Towards a Theory of Volatility Trading

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Introduction

- Three methods have evolved for trading vol:
    1. static positions in options eg. straddles
    2. delta-hedged option positions
    3. volatility swaps
- The purpose of this talk is to explore the advantages and disadvantages of each approach.
- I'll show how the first two methods can be combined to create the third.
- I'll also show the link between some "exotic" volatility swaps and some recent work by Dupire[3] and Derman, Kani, and Kamal[2].

Part I

Static Positions in Options

Trading Vol via Static Positions in Options

- The classic position for trading vol is an at-the-money straddle.
- Unfortunately, the position loses sensitivity to vol as the underlying moves away from the strike.
- Is there a static options position which maintains its sensitivity to vol as the underlying moves?
- To answer this question, we first need to develop a theory of static replication using options.
- We assume the following:
  1. frictionless markets
  2. no arbitrage
  3. underlying futures matures at $T' \geq T$
  4. continuum of European futures option strikes; single maturity T
- Note that we do not restrict the price process in any way!

Spanning a Payoff

- Consider a terminal cash flow $f(F_T)$, which is a twice differentiable function of the final futures price F_T.
- We will show that only the second derivative of the payoff is relevant for generating volatility-based payoffs. Accordingly, we can restrict attention to payoffs whose value and slope vanish at an arbitrary point κ.
- The paper proves that for any such payoff:

$$f(F_T) = \int_0^{\kappa} f''(K)(K - F_T)^+ dK + \int_{\kappa}^{\infty} f''(K)(F_T - K)^+ dK.$$

- In words, to create a twice differentiable payoff $f(\cdot)$ with value and slope vanishing at a given point κ, buy $f''(K)dK$ puts at all strikes K less than κ and buy $f''(K)dK$ calls at all strikes K greater than κ.
- The absence of arbitrage requires that the initial value $V_0^f(T)$ of the final payoff $f(\cdot)$ can be expressed in terms of the initial prices of puts $P_0(K,T)$, and calls $C_0(K,T)$ respectively:

$$V_0^f(T) = \int_0^{\kappa} f''(K)P_0(K,T)dK + \int_{\kappa}^{\infty} f''(K)C_0(K,T)dK.$$

Variance of Terminal Futures Price

- The variance of the terminal futures price is:

$$\mathrm{Var}_0(F_T) = E_0\{[F_T - E_0(F_T)]^2\}.$$

- If we use risk-neutral expectations with the money market account as numeraire, then all futures prices are martingales, and so:

$$E_0(F_T) = F_0.$$

- Thus, the variance of F_T is just the futures price of the portfolio of options which pays off $[F_T - F_0]^2$ at T (see Figure 0.1):

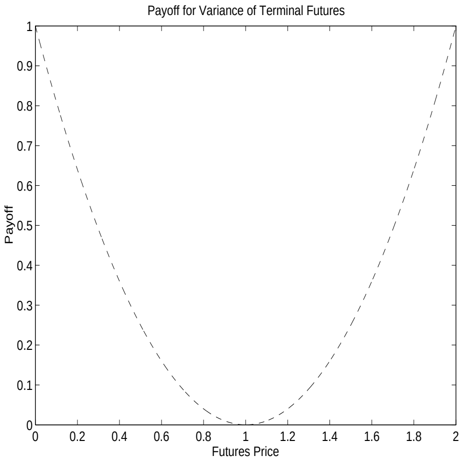

Figure 0.1: Payoff for Variance of Terminal Futures Price($F_0 = 1$).

Variance of Terminal Futures Price(con'd)

- Recall that the spot value of an arbitrary payoff $f(\cdot)$ with value and slope vanishing at some point κ was given by:

$$V_0(T) = \int_0^{\kappa} f''(K)P_0(K,T)dK + \int_{\kappa}^{\infty} f''(K)C_0(K,T)dK.$$

- For $f(F) = (F - F_0)^2$, the value and slope vanish at F_0 and $f''(K) = 2$.
- Thus, the risk-neutral variance of the terminal futures price can be expressed in terms of the futures prices $\hat{P}$ and $\hat{C}$ of puts and calls respectively:

$$\mathrm{Var}_0(F_T) = 2\left[\int_0^{F_0} \hat{P}_0(K,T)dK + \int_{F_0}^{\infty} \hat{C}_0(K,T)dK\right].$$

- We can similarly calculate the risk-neutral variance of the log futures price relative by finding the futures price of the portfolio of options which pays off $\left\{\ln\left(\frac{F_T}{F_0}\right) - E_0\left[\ln\left(\frac{F_T}{F_0}\right)\right]\right\}^2$, where $E_0\left[\ln\left(\frac{F_T}{F_0}\right)\right]$ is the futures price of the portfolio of options which pays off $\ln\left(\frac{F_T}{F_0}\right)$ at T.

