The Reduction Method for Valuing Derivative Securities

Peter Carr
New York University
Courant Institute
New York, NY 10012
(212) 260-3765
pcarr@nyc.rr.com

Alex Lipton
Deutsche Bank
31 West 52nd Street
New York, NY 10006
(212) 469-4728
alex.lipton@db.com

Dilip Madan
University of Maryland
Robert H. Smith School of Business
College Park, MD 20742
(301) 405-2127
dbm@rhsmith.umd.edu

Current Version: July 9, 2002
File Reference: symmetry17.tex

Abstract

It is well known that derivative security valuation often reduces to solving a certain linear partial differential equation with variable coefficients. We derive a complicated expression which the three coefficients must satisfy in order that this PDE can be transformed into the heat equation. We also present a technique for constructing a triplet of coefficients which solve this expression. We thereby exhibit a technique for generating closed form solutions for derivative security values in a wide array of models.

We thank the participants of presentations at the Cornell Theory Center (CTC) and the 2001 INFORMS conference, both in Manhattan. We also thank Claudio Albanese, Majid Borumand, Ali Hirsa, Peter Laurence, Alex Levin, Keith Lewis, Walter Muller, Jeremy Staum, Michael Tari, Robert Thrash, Puru Voruganti, and Thaleia Zariphopolou for comments. Any errors are ours. This version is very preliminary so comments are appreciated.

The Reduction Method for Valuing Derivative Securities

In order to solve a differential equation you look at it till a solution occurs to you – George Pólya, How to Solve It (p. 181).

1 Introduction

In their landmark work, Merton[18] and Black-Scholes[1] derived a fundamental partial differential equation (PDE) which all derivative security values must satisfy. Working in a lognormal framework, these authors transformed this PDE to the heat equation, thereby generating analytic solutions for European option values. Recognizing that the stock's volatility is typically (negatively) correlated with the stock price, subsequent authors relaxed the lognormality assumption by letting the volatility be a function of stock price and time. This simple modification of the original approach retains the important property that an option's payoff can be replicated by dynamic trading in its underlying stock and in the riskless asset. While more sophisticated approaches to option valuation have also been introduced (eg. stochastic volatility and jumps), these approaches all require that one or more options be part of the hedge. Unfortunately, in many options markets, the width of the bid-ask spread renders dynamic trading in options economically unviable.

When the instantaneous volatility is a function of the stock price and time, the form of the volatility function can either be imposed ex-ante as in Cox and Ross[9], or it can be determined from the option prices themselves as in Dupire[10]. In either case, there is a computational advantage in further specifying a parametric form for the volatility function which is consistent with a closed form solution for option prices. Given a functional form for the volatility which has several unknown parameters, a closed form formula for option prices allows these parameters to be determined from a liquid set of option prices in rapid fashion.

While this computational advantage is well known, only a few volatility function specifications have historically yielded closed form solutions for options prices. One important class is the constant elasticity of variance (CEV) model pioneered in Cox[7] (also see Schroeder[24] and Linetsky and Davidov[16]), which contains the constant, square root, and proportional volatility models as special cases. Bouchouev and Isakov[4], Ingersoll[13], Li[15], and Rady[21] all develop new option pricing formulas for various volatility function specifications. Goldenberg[12], and Carr, Tari, and Zariphopolou[5], also develop new option pricing formulas by mapping the stock price process to more tractable processes, such as the square root process or standard Brownian motion. Similarly, in work related to ours, Lipton[17] studies transformations of the standard valuation PDE to the heat equation.

The map to the heat equation used by Black-Scholes[1] assumed constant volatility and that interest rates and dividends were deterministic. Motivated by the need to value interest rate derivatives, a common generalization is to let the spot interest rate be a function of a driving state variable S and time t. For example, Jamshidian[14] considers one factor interest rate derivative models when the spot interest rate is a function of a univariate standard Brownian motion and time. Schmidt[23] extends this work to further allow a deterministic time change. In general, if the driving variable follows a diffusion process, then the value of many interest rate derivative securities satisfies the following second order linear PDE with three variable coefficients:

$$\frac{a^2(S,t)}{2}\frac{\partial^2 V}{\partial S^2}(S,t) + b(S,t)\frac{\partial V}{\partial S}(S,t) + \frac{\partial V}{\partial t}(S,t) = c(S,t)V(S,t), \quad S \in \Re, t \in (0,T). \tag{1}$$

It is well known that the coefficient $a(S,t)$ is the state variable's volatility function, while the coefficient $b(S,t)$ is the state variable's (absolute) risk-neutral drift. The coefficient $c(S,t)$ is the derivative security's (relative) risk-neutral drift, or equivalently, the cost of carrying the claim once a particular asset is chosen to finance the premium. If this asset is a money market account, then $c(S,t)$ describes the functional relationship between the spot interest rate and the driving state variable in the absence of both default and any continuous cash payouts from the claim.

In equity derivative models, the state variable S is taken to be the stock price, so that the risk-neutral drift $b(S,t)$ takes the form $[c_s(S,t) - q(S,t)]S$, where $c_s(S,t)$ is the proportional cost of financing positions in the stock and $q(S,t)$ is the dividend yield on the stock. For claims with no intermediate cash flow and in markets with no imperfections or credit risk, the net cost of carry for the derivative security position $c(S,t)$ reduces to the spot interest rate, which is typically assumed to not depend on the stock price. However, correlation between these variables can be captured in a crude way by allowing this functional dependence. Furthermore, credit risk and market imperfections can induce a dependence of the derivative's carrying cost on the stock price. For these reasons, it is worthwhile considering the general form of (1) for both interest rate and equity derivative models.

The purpose of this paper is to characterize the set of variable coefficients which permit the one state variable valuation PDE (1) to be transformed to the (univariate) heat equation. We derive a complicated expression which the three coefficients in (1) must satisfy in order that such a transformation exists. The motivation for determining this restriction is three-fold. First, since the heat equation is one of the most widely studied partial differential equations, the large class of known results on it can be used to analyze the fundamental valuation PDE when one is tranformable into the other. Second, as illustrated by the Black Scholes formula, transformation to the heat equation allows generation of closed form formulas for European option prices expressible in terms of standard normal density or distribution functions. Third, our technique has important implications for numerical methods. By discretizing the time and space derivatives, one can analytically map nodes on the tree for the derivative security price to nodes on the tree for the underlying, which can be further mapped to nodes on the tree for a discretized standard Brownian motion. All three trees are recombining as in Nelson and Ramaswamy[20], but in addition, the state prices in the Brownian tree are all equal to each other. Furthermore, the analytic solutions described above can be used in segments of the tree or finite difference scheme where the propagation of value is unencumbered by boundary conditions.

Nelson and Ramaswamy[20] first showed in a financial context that transformation of the independent variable can be used to obtain unit volatility. We extend this result by showing that the risk-neutral drift of the underlying can always be eliminated by transforming the dependent variable as well. As a consequence, the fundamental valuation PDE can *always* be transformed into the following canonical form:

$$\frac{1}{2}\frac{\partial^2 U}{\partial x^2}(x,t) + \frac{\partial U}{\partial t}(x,t) = \gamma^c(x,t)U(x,t), \quad x \in \Re, t \in (0,T). \tag{2}$$

We also show that a necessary and sufficient condition for transforming (2) to the heat equation is that $\gamma^c(x,t)$ is quadratic in x. As these two results were proved earlier[1] by Bluman[2, 3] using Lie groups, a contribution of this paper is to prove the result by simpler means, and to provide a financial interpretation for the manifest changes of variable which are encountered. A second contribution is to show that the quadratic restriction on γ^c engenders a more complicated restriction on the three coefficients in (1). Despite the apparent intractibility of this expression, we give a constructive method for generating solution triplets. As a result, we obtain many new closed form solutions for derivative security prices. While closed form valuation formulas can also be generated by other means, transformation of the valuation PDE to the heat equation is shown to be a straightforward and flexible approach for generating pricing functions.

The outline for this paper is as follows. Section II presents a series of transformations of the standard valuation PDE which under a key assumption results in the heat equation. The following section translates this assumption into an equivalent condition on the three coefficients in the original PDE. Section IV then presents various sets of sufficient conditions which allow the construction of three coefficients satisfying our restriction. Section V shows how these conditions can be used to generate closed form solutions for many derivative security values in a wide array of models. Section VI illustrates with examples. Section VII spells out the implications of our analysis for numerical methods, while the final section summarizes the paper and discusses extensions. The appendices contain various technical derivations, while the body of the paper focusses on financially interpeting these results.

2 Transformation to the Heat Equation

In this section, we lay out our economic model and present a series of transformations for the standard valuation PDE which under a key assumption results in the heat equation. The next section shows how this assumption translates into a condition on the three coefficients, which is equivalent to the existence of maps transforming the valuation PDE into the heat equation.

