

The Reduction Method for Valuing Derivative Securities

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In order to solve a differential equation you look at it till a solution occurs to you – George Pólya, How to Solve It (p. 181).

Introduction

- Black-Scholes (1973) derived a fundamental partial differential equation (PDE) which all derivative security values must satisfy.
- Working in a lognormal framework, they transformed this PDE to the heat equation, thereby generating analytic solutions for European option values.
- The arguments in Merton (1973) imply more generally that the arbitrage-free value V of many derivative securities satisfies the following second order linear parabolic PDE with three variable coefficients:

$$\frac{a^2(S, t)}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + b(S, t) \frac{\partial V}{\partial S}(S, t) + \frac{\partial V}{\partial t}(S, t) = c(S, t)V(S, t).$$

- The purpose of this paper is to characterize the entire set of variable coefficients $\{a(S, t), b(S, t), c(S, t)\}$ which permit the above one state variable valuation PDE to be transformed to the (univariate) heat equation.
- We derive a complicated expression which the three coefficients must satisfy in order that such a transformation exists.

Why Bother?

- Our motivations for studying “heat transfer” are three-fold:
 1. Since the heat equation is one of the most widely studied partial differential equations, the large class of known results on it can be used to analyze the fundamental valuation PDE when one is transformable into the other.
 2. As illustrated by the Black Scholes formula, transformation to the heat equation allows generation of closed form formulas for European option prices expressible in terms of standard normal density or distribution functions.
 3. Our technique has important implications for numerical methods. By discretizing the time and space derivatives, one can analytically map nodes on the tree for the derivative security price to nodes on the tree for the underlying, which can be further mapped to nodes on the tree for a discretized standard Brownian motion. All three trees are recombining as in Nelson and Ramaswamy (1990), but in addition, the state prices in the Brownian tree are all equal to each other. Furthermore, the analytic solutions described above can be used in segments of the tree or finite difference scheme where the propagation of value is unencumbered by boundary conditions.

Applications to Interest Rate Derivatives

- Recall the fundamental second order linear parabolic PDE:

$$\frac{a^2(S, t)}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + b(S, t) \frac{\partial V}{\partial S}(S, t) + \frac{\partial V}{\partial t}(S, t) = c(S, t)V(S, t).$$

- When valuing interest rate derivatives, this PDE arises if the spot interest rate is assumed to be some function of a diffusing state variable S and time t .
- The coefficient $a(S, t)$ is the state variable's volatility function, while the coefficient $b(S, t)$ is the state variable's (absolute) risk-neutral drift.
- The coefficient $c(S, t)$ is the derivative security's (relative) risk-neutral drift, or equivalently, the cost of carrying the claim once a particular asset is chosen to finance the premium. If this asset is a money market account, then $c(S, t)$ describes the functional relationship between the spot interest rate and the driving state variable in the absence of both default and any continuous cash payouts from the claim.

Applications to Equity Derivatives

- Again recall the fundamental second order linear parabolic PDE:

$$\frac{a^2(S, t)}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + b(S, t) \frac{\partial V}{\partial S}(S, t) + \frac{\partial V}{\partial t}(S, t) = c(S, t)V(S, t).$$

- In equity derivative models, this PDE often arises when the diffusing state variable S is taken to be the stock price.
- The function $a(S, t)$ relates the absolute instantaneous volatility of the stock price at t to the stock price level S at t and to the time t .
- The risk-neutral drift $b(S, t)$ takes the form $[c_s(S, t) - q(S, t)]S$, where $c_s(S, t)$ is the proportional cost of financing positions in the stock and $q(S, t)$ is the dividend yield on the stock.
- For claims with no intermediate cash flow and in markets with no imperfections or credit risk, the net cost of carry for the derivative security position $c(S, t)$ reduces to the spot interest rate, which is typically assumed to not depend on the stock price. However, correlation between these variables can be captured in a crude way by allowing this functional dependence. Furthermore, credit risk and market imperfections can induce a dependence of the derivative's carrying cost on the stock price. For these reasons, it is worthwhile considering the general form of the top PDE for both interest rate and equity derivative models.

Overview of Results

- Again recall the fundamental second order linear parabolic PDE:

$$\frac{a^2(S, t)}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + b(S, t) \frac{\partial V}{\partial S}(S, t) + \frac{\partial V}{\partial t}(S, t) = c(S, t)V(S, t).$$

- We show that this PDE can *always* be transformed into the following canonical form:

$$\frac{1}{2} \frac{\partial^2 U}{\partial x^2}(x, t) + \frac{\partial U}{\partial t}(x, t) = \gamma^c(x, t)U(x, t), \quad x \in \mathfrak{R}, t \in (0, T).$$

- We also show that a necessary and sufficient condition for transforming this canonical PDE to the heat equation is that $\gamma^c(x, t)$ is quadratic in x .
- The quadratic restriction on γ^c engenders a very complicated restriction on the three coefficients in the fundamental PDE.
- Despite the apparent intractability of this expression, we develop conditions under which various triplets of coefficients satisfying our restriction can be generated.
- We believe that as a result, many new closed form solutions for derivative security prices can be generated.