Advantages and Disadvantages of Static Positions in Options

- When compared to an at-the-money straddle, the quadratic payoffs have the advantage of maintaining their sensitivity to volatility (suitably defined), as the underlying moves away from its initial level.
- Unfortunately, like straddles, the quadratic payoffs will have non-zero delta once the underlying moves away from its initial level.
- The solution to this problem is to delta-hedge with the underlying.

Part II

Delta-Hedging Options Positions

Review of Delta-hedging in a Constant Vol World

- The Black model assumes continuous trading, a constant interest rate, and a continuous futures price process with constant volatility.
- Let's review delta-hedging of European-style claims in this model. For future use, we assume that even though the current time is $t = 0$, the claim is sold at $t = T$ and that the hedge occurs over (T, T'), where T' is the maturity of the claim.
- Let $V(F, t)$ be any function of the futures price and time. Applying Itô's Lemma to $V(F, t)e^{r(T-t)}$ gives:

$$\begin{aligned} V(F_{T'}, T') &= V(F_T, T)e^{r(T'-T)} + \int_T^{T'} e^{r(T'-t)} \frac{\partial V}{\partial F}(F_t, t) dF_t \\ &\quad + \int_T^{T'} e^{r(T'-t)} \left[\frac{\partial V}{\partial t}(F_t, t) + \frac{\sigma^2 F_t^2}{2} \frac{\partial^2 V}{\partial F^2}(F_t, t) - rV(F_t, t) \right] dt \end{aligned}$$

Review of Delta-hedging in a Constant Vol World (con'd)

- Recall that for any function $V(F,t)$:

$$\begin{aligned} V(F_{T'},T') &= V(F_T,T)e^{r(T'-T)} + \int_T^{T'} e^{r(T'-t)}\frac{\partial V}{\partial F}(F_t,t)dF_t \\ &+ \int_T^{T'} e^{r(T'-t)}\left[\frac{\partial V}{\partial t}(F_t,t) + \frac{\sigma^2 F_t^2}{2}\frac{\partial^2 V}{\partial F^2}(F_t,t) - rV(F_t,t)\right]dt \end{aligned}$$

- Now consider a function $V(F,t;\sigma)$ which solves:

$$\frac{\partial V}{\partial t}(F,t;\sigma) + \frac{\sigma^2 F^2}{2}\frac{\partial^2 V}{\partial F^2}(F,t;\sigma) - rV(F,t;\sigma) = 0,$$

  and:

$$V(F,T';\sigma) = f(F).$$

- Substitution gives:

$$f(F_{T'}) = V(F_T,T;\sigma)e^{r(T'-T)} + \int_T^{T'} e^{r(T'-t)}\frac{\partial V}{\partial F}(F_t,t;\sigma)dF_t.$$

- Evidently, the payoff $f(F_{T'})$ at T' can be created by investing $V(F_T,T;\sigma)$ dollars in the riskless asset at T and always holding $\frac{\partial V}{\partial F}(F_t,t;\sigma)$ futures contracts over the time interval (T,T') (assuming continuous marking-to-market).

Delta-Hedging at a Constant Vol in a Stochastic Vol World

- Now continue to assume that the price process is continuous, but assume that the true vol is given by some unknown stochastic process σ_t.
- Assume that the claim is sold for an implied vol of σ_h and that delta-hedging is conducted using the Black model delta evaluated at this constant hedge vol.
- Let $V(F, t; \sigma_h)$ be a function satisfying the terminal condition $V(F, T'; \sigma_h) = f(F)$ and the Black p.d.e. with constant volatility σ_h.
- Then the paper shows that:

$$f(F_{T'}) + P\&L = V(F_T, T; \sigma_h)e^{r(T'-T)} + \int_T^{T'} e^{r(T'-t)}\frac{\partial V}{\partial F}(F_t, t; \sigma_h)dF_t,$$

  where:

$$P\&L = \int_T^{T'} e^{r(T'-t)}\frac{F_t^2}{2}\frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h)(\sigma_h^2 - \sigma_t^2)dt.$$