Our economic model assumes frictionless markets and the absence of arbitrage, so that there exists a risk-neutral measure Q. We work in a univariate diffusion setting throughout and derive all our results via PDE methods. Thus, we further append whatever assumptions are sufficient to validate the following second order linear PDE governing an arbitrage-free value V for a derivative security:

$$\frac{a^2(S,t)}{2}\frac{\partial^2 V}{\partial S^2}(S,t) + b(S,t)\frac{\partial V}{\partial S}(S,t) + \frac{\partial V}{\partial t}(S,t) = c(S,t)V(S,t), \quad S \in \Re, t \in (0,T). \tag{3}$$

The above PDE governs arbitrage-free values of both interest rate and equity derivative securities, when the derivative's payoffs and carrying cost at any time t can each be expressed as $C^{2,1}$ functions of the time t and the contemporaneous level S_t of a driving state variable. The PDE reflects the assumption that the state variable is following a univariate diffusion process with risk-neutral drift coefficient $b(S,t)$ and with diffusion coefficient $a(S,t)$. Although the above PDE is valid when the

[1] Also see Cherkasov[6], Gihman and Skorohod[11] pgs 33-39, and Ricciardi[22] for earlier related work.

derivative security has discrete payoffs occuring at a finite set of (possibly random times), we will for simplicity assume a continuous payoff to the the claim which can be captured in the specification of the proprtional carrying cost $c(S,t)$, and that there is a final liquidating payoff occuring at a fixed time T:

$$V(S,T) = m(S), \quad S \in \Re. \tag{4}$$

When the asset used to finance the claim is a money market account, then $c(S,t)$ will be the difference between the spot interest rate and the claim's proportional dividend, provided that default risk is not being reflected in c.

When valuing interest rate derivatives with no intermediate cash flows, we are thus assuming that the spot interest rate is a $C^{2,1}$ function $c(S,t)$ of a state variable S and the time t. The function c can be taken to be the identity map $c(S,t) = S$, but it is useful from a modelling perspective to consider the more general case. The valuation PDE (3) applies to many common interest rate derivatives such as bonds, bond options, caps, floors, swaps, swaptions, and even Asian options, but it would not apply to all path-dependent derivatives.

When valuing equity derivatives, the single source of uncertainty will be taken to be the stock price. On the appropriate domain, the valuation PDE (3) thus applies to many common equity derivatives such as European or American options, but it would not apply to strongly path-dependent derivatives such as continuously monitored lookback or Asian options. Our diffusion assumption implies that the stock's net carrying cost per share, b, and the stock's absolute volatility, a, are functions of at most the stock price and time. In most valuation problems, the proportional cost of carrying the stock is usually modelled as the difference between the interest rate and the dividend yield, which are both taken to be deterministic. While deterministic interest rates is a time-honored assumption, there is little evidence to support deterministic dividend yields, and in this case the stock carrying cost assumes a more complicated form. Even when interest rates and dividend yields are deterministic, then there are other reasons why the stock carrying cost may not be constant. For example, if the underlying's value is specified in a different currency than the option's payoff (due to quantoing), then the carrying cost of this underlying is also affected by the covariance between relative changes in the foreign currency and price changes in the underlying. Thus, when the volatility of the underlying depends on the underlying's price and time, the stock carrying cost will depend on these quantities as well. As a second example, if the underlying has positive probability of jumping to zero, then the continuous portion of the carrying cost of the stock will rise to compensate for this catastrophe. If the arrival rate of the jump is a function of the stock price and time, then the continuous portion of the stock carrying cost will depend on these quantities as well. As a third example, if the underlying is a stock index such as the S&P500, then it is not beyond reason to assume that the riskfree rate is a function of the level of the index and time. While imperfect correlation between interest rates and index levels is surely more economically plausible than perfect correlation, the resulting computational burden may reduce the choice set down to assuming either that interest rates are deterministic, or that changes in interest rates are locally perfectly correlated with changes in the underlying's price, either positively or negatively. If the underlying asset is a stock rather than an index, strong correlation of the stock with the index suggests that it is plausible to assume that the interest rate is a function of the stock price and time. When the interest rate is a function of the underlying's price and time, then the cost of carrying the overlying derivative will also depend on these quantities. The possibility of default by the issuer is a second reason why the the cost of carrying the overlying may vary with these state variables. Finally, we permit the claim to pay continuous dividends expressed as a state-dependent proportion of the claim's value. For all of these reasons, we consider the general form of (3) when valuing equity derivatives.

The Black-Scholes PDE for a non-dividend paying *claim* arises as the special case of (3) when $a(S,t) = \sigma S, b(S,t) = (r-q)S$, and $c(S,t) = r$. In this case, Black-Scholes[1] transformed (3) into the standard heat equation:

$$\frac{\partial^2 u}{\partial w^2}(w, \Upsilon) = \frac{\partial u}{\partial \Upsilon}(w, \Upsilon). \tag{5}$$

When the three coefficients do not take this simple form, the transformation given in Black-Scholes[1] does not result in the heat equation. However, we have at least three degrees of freedom in our problem corresponding to our three yardsticks for measuring time and for measuring value in the underlying and in the overlying. The largest class of transformations of these yardsticks is:

$$\begin{aligned} \tau &= \phi_t(V, S, t), \\ w &= \phi_S(V, S, t), \\ u &= \phi_V(V, S, t), \end{aligned}$$

for some functions ϕ_t, ϕ_S, and ϕ_V. Since the fundamental PDE (3) and the target PDE (5) are both linear, we may restrict attention to the class of transformations which preserve linearity:

$$\begin{aligned} \tau &= f(S, t), \\ w &= g(S, t), \\ u &= h(S, t)V(S, t), \end{aligned}$$

where f, g, and h are functions[2] to be determined. Under a certain restriction on the three coefficients a, b, and c to be derived later, we will show that the above class of transformations can be used to effectively rescale the volatility to one and the carrying costs to zero. Applying these changes to (3) results in the backward diffusion equation:

$$\frac{1}{2}\frac{\partial^2 u}{\partial w^2}(w, \tau) + \frac{\partial u}{\partial \tau}(w, \tau) = 0, \tag{6}$$

which is easily transformed into the heat equation (5) by letting $\Upsilon = \frac{T-\tau}{2}$. The next three subsections each examine the consequences of being able to respecify a yardstick for measuring time and for measuring value in the underlying and in the overlying.

2.1 Change of Spatial Independent Variable

This section shows that our flexibility in specifying the value of the underlying state variable can be exploited to induce unit absolute volatility in our new underlying. As expected, our new underlying

[2] The requirement that time not run backward implies that $f(\cdot)$ cannot depend on S. Furthermore, $g(\cdot)$ and $h(\cdot)$ should be bounded on bounded domains. The transformations we employ respect both these properties.

has a different carrying cost than our old one, while the cost of carrying the overlying is invariant. Let[3]:

$$x(S,t) \equiv \int_{S_0}^{S} \frac{1}{a(Z,t)} dZ, \quad S \in \Re, t \in [0,T], \tag{7}$$

be a change in the independent spatial variable. Financially, the new spatial variable can be interpreted as a new underlying asset with a specified value function $x(S,t)$. By Itô's lemma, the absolute volatility of any asset with value function $W(S,t)$ is $\frac{\partial W}{\partial S}(S,t)a(S,t)$. For our new underlying, the value function $x(S,t)$ has been specifically chosen so that its absolute volatility is one. Note from (7) that the final payoff of our new derivative security is given by:

$$X_T = x(S_T,T) = \int_{S_0}^{S_T} \frac{1}{a(Z,T)} dZ. \tag{8}$$

The new derivative security's description as an asset is completed once we determine the intermediate dividends from the asset. Given that the value function of *any* claim is $W(S,t)$, the analysis in Merton[19] implies that the net carrying cost of any claim is given by its generator:

$$c_w(S,t)W(S,t) = \frac{a^2(S,t)}{2}\frac{\partial^2 W}{\partial S^2}(S,t) + b(S,t)\frac{\partial W}{\partial S}(S,t) + \frac{\partial W}{\partial t}(S,t), \quad S > 0, t \in (0,T),$$

where $c_w(S,t)$ is the proportional net carrying cost. For our new derivative security with value function given in (7), differentiating twice in S and once in t implies that the claim's net carrying cost is:

$$c_x(S,t)x(S,t) = -\frac{1}{2}\frac{\partial a(S,t)}{\partial S} + \frac{b(S,t)}{a(S,t)} - \int_{S_0}^{S} \frac{1}{a^2(Z,t)}\frac{\partial a(Z,t)}{\partial t} dZ. \tag{9}$$

To express this carrying cost in terms of x, note that the positivity of the function $a(S,t)$ ensures that the function $x(S,t)$ defined in (7) is increasing in S. The inverse map $S = S(x,t)$ can then be defined on the range of x and substituted into (9).

