 500 index and Sony options off the Nikkei index are presented in section 3. Section 4 concludes.

2 THE FACTOR OPTION PRICING MODEL

We suppose that the risk-neutral process for the value of the index or traded factor risk at time t, $I(t)$, is given by:

$$I(t) = I(0) \exp\left((r-q)t + X(t) + \omega t\right), \tag{1}$$

where r is the riskfree rate of interest, q is the dividend yield on the index, $X(t)$ is a purely discontinuous Lévy process and ω compensates the exponential Lévy process, and is given by:

$$\omega = -\int_{-\infty}^{\infty} (e^x - 1)k(x)dx,$$

where k is the Lévy density for the process $X(t)$.

The model of equation (1) is a typical consequence of time-changing Brownian motion in the original Black-Merton-Scholes [4], [16], theory by an increasing random process with a martingale component to avoid locally deterministic time changes. This argument is developed in Geman, Madan, and Yor [10]. The resulting model has considerable flexibility in that it permits all moments to be altered between the statistical and risk-neutral probabilities, a fact in accord with time series and option price data. The details of this robustness are described in Carr, Geman, Madan, and Yor [6]. Four interesting cases for $X(t)$ are 1) the variance gamma model of Madan, Carr, and Chang [14], 2) its generalization in Carr, Geman, Madan and Yor [6], 3) the use of an inverse Gaussian time change and the normal inverse Gaussian model of Barndorff-Nielsen [3], or 4) its generalizations to the class of generalized hyperbolic distributions [9], [8]. These models provide high quality calibrations of the options data. They also treat index options as primary assets, a perspective in line with the factor pricing approach of this paper.

Options on the index for maturity T and strike K when the spot is at $I(0)$ may be easily priced using the Carr and Madan [7] methodology and we suppose that this option price is given by the pricing function:

$$w = W(I(0), K, r, q, \Theta, T, u), \tag{2}$$

where Θ is the vector of model parameters, and u indicates the type of option, say 1 for a call and 0 for a put.

We will illustrate our methods using the variance gamma process for the risk-neutral dynamics of the index. Under this specification, the process $X(t)$ is described by three parameters, σ, ν, θ, and one may write:

$$X(t) = \theta G(t;\nu) + \sigma W\left(G(t;\nu)\right),$$

whereby $X(t)$ is seen as Brownian motion with drift θ and variance rate σ^2, time-changed by a gamma process with unit mean rate and variance rate ν. In this case, the vector $\Theta = (\sigma, \nu, \theta)$.

Our factor model approach begins by supposing that continuously compounded returns on the stock are linearly related to index returns with a zero mean idiosyncratic residual. If this residual is modeled by a normal distribution, then in continuous time we would describe it by the increment in an independent Brownian motion. To allow for potential higher moment departures from normality in the residuals and yet enforce their zero mean property, we model them by the increment in a symmetric variance gamma process, in that the original Brownian motion being time-changed by a gamma process had a zero drift.

If $S(t)$ is the stock price at time t, specifically we write:

$$\log\left(S(t+h)/S(t)\right) = \alpha + \beta \log\left(I(t+h)/I(t)\right) + Y(t+h; s, \kappa) - Y(t; s, \kappa), \quad (3)$$

where $Y(t)$ is an independent symmetric variance gamma process given by Brownian motion with variance rate s^2, time-changed by a gamma process with unit mean rate and variance rate κ. If we let $W(t)$ denote the standard Brownian motion then:

$$Y(t; s, \kappa) = sW(g(t; \kappa)),$$

where $g(t; \kappa)$ is the gamma process with variance rate parameter κ.

We comment briefly on the stock return specification (3) by noting that it is a single factor return model with the index return being the factor driving asset returns. The residual is a zero mean firm-specific disturbance, which is often modeled by a normal random variable, but is here generalized to a continuous time specification that can have excess kurtosis given by the parameter κ. The process (3) is the statistical return generating mechanism for the stock. It could be estimated from time series data on the stock and the index by maximum likelihood estimation methods, given that there exists a closed form expression for all the densities involved as specified in Madan, Carr, and Chang (1998). However, as shown in Madan, Carr, and Chang (1998), the results of such an estimation cannot be directly applied to pricing options, as the risk-neutral density is generally substantially different from the statistical distribution of the return process.