- In words, when we sell the claim for an implied vol of σ_h at T, the instantaneous P&L rate from delta-hedging with the constant vol σ_h over (T, T') is half the dollar gamma weighted average of the difference between the hedge variance and the true variance.
- Note that the P&L vanishes if $\sigma_t = \sigma_h$.
- If $\frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h) \geq 0$ as is true for options, and if $\sigma_t > \sigma_h$ for all $t \in [T, T']$, then you sold the claim for too low a vol and a loss results, regardless of the path. Conversely, if you manage to sell the claim for an implied vol σ_h which dominates the subsequent realized vol at all times, then delta-hedging at σ_h guarantees a positive P&L.

Advantages and Disadvantages of Delta-hedging Options

- When compared with static options positions, delta-hedging appears to have the advantage of being insensitive to the price of the underlying.
- However, recall the expression for the P&L at T':

$$P\&L = \int_T^{T'} e^{r(T'-t)} \frac{F_t^2}{2} \frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h)(\sigma_h^2 - \sigma_t^2) dt.$$

- In general, this expression depends on the path of the price.
- One solution is to use a stochastic vol model to conduct the delta-hedging. However, this requires specifying the volatility process and dynamic trading in options.
- A better solution is to choose the payoff function $f(\cdot)$, so that the path dependence can be removed or managed.
- For example, Neuberger[4] recognized that if $f(F) = 2 \ln F$, then $\frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h) = e^{-r(T'-t)} \frac{-2}{F_t^2}$ and the cumulative P&L at T' is the payoff of a variance swap $\int_T^{T'} (\sigma_t^2 - \sigma_h^2) dt$.

Part III

Volatility Contracts

Delta-hedging with Zero Vol

- Recall the expression for the final portfolio value when delta-hedging at a constant vol σ_h:

$$f(F_{T'}) + P\&L = V(F_T, T; \sigma_h)e^{r(T'-T)} + \int_T^{T'} e^{r(T'-t)}\frac{\partial V}{\partial F}(F_t, t; \sigma_h)dF_t,$$

where:

$$P\&L \equiv \int_T^{T'} e^{r(T'-t)}\frac{F_t^2}{2}\frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h)(\sigma_h^2 - \sigma_t^2)dt.$$

- Setting $\sigma_h = 0$ implies:

$$\begin{aligned} V(F, t; 0) &= e^{-r(T'-t)}f(F), \\ \frac{\partial V}{\partial F}(F, t; 0) &= e^{-r(T'-t)}f'(F), \\ \frac{\partial^2 V}{\partial F^2}(F, t; 0) &= e^{-r(T'-t)}f''(F). \end{aligned}$$

- Substituting into the top equation and re-arranging gives:

$$\int_T^{T'} f''(F_t)\frac{F_t^2}{2}\sigma_t^2 dt = f(F_{T'}) - f(F_T) - \int_T^{T'} f'(F_t)dF_t.$$

Delta-hedging with Zero Vol (con'd)

- Recall the expression for the hedging error/P&L when delta-hedging at zero vol:

$$\int_T^{T'} f''(F_t)\frac{F_t^2}{2}\sigma_t^2 dt = f(F_{T'}) - f(F_T) - \int_T^{T'} f'(F_t)dF_t.$$

- The left hand side is a payoff dependent on both the realized instantaneous volatility σ_t and the futures price F_t.
- The dependence on the payoff $f(F)$ occurs only through its second derivative. Thus, we can and will restrict attention to payoffs whose value and slope vanish at a given point κ.
- The right hand side results from adding the following three payoffs:
  1. The payoff from a static position in options maturing at T' paying $f(F_{T'})$ at T'.
  2. The payoff from a static position in options maturing at T paying $-e^{-r(T'-T)}f(F_T)$ and future-valued to T'
  3. The payoff from maintaining a dynamic position in $-e^{-r(T'-t)}f'(F_t)$ futures contracts (assuming continuous marking-to-market).