The Fundamental PDE

- As usual, we assume frictionless markets and no arbitrage.
- We work in a univariate diffusion setting throughout and choose to derive all our results via PDE methods.
- Thus, we further append whatever assumptions are sufficient to validate the following fundamental PDE:

$$\frac{a^2(S, t)}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + b(S, t) \frac{\partial V}{\partial S}(S, t) + \frac{\partial V}{\partial t}(S, t) = c(S, t)V(S, t).$$

- Although the above PDE is valid for many contingent claims when the claim has intermediate discrete payoffs occurring at a finite set of (possibly random times), we will for simplicity assume a continuous payoff to the claim which can be captured in the specification of the proportional carrying cost $c(S, t)$.
- The PDE reflects the assumption that the diffusing state variable has diffusion coefficient $a(S, t)$ and risk-neutral drift coefficient $b(S, t)$. When the asset used to finance the claim is a money market account, then $c(S, t)$ will be the difference between the spot interest rate and the claim's proportional dividend, provided that default risk is not being reflected in c .
- Finally, for simplicity we assume that there is a terminal liquidating payoff occurring at a fixed time T :

$$V(S, T) = m(S), \quad S \in \mathfrak{R}.$$

An Insight

- Recall the fundamental PDE:

$$\frac{a^2(S, t)}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + b(S, t) \frac{\partial V}{\partial S}(S, t) + \frac{\partial V}{\partial t}(S, t) = c(S, t)V(S, t).$$

- The Black-Scholes PDE is the special case when $a(S, t) = \sigma S$, $b(S, t) = (r - q)S$, and $c(S, t) = r$. In this case, Black-Scholes (1973) transformed their PDE into the standard heat equation:

$$\frac{\partial^2 u}{\partial w^2}(w, \Upsilon) = \frac{\partial u}{\partial \Upsilon}(w, \Upsilon).$$

- Their transformation fails in the general case. However, our problem has 3 degrees of freedom corresponding to 3 yardsticks for measuring time and for measuring value in the underlying and overlying.
- The largest class of transformations of these yardsticks is:

$$\begin{aligned}\tau &= \phi_t(V, S, t), \\ w &= \phi_S(V, S, t), \\ u &= \phi_V(V, S, t),\end{aligned}$$

for some functions ϕ_t , ϕ_S , and ϕ_V .

- Since the fundamental PDE and the target heat equation are both linear, we may restrict attention to the class of transformations which preserve linearity:

$$\begin{aligned}\tau &= f(S, t), \\ w &= g(S, t), \\ u &= h(S, t)V(S, t),\end{aligned}$$

where f , g , and h are functions to be determined.

Heat Transfer

- Recall the fundamental PDE:

$$\frac{a^2(S, t)}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + b(S, t) \frac{\partial V}{\partial S}(S, t) + \frac{\partial V}{\partial t}(S, t) = c(S, t)V(S, t).$$

- Under a certain restriction on the three coefficients a , b , and c to be derived later, we will show that the aforementioned linearity preserving transformations:

$$\begin{aligned}\tau &= f(S, t), \\ w &= g(S, t), \\ u &= h(S, t)V(S, t),\end{aligned}$$

can be used to rescale the volatility to 1 and the carrying costs to 0.

- Applying these changes to the fundamental PDE results in the backward diffusion equation:

$$\frac{1}{2} \frac{\partial^2 u}{\partial w^2}(w, \tau) + \frac{\partial u}{\partial \tau}(w, \tau) = 0,$$

which is easily transformed into the heat equation

$$\frac{\partial^2 u}{\partial w^2}(w, \Upsilon) = \frac{\partial u}{\partial \Upsilon}(w, \Upsilon).$$

by letting $\Upsilon = \frac{T-\tau}{2}$.

- We next examine the consequences of being able to respecify a yardstick for measuring time and for measuring value in the underlying and in the overlying.

Change of Spatial Independent Variable

- Our flexibility in specifying the value of the underlying state variable can be exploited to induce unit absolute volatility in our new underlying. As expected, our new underlying will have a different carrying cost than our old one, while the cost of carrying the overlying will be invariant.