The stock return model (3) implies that the stock price process at time t is:

$$S(t) = S(0) \exp\left(\alpha t + \beta \log(I(t)/I(0)) + Y(t)\right), \quad (4)$$

where we have dropped the dependence of Y on the parameters for notational convenience. For pricing options on the stock, we must identify the risk-neutral density for the stock price. For this purpose, we suppose that the risk-neutral process for the index may be identified by the prices of index options in a liquid market for such options. The risk-neutral process for the index is then as given by (1). For the firm-specific risk, if we have some options trading on the stock, then we may we allow for a measure change in $Y(t)$, and suppose that the risk-neutral process is in the variance gamma class. Furthermore, the risk-neutral drift on the stock is set at the interest rate less the stock dividend yield. We denote the risk-neutral parameters of the process $Y(t)$ by σ_f, ν_f, θ_f. We may then write the risk-neutral process for the stock as:

$$S(t) = S(0)\exp\left((r-\delta)t + \beta X(t;\sigma,\nu,\theta) + Y(t;\sigma_f,\nu_f,\theta_f) + \lambda t\right), \tag{5}$$

where δ is the stock dividend yield and λ is the compensator for the jump processes in the exponential, and is determined explicitly below. We note that if the firm-specific risk is not priced, then the risk neutral process for $Y(t)$ should be the same as the statistical process and thus $\theta_f = 0, \sigma_f = s, \nu_f = \kappa$.

In the absence of data on individual stock options, or conditional on supposing that idiosyncratic risk is not priced, we may infer the regression coefficient β by standard time series methods and set $(\sigma_f, \nu_f, \theta_f)$ to their statistical counterparts $(s, \kappa, 0)$, as identified by a statistical analysis. Alternatively, if we have some limited data on stock options, we may infer these parameters from such information.

For pricing options, we follow the methodology outlined in Carr and Madan (1998) that identifies analytically the Fourier transform in log strike of the modified call prices in terms of the log characteristic function of the risk-neutral density at maturity for the log stock price. The option prices may then be obtained by direct Fourier inversion, using the FFT (Fast Fourier Transform), if desired, for enhancing the speed of the computation. The option pricing problem is then completed on identifying this characteristic function. The coefficient λ is also easily identified by an evaluation of this characteristic function.

By the independence of the processes X and Y, we determine that:

$$\begin{aligned} E\left[e^{iu\log(S(t))}\right] &= \exp\left(iu(\log(S(0)) + (r-\delta+\lambda)t)\right) X \\ &\quad \left(1 - i\theta\nu\beta u + \sigma^2\beta^2u^2\nu/2\right)^{-t/\nu} X \\ &\quad \left(1 - i\theta_f\nu_f u + \sigma_f^2\nu_f u^2/2\right)^{-t/\nu_f}. \end{aligned} \tag{6}$$

The coefficient λ is determined on noting that (6) evaluated at $u = -i$, must be $\exp((r-\delta)t)$. It follows that:

$$\lambda = \frac{1}{\nu}\log\left(1 - \theta\nu\beta - \sigma^2\beta^2\nu/2\right) + \frac{1}{\nu_f}\log\left(1 - i\theta_f\nu_f - \sigma_f^2\nu_f/2\right).$$

We christen the pricing methodology described above as VGSI for Variance Gamma Stock Index. The matlab program *vgsi.m*, included in the appendix, uses the above characteristic function to computes stock option prices via the Carr and Madan (1998) methodology. A typical call to the pricing function is given by:

$$w = vgsi(S(0), K, r, \delta, \sigma, \nu, \theta, \beta, \sigma_f, \nu_f, \theta_f, T, u), \tag{7}$$

where u is one for a call and zero for a put.

3 APPLICATIONS OF VGSI TO IBM AND SONY

In this section, we present the results of two experiments conducted to evaluate the VGSI pricing methodology. First, we consider the case where we have liquid prices on stock options, the case of IBM, but we ignore these and price the IBM options off the implicit SPX index risk embedded in holding IBM risk with risk-neutral residual. Second, we consider the pricing of SONY options off the NIKKEI, using the few SONY option prices we have to price the firm-specific risk.

3.1 Pricing IBM off the SPX

The data employed for this experiment were for July 1, 1998. The IBM stock price was at 116.8125 and we consider options of maturity .296 years. The risk-free rate for this maturity was 5.75 percent and the dividend yield was estimated at .63 percent. For this maturity, we had available 10 IBM put prices for strikes ranging from 95 dollars to 140 dollars and 9 IBM call prices with strikes ranging from 100 to 140 dollars. The VG model was estimated on this stock option data using only out-of-the-money prices, and this gave parameter estimates of $.296, .1229, -.2882$ for σ, ν, θ respectively. We shall compare these results of direct VG estimation on stock option data with those obtained by applying VGSI and treating stock risk as indirect index risk.