Three Interesting Vol Contracts

- Recall the equivalence between a volatility-based payoff and 3 price-based payoffs:

$$\frac{1}{2}\int_T^{T'} f''(F_t)F_t^2\sigma_t^2 dt = f(F_{T'}) - f(F_T) - \int_T^{T'} f'(F_t)dF_t.$$

- We can choose $f(\cdot)$ so that the dependence of the volatility-based payoff on the price path is to our liking.

- We next consider the following 3 second derivatives of payoffs at T' and work out the $f(\cdot)$ which leads to them:

$f''(F_t)$	Payoff at T'
$\frac{2}{F_t^2}$	$\int_T^{T'} \sigma_t^2 dt$
$\frac{2}{F_t^2}1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]$	$\int_T^{T'} 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]\sigma_t^2 dt$
$\frac{2}{\kappa^2}\delta(F_t - \kappa)$	$\int_T^{T'} \delta(F_t - \kappa)\sigma_t^2 dt.$

Contract Paying Future Variance

- Recall the following equivalence between a volatility-based payoff and 3 price-based payoffs:

$$\frac{1}{2}\int_T^{T'} f''(F_t)F_t^2\sigma_t^2 dt = f(F_{T'}) - f(F_T) - \int_T^{T'} f'(F_t)dF_t.$$

- Consider the following function $\phi(F)$ (see Figure 0.2):

$$\phi(F_t) = 2\left[\ln\left(\frac{\kappa}{F_t}\right) + \frac{F_t}{\kappa} - 1\right],$$

where κ is an arbitrary finite positive number.

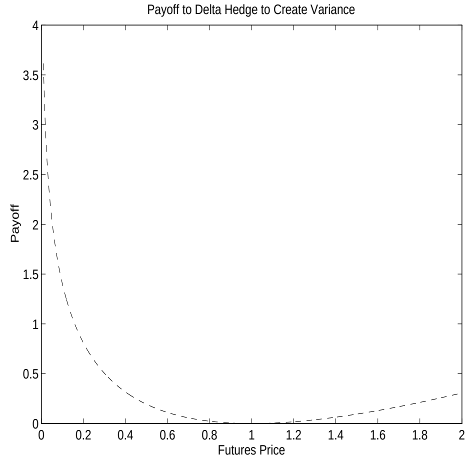

Figure 0.2: Payoff to Delta-Hedge to Create Contract Paying Variance ($\kappa = 1$).

- The first derivative is $\phi'(F_t) = 2\left[\frac{1}{\kappa} - \frac{1}{F_t}\right]$. Note that the value and slope vanish at $F = \kappa$. The second derivative is $\phi''(F_t) = \frac{2}{F_t^2}$.

- Substitution gives $\int_T^{T'} \sigma_t^2 dt =$

$$2\left[\ln\left(\frac{\kappa}{F_{T'}}\right) + \frac{F_{T'}}{\kappa} - 1\right] - 2\left[\ln\left(\frac{\kappa}{F_T}\right) + \frac{F_T}{\kappa} - 1\right] - 2\int_T^{T'}\left[\frac{1}{\kappa} - \frac{1}{F_t}\right]dF_t.$$

Contract Paying Future Variance (Con'd)

- Recall the following equivalence between the variance over (T, T') and 3 price-based payoffs:

$$\begin{aligned}\int_T^{T'} \sigma_t^2 dt &= 2\left[\ln\left(\frac{\kappa}{F_{T'}}\right) + \frac{F_{T'}}{\kappa} - 1\right] - 2\left[\ln\left(\frac{\kappa}{F_T}\right) + \frac{F_T}{\kappa} - 1\right] \\ &\quad -2\int_T^{T'} \left[\frac{1}{\kappa} - \frac{1}{F_t}\right] dF_t.\end{aligned}$$

- Since the value and slope of ϕ vanish at κ:

$$\phi(F) = \int_0^\kappa \phi''(K)(K-F)^+ dK + \int_\kappa^\infty \phi''(K)(F-K)^+ dK.$$