- Let:

$$x(S, t) \equiv \int_{S_0}^S \frac{1}{a(Z, t)} dZ, \quad S \in \mathfrak{R}, t \in [0, T],$$

be a change in the independent spatial variable.

- Financially, the new spatial variable can be interpreted as a new underlying asset with a specified value function $x(S, t)$ and whose payoffs are given in the paper.
- By Itô's lemma, the absolute volatility of any asset with value function $W(S, t)$ is $\frac{\partial W}{\partial S}(S, t)a(S, t)$. For our new underlying, the value function $x(S, t)$ has been specifically chosen so that its absolute volatility is one.

Transformed PDE

- For the last time, recall the fundamental PDE:

$$\frac{a^2(S, t)}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + b(S, t) \frac{\partial V}{\partial S}(S, t) + \frac{\partial V}{\partial t}(S, t) = c(S, t)V(S, t).$$

- Let $U(x, t) \equiv V(S, t)$ where recall $x(S, t) \equiv \int_{S_0}^S \frac{1}{a(Z, t)} dZ$.

- Then the paper shows that the fundamental PDE transforms to:

$$\frac{1}{2} \frac{\partial^2 U}{\partial x^2}(x, t) + \beta_1(x, t) \frac{\partial U}{\partial x}(x, t) + \frac{\partial U}{\partial t}(x, t) = \gamma(x, t)U(x, t), \quad x \in \mathfrak{R}, t \in (0, T),$$

where $\beta_1(x, t)$ is the absolute risk-neutral drift of the new underlying derivative security:

$$\beta_1(x, t) \equiv - \int_{S_0}^S \frac{1}{a^2(Z, t)} \frac{\partial a(Z, t)}{\partial t} dZ + \frac{b(S, t)}{a(S, t)} - \frac{1}{2} \frac{\partial a(S, t)}{\partial S},$$

and where $\gamma(x, t) \equiv c(S(x, t), t)$ is the function relating the proportional cost of carrying the claim to the new underlying x .

Change of Dependent Variable

- Our flexibility in specifying the value of the overlying can be exploited to induce zero *risk-neutral* drift in the underlying. As expected, our new overlying will have a different carrying cost than our old one, while the volatility of the underlying will be invariant to the change.

- Recall the PDE with unit vol derived on the last overhead:

$$\frac{1}{2} \frac{\partial^2 U}{\partial x^2}(x, t) + \beta_1(x, t) \frac{\partial U(x, t)}{\partial x} + \frac{\partial U}{\partial t}(x, t) = \gamma(x, t)U(x, t), \quad x \in \mathfrak{R}, t \in (0, T),$$

- Now consider the change in the dependent variable given by:

$$U^c(x, t) \equiv R(x, t)U(x, t), \quad x \in \mathfrak{R}, t \in (0, T),$$

where:

$$R(x, t) \equiv e^{\int_0^x \beta_1(z, t) dz}, \quad x \in \mathfrak{R}, t \in (0, T).$$

- Financially, we can interpret $R(x, t)$ as an exchange rate and $U^c(x, t)$ as the converted value of our original derivative security.
- The cash flows characterizing the exchange rate are given in the paper. Note that the functional form of the exchange rate $R(x, t)$ is such that its relative volatility is $\frac{\partial R(x, t)}{R(x, t)} = \beta_1(x, t)$.
- The risk-neutral drift β_1 is the net dollar cost of financing positions in the underlying expressed as a function of the new state variable and time. Since the converted overlying U^c is measured in units of the new currency, while the underlying asset x is measured in the old one, a “convexity correction” of $-\beta_1(x, t)$ has been created to neutralize the drift.

On to the Canonical PDE

- Recall the PDE with unit vol:

$$\frac{1}{2} \frac{\partial^2 U}{\partial x^2}(x, t) + \beta_1(x, t) \frac{\partial U(x, t)}{\partial x} + \frac{\partial U}{\partial t}(x, t) = \gamma(x, t)U(x, t), \quad x \in \mathfrak{R}, t \in (0, T),$$

- Also recall that the new dependent variable is:

$$U^c(x, t) \equiv R(x, t)U(x, t), \quad x \in \mathfrak{R}, t \in (0, T),$$

where:

$$R(x, t) \equiv e^{\int_0^x \beta_1(z, t) dz}, \quad x \in \mathfrak{R}, t \in (0, T).$$

- Under this change of dependent variable, the paper shows that the top PDE transforms to the canonical PDE:

$$\mathcal{L}U^c \equiv \frac{1}{2} \frac{\partial^2 U^c}{\partial x^2}(x, t) + \frac{\partial U^c}{\partial t}(x, t) = \gamma^c(x, t)U^c(x, t), \quad x \in \mathfrak{R}, t \in (0, T),$$

where the net proportional cost of carrying the converted claim is:

$$\gamma^c(x, t) \equiv \gamma(x, t) + \frac{1}{2} \frac{\partial \beta_1(x, t)}{\partial x} + \int_0^x \frac{\partial \beta_1(y, t)}{\partial t} dy + \frac{\beta_1^2(x, t)}{2}.$$