Letting $U(x,t) \equiv V(S,t)$ where $x(S,t)$ is given in (7), Appendix 1 shows that the PDE (3) transforms to:

$$\frac{1}{2}\frac{\partial^2 U}{\partial x^2}(x,t) + \beta_1(x,t)\frac{\partial U(x,t)}{\partial x} + \frac{\partial U}{\partial t}(x,t) = \gamma(x,t)U(x,t), \quad x \in \Re, t \in (0,T), \tag{10}$$

where $\beta_1(x,t)$ is the absolute risk-neutral drift of the new underlying derivative security:

$$\beta_1(x,t) \equiv c_x(S,t)x(S,t) = -\int_{S_0}^{S} \frac{1}{a^2(Z,t)}\frac{\partial a(Z,t)}{\partial t} dZ + \frac{b(S,t)}{a(S,t)} - \frac{1}{2}\frac{\partial a(S,t)}{\partial S}, \tag{11}$$

from (9), and where:

$$\gamma(x,t) \equiv c(S(x,t),t) \tag{12}$$

is the function relating the proportional cost of carrying the claim to the new underlying x. The inverse map $S = S(x,t)$ should be substituted in the RHS of (11) to complete the specification for

[3] This transformation first appeared in the finance literature in Nelson and Ramaswamy[20]. Note that the lower limit of the integral can actually be any function of time. The use of S_0 ensures that $x(S_0,t) = 0, t \in [0,T]$.

the generator of the process for the new underlying, but it will prove to be convenient to suspend this substitution.

Assuming a constant interest rate r and dividend yield q, Carr, Tari, and Zariphopolou[5] set $b(S,t) = (r-q)S$ and $\beta_1(x,t) = 0$ in (11) to find the entire class of volatility functions $a(S,t)$ permitting the original PDE (3) to be transformed to the heat equation using scale changes alone. The next subsection explores an alternative approach for eliminating the risk-neutral drift of the underlying, which ultimately determines the weakest possible restrictions on the three coefficients permitting transformation to the heat equation.

2.2 Change of Dependent Variable

This subsection shows that our flexibility in specifying the value of the overlying can be exploited to induce zero *risk-neutral* drift in the underlying. As expected, our new overlying has a different carrying cost than our old one, while the volatility of the underlying is invariant[4] to the change. Consider the change in the dependent variable given by:

$$U^c(x,t) \equiv R(x,t)U(x,t), \quad x \in \Re, t \in (0,T), \tag{13}$$

where:

$$R(x,t) \equiv e^{\int_0^x \beta_1(z,t)dz}, \quad x \in \Re, t \in (0,T). \tag{14}$$

Financially, we can interpret $R(x,t)$ as an exchange rate and $U^c(x,t)$ as the converted value of our original derivative security. We will refer to the currency in which the claim is originally denminated as the base currency, and we will refer to the currency into which the claim is converted as the target currency.

The motivation for the particular functional form of the exchange rate $R(x,t)$ in (14) is the observation that the relative volatility of this exchange rate is $\frac{\frac{\partial R}{\partial x}(x,t)}{R(x,t)} = \beta_1(x,t)$. As indicated in (11), β_1 is interpreted as the net dollar cost of financing positions in the underlying expressed as a function of the new state variable and time. Since the converted overlying U^c is measured in units of the new currency, while the underlying asset x is measured in the old one, a "convexity correction" of $-\beta_1(x,t)$ has been created to neutralize the drift. Hence, performing the change of variables in (13), Appendix 1 shows that the PDE (10) transforms to the canonical PDE:

$$\mathcal{L}U^c \equiv \frac{1}{2}\frac{\partial^2 U^c}{\partial x^2}(x,t) + \frac{\partial U^c}{\partial t}(x,t) = \gamma^c(x,t)U^c(x,t), \quad x \in \Re, t \in (0,T), \tag{15}$$

where the net proportional cost of carrying the converted claim is:

$$\gamma^c(x,t) \equiv \gamma(x,t) + \frac{1}{2}\frac{\partial \beta_1(x,t)}{\partial x} + \int_0^x \frac{\partial \beta_1(y,t)}{\partial t}dy + \frac{\beta_1^2(x,t)}{2}. \tag{16}$$

[4]The elimination of drift and invariance of volatility by this change of variable is well known and goes by several names. In probability theory, it is a consequence of Girsanov's Theorem. In our Markov context, it is also known as Doob's h transform where h is related to the *scale density.* Finally, it is called *Liouville's transformation* in the differential equations literature.

We now finish the financial interpretation of our change of dependent variable by describing the cash flows received by the holder of the new currency. From (14), the final payoff from the new currency is:

$$R(X_T, T) = e^{\int_0^{X_T} \beta_1(y,T)dy}, \tag{17}$$

where X_T is related to S_T in (8). The net cost of carrying the new currency is given by (9) with the new dependent and independent variables, and with $a(S,t)$ and $b(S,t)$ replaced with 1 and $\beta_1(x,t)$ respectively. Differentiating the exchange rate definition (14) twice in x and once in t implies that the net carrying cost of the new currency is $\gamma^c(x,t)R(x,t)$, so that the new proportional claim carrying cost is $\gamma^c(x,t)$ given in (16).

2.3 Time Change

This subsection shows that the flexibility in specifying the clock allows further elimination of the risk-neutral drift in the converted value of our overlying, provided that the just introduced carrying cost $\gamma^c(x,t)$ is quadratic in x, i.e.:

$$\gamma^c(x,t) = q_0(t) + q_1(t)x + q_2(t)\frac{x^2}{2}, \tag{18}$$

where $q_0(\cdot), q_1(\cdot)$, and $q_2(\cdot)$ are arbitrary functions of time.

Thus, consider the stochastic time change:

$$\tau = \tau(x,t), \quad x \in \Re, t \in [0,T], \tag{19}$$

and let $\hat{u}(x,\tau) \equiv U^c(x,t)$ be the new value function. Since X is following a diffusion and the claim value will in general be a nonlinear function of the new time variable, the term $\frac{1}{2}\frac{\partial^2 \hat{u}}{\partial \tau^2}(x,\tau)\left(\frac{\partial \tau}{\partial x}(x,t)\right)^2$ will be irrevocably introduced into the PDE, unless $\frac{\partial \tau}{\partial x}(x,t) = 0$. Enforcing this condition implies that the cross partial $\frac{\partial^2 \hat{u}}{\partial x \partial \tau}(x,\tau)$ also drops out of the PDE, and that the time change is deterministic:

$$\tau = \tau(t), \quad t \in [0,T]. \tag{20}$$

To be a proper time change, we further require $\tau'(t) > 0$ and:

$$\tau(0) = 0. \tag{21}$$

In our new deterministic time scale, the volatility rate will change from one to:

$$F_3(t) \equiv \frac{1}{\sqrt{\tau'(t)}}, \tag{22}$$

unless we also change the spatial variable, as in subsection 2.1. Appendix 1 verifies that volatility remains at one if (20) is coupled with the change of spatial variable:

$$w = w_0(t) + e^{\int_0^t F_2(s)ds}x, \tag{23}$$

where $w_0(t)$ is an arbitrary function of time and $F_2(\cdot)$ is defined so that $e^{\int_0^t F_2(s)ds} = \frac{1}{F_3(t)}$.

In our new time scale and (second) new spatial scale, the risk-neutral drift in the underlying will change from zero to $F_3(t)\mathcal{L}w$, unless we also convert the overlying to a new currency, as in subsection 2.2. Appendix 1 verifies that the underlying remains driftless if (23) is coupled with the following conversion for the overlying:

$$u(w,\tau) = e^{F_0(t)+\int_0^x \beta_2(z,t)dz} U^c(x,t), \tag{24}$$

where $F_0(t)$ is to be determined, and:

$$\beta_2(x,t) \equiv F_3(t)\mathcal{L}w = F_3(t)\mathcal{L}\left[w_0(t) + e^{\int_0^t F_2(s)ds}x\right] = F_3(t)w_0'(t)+F_3(t)F_2(t)e^{\int_0^t F_2(s)ds}x = F_1(t)+F_2(t)x \tag{25}$$

is linear in x with vertical intercept:

$$F_1(t) \equiv F_3(t)w_0'(t). \tag{26}$$

In our new time scale and in our second new spatial scale and currency, the new net proportional carrying cost for the claim is:

$$F_3(t)\left[\gamma^c(x,t) + F_0'(t) + \frac{1}{2}\frac{\partial\beta_2(x,t)}{\partial x} + \int_0^x \frac{\partial\beta_2(z,t)}{\partial t}dz - \frac{\beta_2^2(x,t)}{2}\right],$$

as shown in Appendix 1. Setting this quantity to zero, substituting in (25), and simplifying implies that $\gamma^c(x,t)$ must be quadratic in x:

$$\gamma^c(x,t) = \left[F_2^2(t) - F_2'(t)\right]\frac{x^2}{2} + \left[F_2(t)F_1(t) - F_1'(t)\right]x + \left[-\frac{F_2(t)}{2} + \frac{F_1^2(t)}{2} - F_0'(t)\right]. \tag{27}$$