- Since $\phi''(F) = \frac{2}{F^2}$, substitution gives:

$$\begin{aligned}\int_T^{T'} \sigma_t^2 dt &= \int_0^\kappa \frac{2}{K^2}(K - F_{T'})^+ dK + \int_\kappa^\infty \frac{2}{K^2}(F_{T'} - K)^+ dK \\ &\quad + \int_0^\kappa \frac{2}{K^2}(K - F_T)^+ dK + \int_\kappa^\infty \frac{2}{K^2}(F_T - K)^+ dK \\ &\quad -2\int_T^{T'} \left[\frac{1}{\kappa} - \frac{1}{F_t}\right] dF_t.\end{aligned}$$

Contract Paying Future Variance(Con'd Again)

- Recall the decomposition:

$$\begin{aligned}\int_T^{T'} \sigma_t^2 dt &= \int_0^{\kappa} \frac{2}{K^2}(K-F_{T'})^+ dK + \int_{\kappa}^{\infty} \frac{2}{K^2}(F_{T'}-K)^+ dK \\ &\quad + \int_0^{\kappa} \frac{2}{K^2}(K-F_T)^+ dK + \int_{\kappa}^{\infty} \frac{2}{K^2}(F_T-K)^+ dK \\ &\quad -2\int_T^{T'} \left[\frac{1}{\kappa}-\frac{1}{F_t}\right] dF_t.\end{aligned}$$

- To create the contract paying $\int_T^{T'} \sigma_t^2 dt$ at T', at $t=0$, buy:

$$\begin{aligned}&\int_0^{\kappa} \tfrac{2}{K^2} P_0(K,T')dK + \int_{\kappa}^{\infty} \tfrac{2}{K^2} C_0(K,T')dK \\ &-e^{-r(T'-T)}[\int_0^{\kappa} \tfrac{2}{K^2} P_0(K,T)dK + \int_{\kappa}^{\infty} \tfrac{2}{K^2} C_0(K,T)dK].\end{aligned}$$

- At $t = T$, borrow to finance the payout of $2e^{-r(T'-T)}\left[\ln\left(\frac{\kappa}{F_T}\right)+\frac{F_T}{\kappa}-1\right]$ from having initially written the T maturity options. Also start a dynamic strategy in futures, holding $-2e^{-r(T'-t)}\left[\frac{1}{\kappa}-\frac{1}{F_t}\right]$ futures for each $t \in [T, T']$.

- The net payoff at T' is:

$$\begin{aligned}&\int_0^{\kappa} \frac{2}{K^2}(K-F_{T'})^+ dK + \int_{\kappa}^{\infty} \frac{2}{K^2}(F_{T'}-K)^+ dK \\ &+ \int_0^{\kappa} \frac{2}{K^2}(K-F_T)^+ dK + \int_{\kappa}^{\infty} \frac{2}{K^2}(F_T-K)^+ dK \\ &-2\int_T^{T'} \left[\frac{1}{\kappa}-\frac{1}{F_t}\right] dF_t = \int_T^{T'} \sigma_t^2 dt,\end{aligned}$$

as desired.

Contract Paying Future Corridor Variance

- Consider a corridor $(\kappa - \triangle\kappa, \kappa + \triangle\kappa)$ and suppose that we wish to generate a payoff at T' of $\int_T^{T'} 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]\sigma_t^2 dt$.
- Define:

$$\underline{\bar{F}_t} \equiv \max[\kappa - \triangle\kappa, \min(F_t, \kappa + \triangle\kappa)]$$

as the futures price floored at $\kappa - \triangle\kappa$ and capped at $\kappa + \triangle\kappa$ (see Figure 0.3):

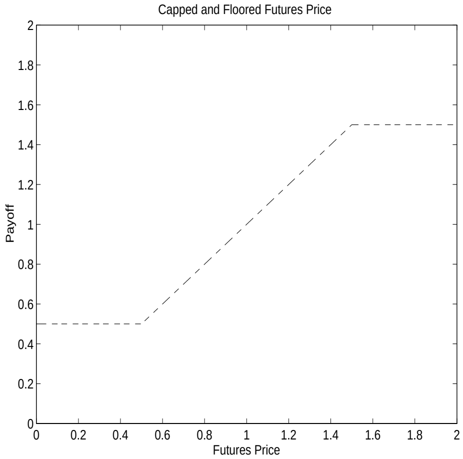

Figure 0.3: Futures Price Capped and Floored($\kappa = 1, \triangle\kappa = 0.5$).