In applying VGSI, we first estimate the time series model (3). For this purpose, we regressed daily log returns for IBM on SPX log returns over the period December 31, 1993 to July 1, 1998, obtaining the result that:

$$\log(S(t+1)/S(t)) = .003 + 1.1920 \log(I(t+1)/I(t)) + \varepsilon_{t+1}. \tag{8}$$

The VG parameters for the residual may be estimated by the method of moments, with an estimate for s being the standard deviation of the residuals, and an estimate of κ is given by the ratio of the fourth moment to three times s^4 less 1. These estimates were annualized using the scaling described in Madan, Carr, and Chang (1998). The resulting annualized estimates for s and κ were .2688 and .0124 respectively (assuming 250 days in a year).

The next step in the implementation of VGSI is the estimation of the risk-neutral density for the index risk. For this step, we obtained data on 25 SPX index options, with 6 calls and 19 puts at a maturity of .219 years. The risk-free rate was 5.75 percent and the dividend yield on the index was estimated at 1.45 percent. The level of the index at this date was $1,145.5643$. The risk-neutral VG parameter estimates for the SPX on this day for this maturity were $\sigma = .1704$, $\nu = .2363$, and $\theta = -.2143$.

If we now suppose that the risk-neutral estimate for the IBM beta is the same as the statistical estimate of 1.1920 and that the residual risk in the stock is not priced, then we have all the information needed to construct VGSI option prices for IBM. The results for these option prices are presented for puts in Figure 1 and for calls in Figure 2. We observe that the VG models fit is very good. Although the VGSI price curve follows the general shape of the other two curves, it overprices both the puts and the calls.

We note that for pure jump processes like the VG, the statistical and risk-neutral volatilities and hence β estimates need not be the same. We advocate using the at-the-money option price to estimate the risk-neutral β. The dotted line in Figures 1 and 2 is the beta-adjusted VGSI price curve. We observe that we can effectively explain IBM option prices as indirectly pricing index risk with risk-neutral pricing of the residual risk, provided that we adjust risk-neutral betas to fit the at-the-money option price. Thus, VGSI is a potentially feasible methodology for pricing options indirectly as exposure to index factor risk.

3.2 Pricing Sony off the NIKKEI Index

In this subsection, we present the results of applying the methodology of section 2 to pricing options on Sony stock. We use information from the market prices of options on the Nikkei index, as well as the few options we have on Sony stock, to calibrate the risk-neutral parameters of the firm-specific process. The data used was from the market close of July 27, 1998.

Figure 1: IBM call option prices and fitted values using VG, VGSI and beta adjusted VGSI

Figure 2: IBM put option prices and fitted values using VG, VGSI and beta adjusted VGSI

Figure 3: Actual and Fitted VG prices for the Nikkei on July 27 1998.

For the September 10 maturity, we have prices on three Sony options. In particular, we have a put struck at $12,000$ priced at 235, and two calls struck at $13,000$ and $14,000$ priced at 640 and 275 respectively. For these options, the time to expiration is .1205 years. The spot market price of Sony on this day was $13,010$.

For the Nikkei index at a time to maturity of .126 years, with the spot at $15,908$ yen, the risk-free rate at .0062 per cent, and a dividend yield of .0003 per cent, we have prices on 4 index puts and 3 index calls. The puts were struck at $14,000, 14,500, 15,000$, and $15,500$ and were priced at $100, 160, 260$, and 410 respectively. The calls were struck at $16,000, 16,500$, and $17,000$ with prices $565, 335$, and 200 respectively. The VG parameter estimates were $\sigma = .2658, \nu = .0505$, and $\theta = -.3855$. Figure 3 presents a graph of the actual out-of-the-money option prices and the VG fitted curve. The implied level of volatility, skewness, and kurtosis in the risk-neutral density is respectively $.2795, -.2022$, and 3.1792. This information is also presented in the form of implied volatilities in Figure 4.

The VGSI option pricing model has four additional parameters associated

Figure 4: Actual and Fitted VG implied volatilities for the NIKKEI for July 27 1998.

with the residual risk. These are the stock beta, residual volatility, skewness, and kurtosis. The statistical beta was estimated from time series at .7593, while the residual volatility is .2261, with a kurtosis of .0045. We supposed a statistical skewness of zero in these calculations. For the corresponding risk-neutral values, we suppose that the residual risk may be treated risk-neutrally, and let the risk-neutral residual skew also be zero. However, we estimate the risk-neutral beta, volatility, and kurtosis using the three Sony options which we have available for this maturity. The estimated risk-neutral values were $\beta = .1931$, with a residual risk-neutral volatility of .3031,and an excess kurtosis parameter of 0.001. Figures 5 presents the actual Sony prices and the price curve as estimated by the VGSI option pricing model. The same information is presented in the form of Black Scholes implied volatilities in Figure 6.