Thus, if $\gamma^c(x,t)$ is specified as in (18), then equating coefficients of x^2 implies that the function $F_2(t)$ solves the Riccati ordinary differential equation (ODE):

$$F_2'(t) - F_2^2(t) + q_2(t) = 0, \quad t \in [0,T]. \tag{28}$$

Changing the dependent variable from $F_2(t)$ to $F_3(t) = e^{-\int_0^t F_2(s)ds}$ implies that $F_3(t)$ solves the normal form for the class of linear second order homogeneous ODE's.

$$F_3''(t) - q_2(t)F_3(t) = 0, \quad t \in [0,T]. \tag{29}$$

Although there is no general explicit solution to (28) or (29), the solution is known explicitly for a wide host of specifications of $q_2(t)$. Treating $F_2(t)$ and $q_1(t)$ as known, equating coefficients of x in (18) and (27) implies that $F_1(t)$ solves the first order linear ODE:

$$F_1'(t) - F_2(t)F_1(t) + q_1(t) = 0, \quad t \in [0,T]. \tag{30}$$

The general solution is expressed in terms of the assumed known functions $F_3(\cdot)$ and $q_1(\cdot)$ as:

$$F_1(t) = \frac{F_1(0) - \int_0^t F_3(s)q_1(s)ds}{F_3(t)}, \quad t \in [0,T], \tag{31}$$

where $F_1(0)$ is an arbitrary constant. Finally, given solutions for $F_2(t)$ and $F_1(t)$, equating the constant terms in (18) and (27) implies that $F_0(t)$ is given by:

$$F_0(t) = \int_0^t \left[-\frac{F_2(s)}{2} + \frac{F_1^2(s)}{2} - q_0(s) \right] ds, \quad t \in [0,T]. \tag{32}$$

Summarizing the results of this subsection, if a PDE has the form:

$$\frac{1}{2}\frac{\partial^2 U^c}{\partial x^2}(x,t) + \frac{\partial U^c}{\partial t}(x,t) = \gamma^c(x,t)U^c(x,t), \quad x \in \Re, t \in (0,T), \tag{33}$$

then it is necessary that:

$$\gamma^c(x,t) = q_0(t) + q_1(t)x + q_2(t)\frac{x^2}{2} \tag{34}$$

in order that it can be further transformed to the backward diffusion equation via the kinds of maps considered. The necessity of the quadratic condition (34) stems from the requirement that the time change be deterministic. If the new time variable τ depends on either the dependent variable U^c or the spatial independent variable x, then the new time process would not be increasing, i.e. time could run backwards. Maintaining the volatility of the new underlying at one thus requires that the new spatial variable w be linear in the old one x. Maintaining the risk-neutral drift of the new underlying at zero by converting the overlying further implies that the overlying carrying cost in the new currency be the sum of the overlying carrying cost γ^c in the old currency and a quadratic expression in x. Further requiring no overlying carrying cost in the new currency forces γ^c to be quadratic.

The quadratic condition (34) is also sufficient for transforming (33) to the backward diffusion equation. Substituting (25) in (24) implies that the change of dependent variable is:

$$u(w,\tau) \equiv e^{F_2(t)\frac{x^2}{2} + F_1(t)x + F_0(t)} U^c(x,t). \tag{35}$$

From (23) and (26), the new spatial variable is:

$$w = \frac{x}{F_3(t)} + \int_0^t \frac{F_3(s)}{F_1(s)} ds + c_w \equiv w(x,t), \tag{36}$$

where c_w is an arbitrary constant, while from (20),(21), and (22), the new time variable is:

$$\tau = \int_0^t \frac{1}{F_3^2(s)} ds \equiv \tau(t). \tag{37}$$

The resulting backward diffusion equation is:

$$\frac{1}{2}\frac{\partial^2 u}{\partial w^2}(w,\tau) + \frac{\partial u}{\partial \tau}(w,\tau) = 0, \quad w \in \Re, \tau \in (0,\bar{\tau}), \tag{38}$$

where $\bar{\tau} \equiv \tau(T)$. Since the backward diffusion equation is easily transformed into the heat equation, this section has shown that the quadratic condition (18) on the converted claim's carrying cost is both necessary and sufficient for transforming the canonical PDE (15) to the heat equation via the maps indicated.

3 Transformation Condition

This section finds a condition on the variable coefficients $a(S,t)$, $b(S,t)$, and $c(S,t)$ appearing in the original valuation PDE (3). This condition is shown to be equivalent to the transformability of (3) to the heat equation via the maps indicated. The next section then discusses various methods for constructing triplets which satisfy our condition.

We begin by exploring the implications of the quadratic condition (18) for the coefficients $\beta_1(x,t)$ and $\gamma(x,t)$ appearing in (10). Equating (16) with (18) implies that $\gamma(x,t)$ and $\beta_1(x,t)$ jointly solve:

$$\gamma(x,t) + \frac{1}{2}\frac{\partial \beta_1(x,t)}{\partial x} + \int_0^x \frac{\partial \beta_1(y,t)}{\partial t} dy + \frac{\beta_1^2(x,t)}{2} = q_0(t) + q_1(t)x + q_2(t)\frac{x^2}{2}. \tag{39}$$

Hence, a specification of $\beta_1(x,t)$ and the three quadratic coefficients $q_0(t)$, $q_1(t)$, $q_2(t)$ determines $\gamma(x,t)$ explicitly. Conversely, given some specification for $q_1(t)$, $q_2(t)$, and $\gamma(x,t)$, differentiating (39) w.r.t. x implies that β_1 solves Burger's equation with an arbitrary forcing term:

$$\frac{1}{2}\frac{\partial^2 \beta_1(x,t)}{\partial x^2} + \frac{\partial \beta_1(x,t)}{\partial t} + \beta_1(x,t)\frac{\partial \beta_1(x,t)}{\partial x} = q_1(t) + q_2(t)x - \frac{\partial \gamma(x,t)}{\partial x}. \tag{40}$$

Recall that $\beta_1(x,t)$ and $\gamma(x,t)$ were defined in terms of the coefficients $a(S,t)$, $b(S,t)$, and $c(S,t)$ by (11) and (12), which are repeated here for convenience:

$$\beta_1(x,t) \equiv -\int_{S_0}^S \frac{1}{a^2(Z,t)}\frac{\partial a(Z,t)}{\partial t} dZ + \frac{b(S,t)}{a(S,t)} - \frac{1}{2}\frac{\partial a(S,t)}{\partial S}, \tag{41}$$

$$\gamma(x,t) \equiv c(S(x,t),t). \tag{42}$$

Differentiating (42) w.r.t. x implies:

$$\frac{\partial \gamma(x,t)}{\partial x} = \frac{\partial c(S,t)}{\partial S}\frac{\partial S(x,t)}{\partial x} = \frac{\partial c(S,t)}{\partial S}\frac{1}{\frac{\partial x(S,t)}{\partial S}} = \frac{\partial c(S,t)}{\partial S} a(S,t), \tag{43}$$

from (7).

By substituting (7), (41), and (43) in (40), Appendix 1 shows that if the quadratic condition (18) is to be arrived at by the maps (7) and (13), then the three coefficients $a(S,t)$, $b(S,t)$, and $c(S,t)$ must jointly solve:

$$\frac{\partial a(S,t)}{\partial S}\frac{\partial \ln a(S,t)}{\partial t} - \frac{\partial^2 a(S,t)}{\partial S \partial t} - \frac{\partial a(S,t)}{\partial S}\frac{\partial b(S,t)}{\partial S} + \frac{b(S,t)}{a(S,t)}\left(\frac{\partial a(S,t)}{\partial S}\right)^2 - \frac{a^2(S,t)}{4}\frac{\partial^3 a(S,t)}{\partial S^3}$$

$$
\begin{aligned}
&+\frac{1}{a(S,t)}\left[\frac{a^2(S,t)}{2}\frac{\partial^2 b(S,t)}{\partial S^2}+b(S,t)\frac{\partial b(S,t)}{\partial S}+\frac{\partial b(S,t)}{\partial t}\right]\\
&-\frac{2b(S,t)}{a^2(S,t)}\left[\frac{a^2(S,t)}{2}\frac{\partial^2 a(S,t)}{\partial S^2}+\frac{\partial a(S,t)}{\partial t}+\frac{b(S,t)}{2}\frac{\partial a(S,t)}{\partial S}\right]\\
&+\int_{S_0}^{S}\frac{2}{a^3(Z,t)}\left(\frac{\partial a(Z,t)}{\partial t}\right)^2 dZ-\int_{S_0}^{S}\frac{1}{a^2(Z,t)}\frac{\partial^2 a(Z,t)}{\partial t^2}dZ=q_1(t)+q_2(t)\int_{S_0}^{S}\frac{1}{a(Z,t)}dZ-\frac{\partial c(S,t)}{\partial S}a(S,t),
\end{aligned}
\tag{44}
$$

for $S \geq 0, t \in [0,T]$ and for any two functions $q_1(t)$ and $q_2(t)$[5]. Since transformation of (15) to the heat equation requires the quadratic condition (18), (44) is a necessary condition which the three coefficients $a(S,t)$, $b(S,t)$, and $c(S,t)$ must satisfy in order that the maps described in the last section transform the valuation PDE (3) into the heat equation.