- Note that $\lim_{\triangle\kappa \uparrow \infty} \underline{\bar{F}_t} = F$ and $\lim_{\triangle\kappa \downarrow 0} \underline{\bar{F}_t} = \kappa$.

Contract Paying Future Corridor Variance(Con'd)

- Recall the payoff which generates the future variance when delta hedged at zero vol:

$$\phi(F_t) = 2\left[\ln\left(\frac{\kappa}{F_t}\right) + \frac{F_t}{\kappa} - 1\right] = 2\left[\ln\left(\frac{\kappa}{F_t}\right) + F_t\left(\frac{1}{\kappa} - \frac{1}{F_t}\right)\right].$$

- Consider the following generalization of this payoff $\phi(\cdot)$ (see Figure 0.4):

$$\phi_{\triangle\kappa}(F_t) = 2\left[\ln\left(\frac{\kappa}{\bar{\underline{F}}_t}\right) + F_t\left(\frac{1}{\kappa} - \frac{1}{\bar{\underline{F}}_t}\right)\right].$$

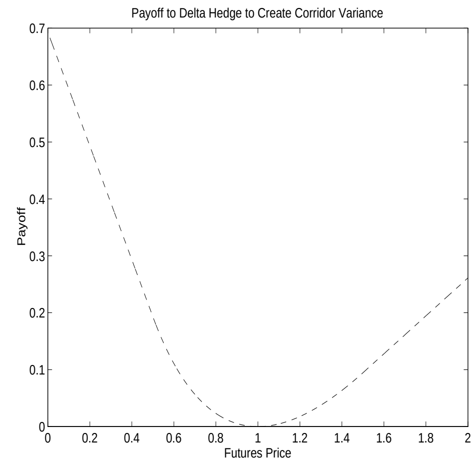

Figure 0.4: Trimming the Log Payoff ($\kappa = 1, \triangle\kappa = 0.5$).

- The first derivative is $\phi'_{\triangle\kappa}(F_t) = 2\left[\frac{1}{\kappa} - \frac{1}{\bar{\underline{F}}_t}\right]$. Once again, the value and slope vanish at $F = \kappa$. The second derivative is $\phi''_{\triangle\kappa}(F_t) = \frac{2}{F_t^2}1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]$.

- Substitution gives $\int_T^{T'} \sigma_t^2 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]dt =$

$$2\left[\ln\left(\frac{\kappa}{\bar{\underline{F}}_{T'}}\right) + \frac{\bar{\underline{F}}_{T'}}{\kappa} - 1\right] - 2\left[\ln\left(\frac{\kappa}{\bar{\underline{F}}_T}\right) + \frac{\bar{\underline{F}}_T}{\kappa} - 1\right] - 2\int_T^{T'}\left[\frac{1}{\kappa} - \frac{1}{\bar{\underline{F}}_t}\right]dF_t.$$

Contact Paying Future Corridor Variance (Con'd)

- Recall the decomposition of the corridor variance:

$$
\begin{aligned}
& \int_T^{T'} \sigma_t^2 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]dt \\
= \ & 2\left[\ln\left(\frac{\kappa}{\bar{F}_{T'}}\right) + \frac{\bar{F}_{T'}}{\kappa} - 1\right] - 2\left[\ln\left(\frac{\kappa}{\bar{F}_T}\right) + \frac{\bar{F}_T}{\kappa} - 1\right] \\
& -2\int_T^{T'} \left[\frac{1}{\kappa} - \frac{1}{\bar{F}_t}\right] dF_t.
\end{aligned}
$$

- Since the value and slope of $\phi_{\triangle\kappa}$ vanish at κ:

$$
\phi_{\triangle\kappa}(F) = \int_0^{\kappa} \phi''_{\triangle\kappa}(K)(K - F)^+ dK + \int_{\kappa}^{\infty} \phi''_{\triangle\kappa}(K)(F - K)^+ dK.
$$