Determining the Time Change

- Recall the canonical PDE:

$$\mathcal{L}U^c \equiv \frac{1}{2} \frac{\partial^2 U^c}{\partial x^2}(x, t) + \frac{\partial U^c}{\partial t}(x, t) = \gamma^c(x, t)U^c(x, t), \quad x \in \mathfrak{R}, t \in (0, T),$$

- The flexibility in specifying the clock will allow transformation to the backward diffusion equation provided that the converted claim's carrying cost $\gamma^c(x, t)$ is quadratic in x , i.e.:

$$\gamma^c(x, t) = q_0(t) + q_1(t)x + q_2(t)\frac{x^2}{2},$$

where $q_0(\cdot)$, $q_1(\cdot)$, and $q_2(\cdot)$ are arbitrary functions of time.

- Consider the stochastic time change:

$$\tau = \tau(x, t), \quad x \in \mathfrak{R}, t \in [0, T],$$

and let $\hat{u}(x, \tau) \equiv U^c(x, t)$ be the new value function.

- Since X is following a diffusion and the claim value will in general be a nonlinear function of the new time variable, the term $\frac{1}{2} \frac{\partial^2 \hat{u}}{\partial \tau^2}(x, \tau) \left(\frac{\partial \tau}{\partial x}(x, t) \right)^2$ will be irrevocably introduced into the PDE, unless $\frac{\partial \tau}{\partial x}(x, t) = 0$. Enforcing this condition implies that the cross partial $\frac{\partial^2 \hat{u}}{\partial x \partial \tau}(x, \tau)$ also drops out of the PDE, and that the time change is deterministic:

$$\tau = \tau(t), \quad t \in [0, T].$$

- To be a proper time change, we further require $\tau'(t) \geq 0$ and:

$$\tau(0) = 0.$$

One More Time

- Recall our plan to transform the canonical PDE:

$$\mathcal{L}U^c \equiv \frac{1}{2} \frac{\partial^2 U^c}{\partial x^2}(x, t) + \frac{\partial U^c}{\partial t}(x, t) = \gamma^c(x, t)U^c(x, t), \quad x \in \mathfrak{R}, t \in (0, T),$$

to the backward diffusion equation by a deterministic time change:

$$\tau = \tau(t), \quad t \in [0, T].$$

- In our new deterministic time scale, the volatility rate will change from one to $\frac{1}{\sqrt{\tau'(t)}}$, unless we also change the spatial variable, as was done previously. The paper verifies that volatility remains at one if the deterministic time change is coupled with the following change in the underlying:

$$w = w_0(t) + e^{\int_0^t F_2(s) ds} x,$$

where $w_0(t)$ is an arbitrary function of time and $e^{\int_0^t F_2(s) ds} = \sqrt{\tau'(t)}$, so that:

$$F_2(t) \equiv \frac{\tau''(t)}{2\tau'(t)}.$$

Threepart

- Let $F_3(t) \equiv e^{-\int_0^t F_2(s)ds} = \frac{1}{\sqrt{\tau'(t)}}$. In our new time scale and (second) new spatial scale, the risk-neutral drift in the underlying will change from zero to $\frac{\mathcal{L}w}{\sqrt{\tau'(t)}} = F_3(t)\mathcal{L}w$, unless we also convert the overlying to a new currency, as was done previously. The paper verifies that the underlying remains driftless if the change in the underlying is coupled with the following conversion for the overlying:

$$u(w, \tau) = e^{F_0(t) + \int_0^x \beta_2(z, t) dz} U^c(x, t),$$

where $F_0(t)$ is to be determined, and:

$$\begin{aligned} \beta_2(x, t) &\equiv \frac{\mathcal{L}w}{\sqrt{\tau'(t)}} = F_3(t)\mathcal{L} \left[w_0(t) + e^{\int_0^t F_2(s)ds} x \right] \\ &= F_3(t)w'_0(t) + F_3(t)F_2(t)e^{\int_0^t F_2(s)ds} x = F_1(t) + F_2(t)x, \end{aligned}$$

is linear in x with vertical intercept:

$$F_1(t) \equiv F_3(t)w'_0(t).$$

Quadratic Claim Carrying Cost

- Recall that the risk-neutral drift of our twice transformed underlying is linear in its value:

$$\beta_2(x, t) = F_1(t) + F_2(t)x.$$

- In our new time scale and in our second new spatial scale and currency, the paper shows that the new net proportional carrying cost for the claim is:

$$F_3(t) \left[\gamma^c(x, t) + F_0'(t) + \frac{1}{2} \frac{\partial \beta_2(x, t)}{\partial x} + \int_0^x \frac{\partial \beta_2(z, t)}{\partial t} dz - \frac{\beta_2^2(x, t)}{2} \right].$$