Figure 5: Actual and fitted VGSI prices for Sony July 27 1998

Figure 6: Actual and fitted VGSI implied volatilities for Sony July 27 1998

4 Conclusion

This paper presents a methodology for pricing options on stocks using a factor model, whereby stock risk is viewed as the sum of indirect index risk and residual risk. The method is particularly suited to developing quotations on stock options when these markets are relatively illiquid and one has a liquid index options market to judge the index risk. The method exploits the time series correlation between the stock and the index and also incorporates a statistical analysis of the higher moments of the regression residuals in the absence of data on individual stock options. Regression coefficients and parameters for the risk-neutral residual risk may be estimated if there is some information on a few stock options. The results provide market consistent quotes on the remaining range of strikes. Generally, we advocate the use of at-the-money options prices to calibrate the choice of the level for the risk-neutral beta coefficient. The pricing strategy is illustrated on IBM options without using the IBM option prices directly. We observe that the strategy of pricing options as an indirect exposure to index risk performs reasonably well. We also illustrate and report results for pricing Sony stock options off the Nikkei index.

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Appendix

Matlab Programs for computing VGSI prices. The procedure computes the inverse Fourier transform of the Fourier transform of the modified call price to obtain call prices which are then transformed by put-call parity for put prices if desired. The inputs are

1. p, the current spot price.
2. x, the option strike.
3. r, the interest rate.
4. q, the dividend yield.
5. sig, nu, th the index VG parameters.
6. bta, the asset beta with respect to the index.
7. sgr, nur, thr, the VG parameters for the idiosyncratic residual.
8. t, the option maturity
9. u, a flag that is 1 for calls and 0 for puts.

The output is the option price. The program calls vgsifr.m, that calls in turn vgsifrf.m and vgsicf.m. The final program is the characteristic function for the log of stock price at maturity.

```
function y = vgsi(p,x,r,q,sig,nu,th,bta,sgr,nur,thr,t,u)
u1=vgsifr(log(x),p,r,q,t,sig,nu,th,bta,sgr,nur,thr);
uu=u1+(1-u).*(x.*exp(-r.*t)-p.*exp(-q.*t)); y=uu;
return

function y=vgsifr(k,p,r,q,t,sig,nu,th,bta,sgr,nur,thr)
aa=[k p r q t];
rni=[sig nu th];
sip=[bta sgr nur thr];
nn1=8; nn2=64; nn3=256;
uu1=quad8('vgsifrf',0,nn1,[],[],aa,rni,sip);
uu2=quad8('vgsifrf',nn1,nn2,[],[],aa,rni,sip);
uu3=quad8('vgsifrf',nn2,nn3,[],[],aa,rni,sip);
uu=uu1+uu2+uu3;
y=2.*uu;
return
```

```
function y = vgsifrf(v,P1,P2,P3)
k=P1(1);
p=P1(2);
r=P1(3);
q=P1(4);
t=P1(5);
sig=P2(1);
nu=P2(2);
th=P2(3);
bta=P3(1);
sgr=P3(2);
nur=P3(3);
thr=P3(4);
aa=.03;
u1=exp(-r.*t);
u2=vgsicf(v-(aa+1).*i,p,r,q,t,sig,nu,th,bta,sgr,nur,thr);
u3=aa.^2+aa-v.^2+i.*v.*(2.*aa+1);
u4=exp(-aa.*k);
u5=exp(-i.*v.*k);
u6=1./(2.*pi);
uu=u4.*u6.*u5.*u1.*u2./u3;
y=real(uu);
return

function y = vgsicf(u,p,r,q,t,sig,nu,th,bta,sgr,nur,thr)
h1=(1./nu).*log(1-bta.*th.*nu-sig.^2.*bta.^2.*nu./2);
h2=(1./nur).*log(1-thr.*nur-sgr.^2.*nur./2);
lda=h1+h2;
u1=log(p)+(r-q).*t+lda.*t;
u2=exp(i.*u.*u1);
u3=1-i.*th.*nu.*bta.*u+(sig.^2.*bta.^2.*nu./2).*u.^2;
u4=u3.^(-t./nu);
u5=1-i.*thr.*nur.*u+(sgr.^2.*nur./2).*u.^2;
u6=u5.^(-t./nur);
uu=u2.*u4.*u6;
y=uu;
return
```