- Since $\phi''_{\triangle\kappa}(F) = \frac{2}{F^2} 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]dt$, substitution gives:

$$
\begin{aligned}
& \int_T^{T'} \sigma_t^2 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]dt \\
= \ & \int_{\kappa - \triangle\kappa}^{\kappa} \frac{2}{K^2}(K - F_{T'})^+ dK + \int_{\kappa}^{\kappa + \triangle\kappa} \frac{2}{K^2}(F_{T'} - K)^+ dK \\
& -e^{-r(T'-T)}[\int_{\kappa - \triangle\kappa}^{\kappa} \frac{2}{K^2}(K - F_T)^+ dK + \int_{\kappa}^{\kappa + \triangle\kappa} \frac{2}{K^2}(F_T - K)^+ dK] \\
& -2\int_T^{T'} \left[\frac{1}{\kappa} - \frac{1}{\bar{F}_t}\right] dF_t.
\end{aligned}
$$

Contact Paying Future Corridor Variance (Con'd Again)

- Recall the decomposition of the corridor variance:

$$\begin{aligned}
&\int_T^{T'} \sigma_t^2 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]dt \\
= \; &\int_{\kappa-\triangle\kappa}^{\kappa} \frac{2}{K^2}(K - F_{T'})^+ dK + \int_{\kappa}^{\kappa+\triangle\kappa} \frac{2}{K^2}(F_{T'} - K)^+ dK \\
&-e^{-r(T'-T)}[\int_{\kappa-\triangle\kappa}^{\kappa} \frac{2}{K^2}(K - F_T)^+ dK + \int_{\kappa}^{\kappa+\triangle\kappa} \frac{2}{K^2}(F_T - K)^+ dK] \\
&-2\int_T^{T'} \left[\frac{1}{\kappa} - \frac{1}{\underline{\bar{F}}_t}\right] dF_t.
\end{aligned}$$

- Thus, to create the contract paying $\int_T^{T'} \sigma_t^2 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]dt$ at T', at $t = 0$, buy and sell options struck within the corridor:

$$\begin{aligned}
&\int_{\kappa-\triangle\kappa}^{\kappa} \tfrac{2}{K^2} P_0(K, T')dK + \int_{\kappa}^{\kappa+\triangle\kappa} \tfrac{2}{K^2} C_0(K, T')dK \\
-&e^{-r(T'-T)}[\int_{\kappa-\triangle\kappa}^{\kappa} \tfrac{2}{K^2} P_0(K, T)dK + \int_{\kappa}^{\kappa+\triangle\kappa} \tfrac{2}{K^2} C_0(K, T)dK].
\end{aligned}$$

- At $t = T$, borrow to finance the payout of $2e^{-r(T'-T)}\left[\ln\left(\frac{\kappa}{\underline{F}_T}\right) + F_T\left(\frac{1}{\kappa} - \frac{1}{\underline{F}_T}\right)\right]$ from having initially written the T maturity options. Also start a dynamic strategy in futures, holding $-2e^{-r(T'-t)}\left[\frac{1}{\kappa} - \frac{1}{\underline{\bar{F}}_t}\right]$ futures for each $t \in [T, T']$.
- The net payoff at T' is:

$$\begin{aligned}
&\int_{\kappa-\triangle\kappa}^{\kappa} \frac{2}{K^2}[K - F_{T'}]^+ dK + \int_{\kappa}^{\kappa+\triangle\kappa} \frac{2}{K^2}[F_{T'} - K]^+ dK \\
&-e^{-r(T'-T)}[\int_{\kappa-\triangle\kappa}^{\kappa} \frac{2}{K^2}[K - F_T]^+ dK + \int_{\kappa}^{\kappa+\triangle\kappa} \frac{2}{K^2}[F_T - K]^+ dK] \\
&-2\int_T^{T'} \left[\frac{1}{\kappa} - \frac{1}{\underline{\bar{F}}_t}\right] dF_t \\
= \; &\int_T^{T'} \sigma_t^2 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]dt,
\end{aligned}$$

as desired.