- Setting this quantity to zero, substituting in the linear expression for $\beta_2(x, t)$, and simplifying implies that $\gamma^c(x, t)$ must be quadratic in x :

$$\begin{aligned} \gamma^c(x, t) = & \left[F_2^2(t) - F_2'(t) \right] \frac{x^2}{2} + [F_2(t)F_1(t) - F_1'(t)] x \\ & + \left[-\frac{F_2(t)}{2} + \frac{F_1^2(t)}{2} - F_0'(t) \right]. \end{aligned}$$

- Thus, if $\gamma^c(x, t)$ is instead specified as the following quadratic in x :

$$\gamma^c(x, t) = q_0(t) + q_1(t)x + q_2(t)\frac{x^2}{2},$$

then the paper shows how to determine the functions $F_0(t)$, $F_1(t)$, $F_2(t)$, and $F_3(t)$ from the functions $q_0(t)$, $q_1(t)$, and $q_2(t)$.

Why Quadratic?

- Our results imply that if a PDE has the form

$$\frac{1}{2} \frac{\partial^2 U^c}{\partial x^2}(x, t) + \frac{\partial U^c}{\partial t}(x, t) = \left[q_0(t) + q_1(t)x + q_2(t) \frac{x^2}{2} \right] U^c(x, t),$$

then it can be further transformed to the backward diffusion equation.

- The necessity of the quadratic condition stems from the requirement that the time change be deterministic. If the new time variable τ depends on either the dependent variable U^c or the spatial independent variable x , then the new time process would not be increasing, i.e. time could run backwards.
- Maintaining the volatility of the new underlying at one thus requires that the new spatial variable w be linear in the old one x .
- Maintaining the risk-neutral drift of the new underlying at zero by converting the overlying further implies that the overlying carrying cost in the new currency be the sum of the overlying carrying cost γ^c in the old currency and a quadratic expression in x .
- Further requiring no overlying carrying cost in the new currency forces γ^c to be quadratic.
- The paper shows that the quadratic condition is also sufficient for transforming the canonical PDE to the backward diffusion equation and gives the needed maps.

Transformation Condition

- Recall the original fundamental PDE:

$$\frac{a^2(S, t)}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + b(S, t) \frac{\partial V}{\partial S}(S, t) + \frac{\partial V}{\partial t}(S, t) = c(S, t)V(S, t).$$

and the canonical PDE with quadratic claim carrying cost:

$$\frac{1}{2} \frac{\partial^2 U^c}{\partial x^2}(x, t) + \frac{\partial U^c}{\partial t}(x, t) = \left[q_0(t) + q_1(t)x + q_2(t) \frac{x^2}{2} \right] U^c(x, t).$$

- The paper shows that the following condition on the three coefficients $a(S, t)$, $b(S, t)$, and $c(S, t)$ is both necessary and sufficient in order that the top equation be transformable to the second one:

$$\begin{aligned} & \frac{1}{2} \left\{ -\frac{\partial^2 \ln a(S, t)}{\partial S \partial t} + \frac{\partial^2 b(S, t)}{\partial S^2} - \frac{\partial b(S, t)}{\partial S} \frac{\partial \ln a(S, t)}{\partial S} - b(S, t) \frac{\partial^2 \ln a(S, t)}{\partial S^2} \right. \\ & \left. - \frac{1}{2} \frac{\partial a(S, t)}{\partial S} \frac{\partial^2 a(S, t)}{\partial S^2} - \frac{a(S, t)}{2} \frac{\partial^3 a(S, t)}{\partial S^3} \right\} a(S, t) \\ & - \left[\int_{S_0}^S \frac{1}{a^2(Z, t)} \frac{\partial a(Z, t)}{\partial t} dZ - \frac{b(S, t)}{a(S, t)} + \frac{1}{2} \frac{\partial a(S, t)}{\partial S} - \frac{1}{a(S, t)} \frac{\partial S(x, t)}{\partial t} \Big|_{x=x(S, t)} \right] \times \\ & \left[-\frac{\partial \ln a(S, t)}{\partial t} + \frac{\partial b(S, t)}{\partial S} - b(S, t) \frac{\partial \ln a(S, t)}{\partial S} - \frac{a(S, t)}{2} \frac{\partial^2 a(S, t)}{\partial S^2} \right] \\ & + \int_{S_0}^S \frac{2}{a^3(Z, t)} \left(\frac{\partial a(Z, t)}{\partial t} \right)^2 dZ - \int_{S_0}^S \frac{1}{a^2(Z, t)} \frac{\partial^2 a(Z, t)}{\partial t^2} dZ \\ & + \frac{1}{a(S, t)} \frac{\partial b(S, t)}{\partial t} - \frac{b(S, t)}{a^2(S, t)} \frac{\partial a(S, t)}{\partial t} - \frac{1}{2} \frac{\partial^2 a(S, t)}{\partial S \partial t} + \frac{\partial c(S, t)}{\partial S} a(S, t) = \\ & q_1(t) + q_2(t) \int_{S_0}^S \frac{1}{a(Z, t)} dZ, \end{aligned}$$