The necessary condition on the coefficients is also sufficient for transforming the valuation PDE to the heat equation. To see this, recall that the maps (7) and (13) transform the valuation PDE (3) to the canonical PDE (15). Differentiating the definition (16) of $\gamma^c(x,t)$ w.r.t. x implies:

$$
\frac{\partial \gamma^c(x,t)}{\partial x}=\frac{\partial \gamma(x,t)}{\partial x}+\frac{1}{2}\frac{\partial^2 \beta_1(x,t)}{\partial x^2}+\frac{\partial \beta_1(x,t)}{\partial t}+\beta_1(x,t)\frac{\partial \beta_1(x,t)}{\partial x}. \tag{45}
$$

The various derivatives of $\beta_1(x,t)$ are computed in Appendix 1. Substituting (7), (43), and these derivatives into the necessary condition (44) imply that $\frac{\partial \gamma^c(x,t)}{\partial x}$ is linear in x:

$$
\frac{\partial \gamma^c(x,t)}{\partial x}=q_1(t)+q_2(t)x. \tag{46}
$$

Integation w.r.t. x then produces the quadratic condition (18) for $\gamma_c(x,t)$, which is sufficient to map the canonical PDE (15) to the heat equation.

Since (44) is both necessary and sufficient, we term it the *transformation condition*. Clearly, the equation is quite complicated to both analyze and solve. However, the next section discusses various sets of conditions which permit the construction of a triplet of coefficients $\{a(S,t), b(S,t), c(S,t)\}$ solving the transformation condition (44).

4 Constructing Coefficients

4.1 Specifying a Pair of Coefficients

The simplest approach for finding a triplet satisfying (44) is to specify $a(S,t)$ and $b(S,t)$, since (44) then simplifies into an equation for $\frac{\partial c(S,t)}{\partial S}$, which can be integrated to get $c(S,t)$. In contrast, if one specifies the pair $a(S,t)$ and $c(S,t)$, or the pair $b(S,t)$ and $c(S,t)$, then (44) is difficult to solve analytically for the remaining coefficient.

[5]Denoting the left hand side of (44) by $\mathcal{A}a(S,t)$ where $\mathcal{A}$ is an operator, then (44) can be rewritten as: $\frac{\partial}{\partial S}\left[a(S,t)\frac{\partial}{\partial S}\mathcal{A}a(S,t)\right]=0$. The veracity of this expression for a candidate triplet can be easily checked using a symbolic calculator.

4.2 Quadratic Claim Carrying Costs

This subsection presents an alternative approach for generating solution triplets. Substituting (42) in (39) implies that the coefficient $c(S,t)$ is given by:

$$c(S,t) = q_0(t) + q_1(t)x(S,t) + q_2(t)\frac{x^2(S,t)}{2} - \frac{1}{2}\frac{\partial \beta_1(x(S,t),t)}{\partial x} - \int_0^x \frac{\partial \beta_1(y,t)}{\partial t}dy - \beta_1^2(x(S,t),t), \quad (47)$$

where $x(S,t)$ depends on $a(S,t)$ as in (7). Substituting $x = x(S,t)$ in (41) implies that the coefficients $a(S,t)$ and $b(S,t)$ also depend on $\beta_1(x,t)$:

$$-\int_{S_0}^{S} \frac{1}{a^2(Z,t)}\frac{\partial a(Z,t)}{\partial t}dZ + \frac{b(S,t)}{a(S,t)} - \frac{1}{2}\frac{\partial a(S,t)}{\partial S} = \beta_1(x(S,t),t). \quad (48)$$

If we somehow know the form of $\beta_1(x,t)$, then we can treat (47) and (48) as a weakly coupled system of two PDE's in the 3 coefficients. In this subsection, we assume that the claim's carrying costs are quadratic in x and show that this assumption along with (18) are sufficient to derive the form of β_1. We then consider several alternative conditions supplementing (47) and (48) which allow the construction of coefficients solving the transformation condition.

To determine the form of $\beta_1(x,t)$, let:

$$R(x,t) \equiv \exp\left[\int_0^x \beta_1(y,t)dy\right], \quad x \in \Re, t \in [0,T] \quad (49)$$

be a change of dependent variable in (39). Then:

$$\beta_1(x,t) = \frac{\frac{\partial R}{\partial x}(x,t)}{R(x,t)} = \frac{\partial \ln R(x,t)}{\partial x}, \quad (50)$$

and substituting (50) in (39) implies that:

$$\frac{1}{2}\left[\frac{\frac{\partial^2 R(x,t)}{\partial x^2}}{R(x,t)} - \left(\frac{\frac{\partial R(x,t)}{\partial x}}{R(x,t)}\right)^2\right] + \frac{\partial \ln R(x,t)}{\partial t} + \frac{1}{2}\left(\frac{\frac{\partial R(x,t)}{\partial x}}{R(x,t)}\right)^2 = q_0(t) + q_1(t)x + q_2(t)\frac{x^2}{2} - \gamma(x,t).$$

Cancelling the nonlinear terms and multiplying by $R(x,t)$ implies that $R(x,t)$ solves the linear PDE:

$$\frac{1}{2}\frac{\partial^2 R}{\partial x^2}(x,t) + \frac{\partial R}{\partial t}(x,t) = \left[q_0(t) + q_1(t)x + q_2(t)\frac{x^2}{2} - \gamma(x,t)\right]R(x,t), \quad x \in \Re, t \in (0,T). \quad (51)$$

We can find a solution to this PDE if we specify a terminal condition and if we assume that $\gamma(x,t)$ is quadratic in x:

$$\gamma(x,t) = q_0^u(t) + q_1^u(t)x + q_2^u(t)\frac{x^2}{2}. \quad (52)$$

Substituting (52) in (51) implies that $R(x,t)$ solves a PDE in canonical form:

$$\frac{1}{2}\frac{\partial^2 R}{\partial x^2}(x,t) + \frac{\partial R}{\partial t}(x,t) = \left[q_0^r(t) + q_1^r(t)x + q_2^r(t)\frac{x^2}{2}\right]R(x,t), \quad x \in \Re, t \in (0,T), \quad (53)$$

where $q_i^r(t) \equiv q_i(t) - q_i^u(t), i = 0, 1, 2$. Thus, the assumptions (18) and (52) that the converted and unconverted contingent claim's carrying costs are quadratic in x implies that the target currency's carrying costs are also quadratic in x. Since the potential in (53) is quadratic, Appendix 3 shows that (53) can be mapped to the heat equation. The terminal condition is also known under the map, so $R(x,t)$ is known up to quadrature.

In Appendix 4, we further assume that $q_i^u(t) = q_i(t), i = 0, 1, 2$, so that from (52):

$$\gamma(x,t) = q_0(t) + q_1(t)x + q_2(t)\frac{x^2}{2}. \tag{54}$$

Hence in this special case, (51) implies that $R(x,t)$ solves the backward diffusion equation:

$$\frac{1}{2}\frac{\partial^2 R}{\partial x^2}(x,t) + \frac{\partial R}{\partial t}(x,t) = 0, \quad x \in \Re, t \in (0,T). \tag{55}$$

Once again, if we specify a terminal condition, then $R(x,t)$ is known up to quadrature.

Thus, under either assumption (52) or its special case of (54), one can solve the PDE (51) for $R(x,t)$. Once $R(x,t)$ is known, $\beta_1(x,t) \equiv \frac{\partial \ln R(x,t)}{\partial x}$ is known as well.

4.3 Given Volatility or Given Claim Carrying Cost

In this subsection, we assume that $a(S,t)$ is given and solve for $b(S,t)$ and $c(S,t)$. When $a(S,t)$ is given, then $x(S,t)$ is known by (7). Hence, $b(S,t)$ is given by (48) and $c(S,t)$ is given by (47).