Contract Paying Variance Along a Strike

- Recall that we created a contract paying $\int_T^{T'} \sigma_t^2 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]dt$ at T' by initially spreading options struck within the corridor:

$$\begin{aligned}&\textstyle\int_{\kappa-\triangle\kappa}^{\kappa} \frac{2}{K^2} P_0(K,T')dK + \int_{\kappa}^{\kappa+\triangle\kappa} \frac{2}{K^2} C_0(K,T')dK \\ -e^{-r(T'-T)}&\textstyle[\int_{\kappa-\triangle\kappa}^{\kappa} \frac{2}{K^2} P_0(K,T)dK + \int_{\kappa}^{\kappa+\triangle\kappa} \frac{2}{K^2} C_0(K,T)dK].\end{aligned}$$

- Suppose we re-scale everything by $\frac{1}{2\triangle\kappa}$. The payoff at T' would instead be:

$$\int_T^{T'} \frac{1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]}{2\triangle\kappa} \sigma_t^2 dt.$$

- By letting $\triangle\kappa \downarrow 0$, the variance received can be localized in the spatial dimension: $\int_T^{T'} \delta(F_t - \kappa)\sigma_t^2 dt$.

- Since only options struck within the corridor are used, the initial cost of creating this localized cash flow is:

$$\frac{1}{\kappa^2}[V_0(\kappa, T') - e^{-r(T'-T)} V_0(\kappa, T)],$$

  where $V_0(\kappa, T)$ is the cost of a straddle struck at κ and maturing at T:

$$V_0(\kappa, T) = P_0(\kappa, T) + C_0(\kappa, T).$$

- As usual, at $t = T$, borrow to finance the payout of $\frac{|F_T - \kappa|}{\kappa^2}$ from having initially written the T maturity straddle. One can work out that the dynamic strategy in futures initiated at T involves holding $-\frac{e^{-r(T'-t)}}{\kappa^2} sgn(F_t - \kappa)$ futures contracts, where $sgn(x)$ is the sign function:

$$sgn(x) \equiv \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x > 0. \end{cases}$$

- The dynamic futures strategy is known as the (deferred) stop-loss start-gain strategy investigated by Carr and Jarrow[1].

Advantages and Disadvantages of Volatility Contracts

- When compared to delta-hedging options, volatility contracts offer the user control over the sensitivity to the path.
- Not all volatility based payoffs can be spanned unless one is willing to specify or derive a risk-neutral volatility process.
- It is an open question as to which volatility payoffs can be spanned by static positions in options combined with dynamic trading in the underlying.

Part IV

Connection to Recent Work on Stochastic Vol

Contract Paying Local Variance

- Recall that we were able to create a contract paying the variance along a strike, $\int_T^{T'} \delta(F_t - \kappa)\sigma_t^2 dt$, by initially buying a (ratioed) calendar spread of straddles, $\frac{1}{\kappa^2}[V_0(\kappa, T') - e^{-r(T'-T)}V_0(\kappa, T)]$.
- Suppose we further re-scale this payoff by $\frac{1}{\triangle T}$ where $\triangle T \equiv T' - T$.
- The payoff at T' would instead be:

$$\int_T^{T'} \frac{\delta(F_t - \kappa)}{\triangle T}\sigma_t^2 dt.$$

- The cost of creating this position would be:

$$\frac{1}{\kappa^2}\left[\frac{V_0(\kappa, T') - e^{-r(T'-T)}V_0(\kappa, T)}{\triangle T}\right].$$

- By letting $\triangle T \downarrow 0$, one gets the beautiful result of Dupire[3] that $\frac{1}{\kappa^2}\left[\frac{\partial V_0(\kappa,T)}{\partial T} + rV_0(\kappa, T)\right]$ is the cost of creating the payment $\delta(F_T - \kappa)\sigma_T^2$ at T.
- As shown in Dupire, the forward local variance can be defined as the number of butterfly spreads paying $\delta(F_T - \kappa)$ at T one must sell in order to finance the above option position initially.
- A discretized version of this result can be found in Derman et. al. [2].

Summary

- One can go on to impose a stochastic process on the spot or forward price of local variance as in Dupire[3] and in Derman et. al.[2].
- The approach taken here is to examine the theoretical underpinnings of all such stochastic processes for volatility.
- It is interesting to note how naturally options arise as part of the analysis.
- Copies of the overheads can be downloaded from www.math.nyu.edu/research/carrp/papers

Bibliography

[1] Carr P. and R. Jarrow, 1990, "The Stop-Loss Start-Gain Strategy and Option Valuation: A New Decomposition into Intrinsic and Time Value", *Review of Financial Studies*, **3**, 469–492.

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[3] Dupire B., 1996, "A Unified Theory of Volatility", Paribas working paper.

[4] Neuberger, A. 1990, "Volatility Trading", London Business School working paper.