for $S \geq 0, t \in [0, T]$ and for any two functions $q_1(t)$ and $q_2(t)$.

Mathematica/Maple etc.

- Recall the necessary and sufficient condition for transforming the fundamental valuation PDE to the heat equation:

$$\begin{aligned}
& \frac{1}{2} \left\{ -\frac{\partial^2 \ln a(S, t)}{\partial S \partial t} + \frac{\partial^2 b(S, t)}{\partial S^2} - \frac{\partial b(S, t)}{\partial S} \frac{\partial \ln a(S, t)}{\partial S} - b(S, t) \frac{\partial^2 \ln a(S, t)}{\partial S^2} \right. \\
& \left. - \frac{1}{2} \frac{\partial a(S, t)}{\partial S} \frac{\partial^2 a(S, t)}{\partial S^2} - \frac{a(S, t)}{2} \frac{\partial^3 a(S, t)}{\partial S^3} \right\} a(S, t) \\
& - \left[\int_{S_0}^S \frac{1}{a^2(Z, t)} \frac{\partial a(Z, t)}{\partial t} dZ - \frac{b(S, t)}{a(S, t)} + \frac{1}{2} \frac{\partial a(S, t)}{\partial S} - \frac{1}{a(S, t)} \frac{\partial S(x, t)}{\partial t} \Big|_{x=x(S, t)} \right] \times \\
& \left[-\frac{\partial \ln a(S, t)}{\partial t} + \frac{\partial b(S, t)}{\partial S} - b(S, t) \frac{\partial \ln a(S, t)}{\partial S} - \frac{a(S, t)}{2} \frac{\partial^2 a(S, t)}{\partial S^2} \right] \\
& + \int_{S_0}^S \frac{2}{a^3(Z, t)} \left(\frac{\partial a(Z, t)}{\partial t} \right)^2 dZ - \int_{S_0}^S \frac{1}{a^2(Z, t)} \frac{\partial^2 a(Z, t)}{\partial t^2} dZ \\
& + \frac{1}{a(S, t)} \frac{\partial b(S, t)}{\partial t} - \frac{b(S, t)}{a^2(S, t)} \frac{\partial a(S, t)}{\partial t} - \frac{1}{2} \frac{\partial^2 a(S, t)}{\partial S \partial t} + \frac{\partial c(S, t)}{\partial S} a(S, t) = \\
& q_1(t) + q_2(t) \int_{S_0}^S \frac{1}{a(Z, t)} dZ,
\end{aligned}$$

for $S \geq 0, t \in [0, T]$ and for any two functions $q_1(t)$ and $q_2(t)$.

- Given a candidate triplet of coefficients $\{a(S, t), b(S, t), c(S, t)\}$, the veracity of the above expression can be easily checked using a symbolic calculator. Denoting the left hand side by $\mathcal{A}a(S, t)$ where \mathcal{A} is an operator, then the *transformation condition* can be rewritten as: $\frac{\partial}{\partial S} \left[a(S, t) \frac{\partial}{\partial S} \mathcal{A}a(S, t) \right] = 0$.
- However, finding a triplet of coefficients which solves the equation is quite another matter. Nonetheless, we next discuss various sets of conditions which permit the construction of a triplet of coefficients solving the transformation condition.