Now suppose $a(S,t)$ is not given but $c(S,t)$ is given. From (43) and (52):

$$a(S,t)\frac{\partial c(S,t)}{\partial S} = q_1^u(t) + q_2^u(t)x(S,t). \tag{56}$$

Differentiating w.r.t. S:

$$\frac{\partial a(S,t)}{\partial S}\frac{\partial c(S,t)}{\partial S} + a(S,t)\frac{\partial^2 c(S,t)}{\partial S^2} = \frac{q_2^u(t)}{a(S,t)},$$

from (7). Multiplying by $a(S,t)$ implies that a^2 solves a 1st order linear inhomogeneous ODE in S:

$$\frac{\partial a^2(S,t)}{\partial S}\frac{1}{2}\frac{\partial c(S,t)}{\partial S} + a^2(S,t)\frac{\partial^2 c(S,t)}{\partial S^2} = q_2^u(t).$$

Multiplying both sides by an integrating factor $2\frac{\partial c(S,t)}{\partial S}$ gives:

$$\frac{\partial a^2(S,t)}{\partial S}\left(\frac{\partial c(S,t)}{\partial S}\right)^2 + a^2(S,t)2\frac{\partial c(S,t)}{\partial S}\frac{\partial^2 c(S,t)}{\partial S^2} = 2q_2^u(t)\frac{\partial c(S,t)}{\partial S}.$$

Integrating both sides w.r.t. S implies:

$$a^2(S,t)\left(\frac{\partial c(S,t)}{\partial S}\right)^2 = 2q_2^u(t)c(S,t) + k(t),$$

where $k(t)$ is an arbitrary function of time. Hence, the volatility is determined in terms of the given $c(S,t)$ as:

$$a(S,t) = \pm \frac{\sqrt{2q_2^u(t)c(S,t) + k(t)}}{\frac{\partial c(S,t)}{\partial S}}.$$

Once $a(S,t)$ is known, then $x(S,t)$ is known by (7) and hence, $b(S,t)$ is given by (48).

4.4 Other Approaches

We now suppose that neither $a(S,t)$ nor $c(S,t)$ is specified *ex ante*. If we are given the drift $b(S,t)$, then from the definition (11) of $\beta_1(x,t)$, $a(S,t)$ solves the PIDE:

$$-\int_{S_0}^{S} \frac{1}{a^2(Z,t)} \frac{\partial a(Z,t)}{\partial t} dZ + \frac{b(S,t)}{a(S,t)} - \frac{1}{2} \frac{\partial a(S,t)}{\partial S} = \beta_1(x,t). \tag{57}$$

Differentiating w.r.t. x:

$$\left[-\frac{1}{a^2(S,t)} \frac{\partial a(S,t)}{\partial t} + \frac{\frac{\partial b(S,t)}{\partial S}}{a(S,t)} - \frac{b(S,t)}{a^2(S,t)} \frac{\partial a(S,t)}{\partial S} - \frac{1}{2} \frac{\partial^2 a(S,t)}{\partial S^2} \right] \frac{\partial S(x,t)}{\partial x} = \frac{\partial \beta_1(x,t)}{\partial x}. \tag{58}$$

Now:

$$\frac{\partial S}{\partial x}(x,t) = \frac{1}{\frac{\partial x(S,t)}{\partial S}} = a(S(x,t),t), \quad x \in \Re, t \in [0,T], \tag{59}$$

from (7). Substituting (59) in (58) and multiplying by $-a(S,t)$ implies that when γ is quadratic in x, then $a(S,t)$ satisfies the fully nonlinear PDE:

$$\frac{a^2(S,t)}{2} \frac{\partial^2 a(S,t)}{\partial S^2} + b(S,t) \frac{\partial a(S,t)}{\partial S} + \frac{\partial a(S,t)}{\partial t} = \left[\frac{\partial b(S,t)}{\partial S} - \frac{\partial \beta_1(x(S,t),t)}{\partial x} \right] a(S,t), \tag{60}$$

where $x(S,t)$ is given by (7). We will not be able to solve this nonlinear PDE in complete generality. Rather we derive two sets of conditions on both a and b, either of which is sufficient to obtain a solution.

To motivate these sets of conditions, note that from the PDE (10) arising after the first change of independent variables, the risk-neutral process under Q of $X_t \equiv \int_{S_0}^{S_t} \frac{1}{a(Z,t)} dZ$ is:

$$dX_t = \beta_1(X_t,t)dt + dW_t, \quad t \in [0,T]. \tag{61}$$

Recall that the positivity of the unknown volatility function $a(S,t)$ ensures that the map from S to x in (7) is invertible, and thus the risk-neutral state variable process is given by $S_t = S(X_t,t)$ for some increasing function $S(x,t)$. Itô's Lemma implies that the dynamics of the state variable S_t can be expressed in terms of this function as:

$$dS_t = \left[\frac{1}{2} \frac{\partial^2 S}{\partial x^2}(X_t,t) + \beta_1(X_t,t) \frac{\partial S}{\partial x}(X_t,t) + \frac{\partial S}{\partial t}(X_t,t) \right] dt + \frac{\partial S}{\partial x}(X_t,t) dW_t, \quad t \in [0,T]. \tag{62}$$

However, the PDE (3) implies that the risk-neutral dynamics under Q of the state variable S is also given by:

$$dS_t = b(S_t, t)dt + a(S_t, t)dW_t, \quad t \in [0, T]. \tag{63}$$

Equating drifts in (62) and (63) implies that the state variable pricing function $S(x, t)$ satisfies the semi-linear PDE:

$$\frac{1}{2}\frac{\partial^2 S}{\partial x^2}(x, t) + \beta_1(x, t)\frac{\partial S(x, t)}{\partial x} + \frac{\partial S}{\partial t}(x, t) = b(S(x, t), t), \quad x \in \Re, t \in (0, T). \tag{64}$$

Equating diffusion coefficients in (62) and (63) gives an important constraint on the unknown volatility function:

$$a(S, t) = \frac{\partial S}{\partial x}(x(S, t), t), \quad S > 0, t \in [0, T], \tag{65}$$

where $x(S, t)$ is the inverse of $S(x, t)$. Thus, if one can determine $S(x, t)$ by solving (64), then one can determine the volatility function $a(S, t)$ from (65).

The next two subsections present two different approaches for solving the semi-linear PDE $S(x, t)$. The approaches differ in how one then determines $b(S, t)$ and $c(S, t)$. However, once $S(x, t)$ is determined, the approaches determine the claim value $U(x, t)$ in the same way. Substituting (52) in (10) implies that the derivative security value $U(x, t)$ solves:

$$\frac{1}{2}\frac{\partial^2 U}{\partial x^2}(x, t) + \beta_1(x, t)\frac{\partial U(x, t)}{\partial x} + \frac{\partial U}{\partial t}(x, t) = \left[q_0^u(t) + q_1^u(t)x + q_2^u(t)\frac{x^2}{2}\right] U(x, t), \quad x \in \Re, t \in (0, T). \tag{66}$$

Since $S(x, t)$ is now known for all $t \in [0, T]$, the terminal condition for $U(x, t)$ becomes:

$$U(x, T) = m(S(x, T)), \quad x \in \Re. \tag{67}$$

The map from (66) to the backward diffusion equation is given in (140) to (142) of Appendix 2, where $q_i(t) = q_i^u(t) + q_i^r(t), i = 0, 1, 2$. Using this map, the terminal condition at $\bar{\tau} \equiv \tau(T)$ becomes:

$$u(w, \bar{\tau}) = e^{F_0(T) + \int_0^{X(w,T)} [\beta_1(z,T) + \beta_2(z,T)]dz} m(S(X(w, T), T)), \tag{68}$$

where $F_0(t)$, $\beta_1(z, t)$ and $\beta_2(z, t)$ are defined in (121), (93), and (112) respectively, and where $X(w, T)$ is the spatial inverse of the function $w(x, t)$ defined in (141). Using the fundamental solution defined in (148), the solution of the backward diffusion equation subject to this terminal condition is:

$$u(w, \tau) = \int_{-\infty}^{\infty} p(w, \tau; k, \bar{\tau}) u(k, \bar{\tau}) dk. \tag{69}$$

The maps given in (140) to (142) must be inverted to determine U as a function of x and t. When this is done, the solution to the original PDE is given by:

$$V(S, t) = U(x(S, t), t), \quad S \in \Re, t \in [0, T], \tag{70}$$

where $x(S, t)$ is the inverse of $S(x, t)$.

The next two subsections now indicate two different methods for determining a solution to the semi-linear PDE for $S(x, t)$.