Constructing Coefficients

- Recall the transformation condition:

$$\begin{aligned}
& \frac{1}{2} \left\{ -\frac{\partial^2 \ln a(S, t)}{\partial S \partial t} + \frac{\partial^2 b(S, t)}{\partial S^2} - \frac{\partial b(S, t)}{\partial S} \frac{\partial \ln a(S, t)}{\partial S} - b(S, t) \frac{\partial^2 \ln a(S, t)}{\partial S^2} \right. \\
& \left. - \frac{1}{2} \frac{\partial a(S, t)}{\partial S} \frac{\partial^2 a(S, t)}{\partial S^2} - \frac{a(S, t)}{2} \frac{\partial^3 a(S, t)}{\partial S^3} \right\} a(S, t) \\
& - \left[\int_{S_0}^S \frac{1}{a^2(Z, t)} \frac{\partial a(Z, t)}{\partial t} dZ - \frac{b(S, t)}{a(S, t)} + \frac{1}{2} \frac{\partial a(S, t)}{\partial S} - \frac{1}{a(S, t)} \frac{\partial S(x, t)}{\partial t} \Big|_{x=x(S, t)} \right] \times \\
& \left[-\frac{\partial \ln a(S, t)}{\partial t} + \frac{\partial b(S, t)}{\partial S} - b(S, t) \frac{\partial \ln a(S, t)}{\partial S} - \frac{a(S, t)}{2} \frac{\partial^2 a(S, t)}{\partial S^2} \right] \\
& + \int_{S_0}^S \frac{2}{a^3(Z, t)} \left(\frac{\partial a(Z, t)}{\partial t} \right)^2 dZ - \int_{S_0}^S \frac{1}{a^2(Z, t)} \frac{\partial^2 a(Z, t)}{\partial t^2} dZ \\
& + \frac{1}{a(S, t)} \frac{\partial b(S, t)}{\partial t} - \frac{b(S, t)}{a^2(S, t)} \frac{\partial a(S, t)}{\partial t} - \frac{1}{2} \frac{\partial^2 a(S, t)}{\partial S \partial t} + \frac{\partial c(S, t)}{\partial S} a(S, t) = \\
& q_1(t) + q_2(t) \int_{S_0}^S \frac{1}{a(Z, t)} dZ,
\end{aligned}$$

for $S \geq 0, t \in [0, T]$ and for any two functions $q_1(t)$ and $q_2(t)$.

- The simplest approach for finding a triplet satisfying the transformation condition is to specify $a(S, t)$ and $b(S, t)$, since the transformation condition then simplifies into an equation for $\frac{\partial c(S, t)}{\partial S}$, which can be integrated to get $c(S, t)$.
- In contrast, if one specifies the pair $a(S, t)$ and $c(S, t)$, or the pair $b(S, t)$ and $c(S, t)$, then the transformation condition is difficult to solve analytically for the remaining coefficient.

Another Way

- An alternative approach for generating solution triplets is to specify restrictions on the coefficients which appear in the PDE's for various asset values after changing variables. In particular, it is sufficient to consider only the PDE's which arise after the first change in independent variable.
- The paper shows that three conditions which permit explicit construction of the coefficients satisfying the transformation condition is to require that the functions $R(x, t)$, $S(x, t)$, and $U(x, t)$ respectively solve:

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 R}{\partial x^2}(x, t) + \frac{\partial R}{\partial t}(x, t) &= \left[q_0^r(t) + q_1^r(t)x + q_2^r(t) \frac{x^2}{2} \right] R(x, t), \\ \frac{1}{2} \frac{\partial^2 S}{\partial x^2}(x, t) + \beta_1(x, t) \frac{\partial S(x, t)}{\partial x} + \frac{\partial S}{\partial t}(x, t) &= \left[q_0^s(t) + q_1^s(t)x + q_2^s(t) \frac{x^2}{2} \right] S(x, t), \\ \frac{1}{2} \frac{\partial^2 U}{\partial x^2}(x, t) + \beta_1(x, t) \frac{\partial U(x, t)}{\partial x} + \frac{\partial U}{\partial t}(x, t) &= \left[q_0^u(t) + q_1^u(t)x + q_2^u(t) \frac{x^2}{2} \right] U(x, t), \end{aligned}$$

where recall $\beta_1(x, t) = \frac{\partial R}{\partial x}(x, t) / R(x, t)$ and for each asset, the $q_i(\cdot)$, $i = 0, 1, 2$, are arbitrary functions of time.

- The paper shows how to construct solutions to all three PDE's.
- Financially, the sufficient conditions amount to assuming that carrying costs are quadratic in the x for three different assets. The three assets are the (first) exchange rate, the original state variable S , and the original contingent claim. All three assets are assumed to have a dividend process consistent with whatever functional forms solve the above PDE's.

Solution

- Recall that the transformation condition holds if the functions $R(x, t)$, $S(x, t)$, and $U(x, t)$ respectively solve:

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 R}{\partial x^2}(x, t) + \frac{\partial R}{\partial t}(x, t) &= \left[q_0^r(t) + q_1^r(t)x + q_2^r(t) \frac{x^2}{2} \right] R(x, t), \\ \frac{1}{2} \frac{\partial^2 S}{\partial x^2}(x, t) + \beta_1(x, t) \frac{\partial S(x, t)}{\partial x} + \frac{\partial S}{\partial t}(x, t) &= \left[q_0^s(t) + q_1^s(t)x + q_2^s(t) \frac{x^2}{2} \right] S(x, t), \\ \frac{1}{2} \frac{\partial^2 U}{\partial x^2}(x, t) + \beta_1(x, t) \frac{\partial U(x, t)}{\partial x} + \frac{\partial U}{\partial t}(x, t) &= \left[q_0^u(t) + q_1^u(t)x + q_2^u(t) \frac{x^2}{2} \right] U(x, t), \end{aligned}$$

where recall $\beta_1(x, t) = \frac{\partial R(x, t)}{R(x, t)}$ and for each asset, the $q_i(\cdot)$, $i = 0, 1, 2$, are arbitrary functions of time.

- Given this assumption, the paper shows that the coefficients solving the transformation condition can be expressed in terms of $S(x, t)$ and its inverse $x(S, t)$ as:

$$\begin{aligned} a(S, t) &= \frac{\partial S}{\partial x}(x(S, t), t) \\ b(S, t) &= S \left[q_0^s(t) + q_1^s(t)x(S, t) + q_2^s(t) \frac{x^2(S, t)}{2} \right] \\ c(S, t) &= q_0^u(t) + q_1^u(t)x(S, t) + q_2^u(t) \frac{x^2(S, t)}{2}. \end{aligned}$$

Closed Form Calibration

- Recall that the transformation condition holds if the functions $R(x, t)$, $S(x, t)$, and $U(x, t)$ respectively solve:

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 R}{\partial x^2}(x, t) + \frac{\partial R}{\partial t}(x, t) &= \left[q_0^r(t) + q_1^r(t)x + q_2^r(t) \frac{x^2}{2} \right] R(x, t), \\ \frac{1}{2} \frac{\partial^2 S}{\partial x^2}(x, t) + \beta_1(x, t) \frac{\partial S(x, t)}{\partial x} + \frac{\partial S}{\partial t}(x, t) &= \left[q_0^s(t) + q_1^s(t)x + q_2^s(t) \frac{x^2}{2} \right] S(x, t), \\ \frac{1}{2} \frac{\partial^2 U}{\partial x^2}(x, t) + \beta_1(x, t) \frac{\partial U(x, t)}{\partial x} + \frac{\partial U}{\partial t}(x, t) &= \left[q_0^u(t) + q_1^u(t)x + q_2^u(t) \frac{x^2}{2} \right] U(x, t), \end{aligned}$$

where recall $\beta_1(x, t) = \frac{\frac{\partial R}{\partial x}(x, t)}{R(x, t)}$ and for each asset, the $q_i(\cdot), i = 0, 1, 2$, are arbitrary functions of time.

- We note that a unique solution to the three PDE's was obtained by specifying nine functions of time and three terminal conditions.
- The terminal condition for the claim is assumed to be specified *a priori*, so only two functions of the spatial variable can be specified freely.
- We note that the eleven functions can in principle be chosen so as to match nine term structures and two strike structures of option prices. Thus, one can in principle match the outer boundaries of a strike maturity domain, as well as seven interior term structures of option prices.
- The details for deriving the closed form derivative security values are given in the paper.

Implications for Numerical Methods

- The present analysis also has implications for numerical work.
- If the coefficients only satisfy the transformation condition in some region, then our change of variables can still be used in this region to complement a numerical analysis. For example, if the transformation condition only holds for the last month of a contract, then the analytic solution can be used as a terminal condition for a finite difference scheme.
- If the necessary condition holds globally, but intermittent boundary conditions (eg. discrete barrier monitoring or Bermudan exercise) preclude the recovery of an explicit solution, then one can still discretize time and/or space to solve the heat equation in the appropriate domain.
- Finally, Monte Carlo simulation can be enhanced by simulating standard Brownian motion rather than a complicated state variable process.

Summary and Future Work

- Assuming a general form for the linear parabolic PDE governing many derivative security values, we derived an expression which the three coefficients must satisfy in order that this PDE can be transformed into the heat equation.
- We presented a technique for generating solutions to this expression thereby exhibiting a method for generating closed form solutions for derivative security prices.
- Future work will illustrate our results with several examples.
- We also plan to explore transforming the fundamental PDE to other well known PDE's, such as the Bessel PDE.
- Further extensions would include developing explicit pricing formulas for *path-dependent* options such as American, Bermudan, compound, or barrier options.
- One can also develop the multivariate version of our results, or explore transforming PIDE's or non-linear PDE's to simpler equations.