4.4.1 Quadratic Carrying Costs for S

Recall that the assumptions that the converted contingent claim and the unconverted claim both have carrying costs which are quadratic in x imply that the target currency also has has carrying costs which are quadratic in x. This subsection further assumes that the state variable S has quadratic carrying costs. Hence, the functions $R(x,t)$, $S(x,t)$, and $U(x,t)$ respectively solve:

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 R}{\partial x^2}(x,t) + \frac{\partial R}{\partial t}(x,t) &= \left[q_0^r(t) + q_1^r(t)x + q_2^r(t)\frac{x^2}{2}\right] R(x,t), \\
\frac{1}{2}\frac{\partial^2 S}{\partial x^2}(x,t) + \beta_1(x,t)\frac{\partial S(x,t)}{\partial x} + \frac{\partial S}{\partial t}(x,t) &= \left[q_0^s(t) + q_1^s(t)x + q_2^s(t)\frac{x^2}{2}\right] S(x,t), \\
\frac{1}{2}\frac{\partial^2 U}{\partial x^2}(x,t) + \beta_1(x,t)\frac{\partial U(x,t)}{\partial x} + \frac{\partial U}{\partial t}(x,t) &= \left[q_0^u(t) + q_1^u(t)x + q_2^u(t)\frac{x^2}{2}\right] U(x,t), \qquad (71)
\end{aligned}$$

where recall $\beta_1(x,t) = \frac{\frac{\partial R}{\partial x}(x,t)}{R(x,t)}$ and for each asset, the $q_i(\cdot)$, $i = 0,1,2$, are arbitrary functions of time. Appendix 3 shows how to construct solutions to all three PDE's.

Given that the three conditions in (71) all hold, Appendix 3 shows that the coefficients solving (44) can be expressed in terms of $S(x,t)$ and its inverse $x(S,t)$ as:

$$\begin{aligned}
a(S,t) &= \frac{\partial S}{\partial x}(x(S,t),t) \\
b(S,t) &= S\left[q_0^s(t) + q_1^s(t)x(S,t) + q_2^s(t)\frac{x^2(S,t)}{2}\right] \\
c(S,t) &= q_0^u(t) + q_1^u(t)x(S,t) + q_2^u(t)\frac{x^2(S,t)}{2}. \qquad (72)
\end{aligned}$$

We note that a unique solution to the three PDE's in (71) is obtained by specifying nine functions of time and three terminal conditions. The terminal condition for the claim is assumed to be specified *a priori*, so only two functions of the spatial variable can be specified freely. We note that the eleven functions can in principle be chosen so as to match nine term structures and two strike structures of option prices. Thus, there are enough degrees of freedom to potentially match the outer boundaries of a rectangular strike maturity domain, as well as seven interior term structures of option prices.

4.4.2 Specifying Measure Changes

Recall that we have only two PDE's with which to determine the three coefficients. The last subsection imposed quadratic carrying costs on the state variable as a way to determine the coefficients. In this subsection, we instead impose a structural relationship between the risk-neutral drift and the volatility. This allows us to weaken the requirement on the carrying cost of the state variable. However, we must also strengthen our assumption regarding the carrying cost of the target currency. In particular, Appendix 4 shows that an alternative solution approach is to instead require that the functions $R(x,t)$, $S(x,t)$, and $U(x,t)$ respectively solve:

$$\frac{1}{2}\frac{\partial^2 R}{\partial x^2}(x,t) + \frac{\partial R}{\partial t}(x,t) = 0,$$

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 S}{\partial x^2}(x,t) + \beta_1(x,t)\frac{\partial S(x,t)}{\partial x} + \frac{\partial S}{\partial t}(x,t) &= b(S(x,t),t), \\
\frac{1}{2}\frac{\partial^2 U}{\partial x^2}(x,t) + \beta_1(x,t)\frac{\partial U(x,t)}{\partial x} + \frac{\partial U}{\partial t}(x,t) &= \left[q_0(t) + q_1(t)x + q_2(t)\frac{x^2}{2}\right]U(x,t),
\end{aligned} \tag{73}$$

where again $\beta_1(x,t) = \frac{\frac{\partial R}{\partial x}(x,t)}{R(x,t)}$.

The assumed structural relationship between $a(S,t)$ and $b(S,t)$ is that there exists a positive function $h(S,t)$, monotonic in S, which solves:

$$\frac{a^2(S,t)}{2}\frac{\partial^2 h}{\partial S^2}(S,t) + b(S,t)\frac{\partial h}{\partial S}(S,t) + \frac{\partial h}{\partial t}(S,t) = 0. \tag{74}$$

Note that we are not assuming knowledge of a particular solution to the above PDE for given a and b. Rather, given that a and b satisfy the above constraint, we choose a positive monotonic h and find the a and b which satisfy it and the other structural constraints. Given the choice of some positive monotonic $h(S,t)$, Appendix 4 shows how to construct solutions to all three PDE's in (73). The solution to the semi-linear PDE for S is obtained by showing that $h(S,t) \equiv \frac{1}{R(x(S,t),t)}$ solves (74). Hence, once $R(x,t)$ is known and $h(S,t)$ is specified, $S(x,t)$ is simply $h^{-1}\left(\frac{1}{R(x,t)}, t\right)$. Once $S(x,t)$ is known, $a(S,t)$ is determined from (65), so that $b(S,t)$ is determined from (74). Appendix 4 also shows that $c(S,t)$ is simply given by:

$$c(S,t) = q_0(t) + q_1(t)x(S,t) + q_2(t)\frac{x^2(S,t)}{2}, \tag{75}$$

where again $x(S,t)$ is given by (7).

The imposition of any further structure on $h(S,t)$ imposes further structure on $b(S,t)$ and $a(S,t)$. For example, if we assume that $h(S,t)$ is a linear function of S, then substitution of this assumed form in (74) implies that $b(S,t)$ is linear in S and that $a(S,t)$ is arbitrary. For a second example, if h is assumed to be independent of time, then (74) becomes:

$$\frac{a^2(S,t)}{2}h''(S) + b(S,t)h'(S) = 0. \tag{76}$$

Hence, the further structure implies that the drift $b(S,t)$ is related to the volatility $a(S,t)$ by:

$$b(S,t) = g(S)a^2(S,t), \tag{77}$$

where:

$$g(S) \equiv -\frac{h''(S)}{2h'(S)} \tag{78}$$

is some given function. Hence from (77), the drift is an arbitrary nonlinear function of the spatial variable, but the time dependence is assumed to be captured by a^2. An important special case is when a is independent of time.

5 Derivative Security Valuation

Appendix 3 shows that the deterministic functions $q_i(t), i = 0, 1, 2$ appearing in (18) are given by:

$$q_i(t) = q_i^u(t) + q_i^r(t), \quad t \in [0, T],$$

where $q_i^u(t)$ and $q_i^r(t)$ are the coefficients of the quadratic functions in (71). In Appendix 4, the special case where $q_i^r(t) = 0$ is examined. Having determined the functions $q_i(t)$, this section supposes that one has also identified the corresponding functions $F_i(t), i = 0, 1, 2, 3$ given in (29) to (32), and a corresponding triplet $(a(S,t), b(S,t), c(S,t))$ consistent with the transformation condition (44). Appendix 2 shows that the map from the general parabolic PDE (3) to the backward diffusion equation (6) is explicitly given in terms of these quantities by:

$$v(w,\tau) \equiv e^{F_0(t)+\int_{S_0}^{S}\left[F_2(t)\int_{S_0}^{Y}\frac{1}{a(Z,t)}dZ+F_1(t)-\int_{S_0}^{Y}\frac{1}{a^2(Z,t)}\frac{\partial a(Z,t)}{\partial t}dZ+\frac{b(Y,t)}{a(Y,t)}-\frac{1}{2}\frac{\partial a(Y,t)}{\partial Y}\right]\frac{dY}{a(Y,t)}} V(S,t), \tag{79}$$

where:

$$w = \frac{1}{F_3(t)}\int_{S_0}^{S}\frac{1}{a(Z,t)}dZ + \int_0^t \frac{F_1(s)}{F_3(s)}ds + c_w \equiv W(S,t), \tag{80}$$

with c_w an arbitrary constant and:

$$\tau = \int_0^t \frac{1}{F_3^2(s)}ds. \tag{81}$$

We note that the *form* of the map does not depend on the form of $c(S,t)$, although the map's existence requires that $c(S,t)$ be consistent with the transformation condition. As mentioned previously, it is straightforward to further map (6) into the heat equation.

In our new variables, the original payoff (4) maps to:

$$v(w,\bar{\tau}) = e^{F_0(T)+\int_{S_0}^{s(w,T)}\left[F_2(T)\int_{S_0}^{Y}\frac{1}{a(Z,T)}dZ+F_1(t)-\int_{S_0}^{Y}\frac{1}{a^2(Z,T)}\frac{\partial a(Z,T)}{\partial t}dZ+\frac{b(Y,T)}{a(Y,T)}-\frac{1}{2}\frac{\partial a(Y,T)}{\partial Y}\right]\frac{dY}{a(Y,T)}} m(s(w,T)), \tag{82}$$

where $\bar{\tau} \equiv \tau(T)$ and $s(w,T)$ is the spatial inverse of the function $W(S,t)$ defined in (80) when evaluated at $t = T$. Letting:

$$p(w,\tau;k,\bar{\tau}) \equiv \frac{1}{\sqrt{2\pi[\bar{\tau}-\tau]}} \exp\left\{-\frac{1}{2}\left[\frac{k-w}{\sqrt{\bar{\tau}-\tau}}\right]^2\right\} \tag{83}$$

denote the fundamental solution to the backward diffusion equation (6), the solution of (6) subject to (82) is expressible as:

$$u(w,\tau) = \int_{-\infty}^{\infty} v(k,\bar{\tau})p(w,\tau;k,\bar{\tau})dk, \quad w \in \Re, \tau \in [0,\bar{\tau}],$$

when the integral exists. The maps in (79) to (81) must be inverted to express V in terms of S and t. The next section presents some examples.

6 Examples

This section illustrates the various methods for constructing solutions to the transformation condition. Our first example illustrates the technique of specifying a pair of coefficients and solving for the third. Suppose one is interested in extending the constant volatility Black model for equity derivatives to account for the possibility of default by the writer of the derivative security. We assume that default occurs at the first jump time of a standard Poisson process. In the event of default occuring prior to maturity, the value of the derivative security drops to some fraction of its pre-default value. This recovery value is paid to the owner at the default time. It is well known that the effect of adding this kind of default is to lower the initial claim value and hence increase the proportional rate at which the claim value approaches its stochastic final payout prior to any default. In particular, the claim's proportional carrying cost is the sum of the instantaneous riskfree rate and the credit spread, where the latter is the product of the instantaneous arrival rate of default and the fraction lost in the event of default. We suppose that the instantaneous riskfree rate is deterministic. We assume that the instantaneous credit spread at t depends on only the contemporaneous forward price and time. We now wish to find the most general form for the instantaneous credit spread which retains the property of the standard Black model that the governing PDE can be mapped to the heat equation. Setting $a(S,t) = \sigma S$ and $b(S,t) = 0$ in the transformation condition (44) implies:

$$0 = q_1(t) + q_2(t) \int_{S_0}^{S} \frac{1}{\sigma Z} dZ - \frac{\partial c(S,t)}{\partial S} \sigma S. \tag{84}$$

The integral evaluates to $\frac{\ln(S/S_0)}{\sigma}$. Let:

$$\gamma(x,t) \equiv c(S,t), \tag{85}$$

be the dependent variable resulting from letting $x = \frac{\ln(S/S_0)}{\sigma}$ be a change of independent variable in the ODE (84). Then:

$$\frac{\partial \gamma(x,t)}{\partial x} = q_1(t) + q_2(t)x. \tag{86}$$

Integrating w.r.t. x and substituting (85) implies that:

$$c(S,t) = r(t) + q_0(t) + q_1(t)\frac{\ln(S/S_0)}{\sigma} + q_2(t)\left[\frac{\ln(S/S_0)}{\sigma}\right]^2. \tag{87}$$

Thus, if the credit spread is modelled as quadratic in the log of the forward price, then one can value the credit risky derivative security in closed form by mapping the valuation PDE to the heat equation. It is economically plausible that the forward price should be nonnegative and unbounded above and that the credit spread should be positive and decreasing in the forward price. This is accomodated by setting $q_0(t) > 0$, $q_1(t) < 0$, and $q_2(t) = 0$. If one is willing to bound the forward price above, then one can let $q_2(t) \neq 0$ and retain positive decreasing credit spreads on the allowed domain.

7 Implications for Numerical Methods

The present analysis also has implications for numerical work. If the coefficients only satisfy the transformation condition (44) in some region, then our change of variables can still be used in this region to complement a numerical anlysis. For example, if the the transformation condition only holds for the last month of a contract, then the analytic solution can be used as a terminal condition for a finite difference scheme. If the necessary condition holds globally, but intermittent boundary conditions (eg. discrete barrier monitoring or Bermudan exercise) preclude the recovery of an explicit solution, then one can still discretize time and/or space to solve the heat equation in the appropriate domain. Finally, Monte Carlo simulation can be enhanced by simulating standard Brownian motion rather than a complicated state variable process.

UNFINISHED

8 Summary and Extensions

Assuming a general form for the linear parabolic PDE governing many derivative security values, we derived an expression which the three coefficients must satisfy in order that this PDE can be transformed into the heat equation. We presented a technique for generating solutions to this expression thereby exhibiting a method for generating closed form solutions for derivative security prices. We illustrated our results with several examples.

Further extensions to this analysis would include developing explicit pricing formulas for *path-dependent* options such as American, Bermudan, compound, or barrier options. One can also develop the multivariate version of our results, or explore transforming PIDE's or non-linear PDE's to the heat equation. Finally, one can also explore transforming these equations to other well known PDE's. In the interests of brevity, these extensions are left for future research.

References

[1] Black, F. and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities", *The Journal of Political Economy*, **81**, 637–659.

[2] Bluman, G., 1980, "On the Transformation of Diffusion Processes into the Wiener Process", *Siam Journal of Applied Mathematics*, **39**, 2, 238–47.

[3] Bluman, G., 1980, "On Mapping Linear Partial Differential Equations to Constant Coefficient Equations", *Siam Journal of Applied Mathematics*, **43**, 6, 1259–73.

[4] Bouchouev, I. and V. Isakov, 1999, "Uniqueness, stability and numerical methods for the inverse problem that arises in financial markets", *Inverse Problems*, **15**, forthcoming.

[5] Carr, P., M. Tari, and T. Zariphopolou, 1999, "Closed Form Option Valuation with Smiles," Banc of America Securities working paper.

[6] Cherkasov, I. D., 1957, "On the Transformation of the Diffusion Process to a Wiener Process", *Theor. Probability Appl.*, **2**, 373-77.

[7] Cox, J., 1975, "Notes on Option Pricing I: Constant Elasticity of Variance Diffusions", Stanford University working paper.

[8] Cox, D. and H. Miller, 1965, "The Theory of Stochastic Processes", Chapman and Hall, New York.

[9] Cox, J. and S. Ross, 1976, "The Valuation of Options for Alternative Stochastic Processes", *Journal of Financial Economics*, **3**, 145-166.

[10] Dupire, B., 1994, "Pricing with a Smile", *Risk*, **7**, 1, 18-20.

[11] Gihman I. and A. Skorohod, 1972, Stochastic Differential Equations, Springer-Verlag, Berlin.

[12] Goldenberg D., 1991, "A unified method for pricing options on diffusion processes", *Journal of Financial Economics*, **29**, 3–34.

[13] Ingersoll J. 1996, "Valuing Foreign Exchange Options with a Bounded Exchange Rate Process," *Review of Derivatives Research*, **1**, 2, 159-181.

[14] Jamshidian, F., 1991, "Forward Induction, and Construction of Yield Curve Diffusion Models", *Journal of Fixed Income*, June, 62-74.

[15] Li, A., 1998, "A One Factor Volatility Smile Model with Closed Form Solutions for European Options", *European Financial Management*, forthcoming.

[16] Linetsky, V. and D. Davidov, 1999, "Pricing Options on One Dimensional Diffusions: A Unified Approach", University of Michigan, working paper.

[17] Lipton, A., 1999, On transformations of a general diffusion into a Wiener process", Deutsche Bank Working Paper.

[18] Merton, R., 1973, "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science*, **4**, 141–183.

[19] Merton, R., 1974, "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates", *Journal of Finance*, **29**: 449–70.

[20] Nelson, D. B. and K. Ramaswamy, 1990, "Simple Binomial Processes As Diffusion Approximations in Financial Models," *Review of Financial Studies*, 1990, **3**,3, 393-430.

[21] Rady, S., "Option Pricing in the Presence of Natural Boundaries and a Quadratic Diffusion Term," *Finance and Stochastics*, **1**, 4, 331-344.

[22] Ricciardi, L. , 1976, "On the Transformation of Diffusion Processes into the Wiener Process", *J. Math. Anal. Appl.*, **54**, 185-199.

[23] Schmidt, W., 1997, "On a General Class of One factor Models for the Term Structure of Interest Rates", *Finance and Stochastics*, **1**, 3–24.

[24] Schroder, M., 1989, "Computing the Constant Elasticity of Variance Option Pricing Formula", *Journal of Finance*, **44**, 211-219.

Appendix 1: Derivation of the Transformation Condition

In this appendix, we will change the dependent variable V and the independent spatial variable S in order to transform the PDE (3):

$$\frac{a^2(S,t)}{2}\frac{\partial^2 V}{\partial S^2}(S,t) + b(S,t)\frac{\partial V}{\partial S}(S,t) + \frac{\partial V}{\partial t}(S,t) = c(S,t)V(S,t), \quad S > 0, t \in (0,T), \tag{88}$$

into the following diffusion equation with a potential term:

$$\frac{1}{2}\frac{\partial^2 U^c}{\partial x^2}(x,t) + \frac{\partial U^c}{\partial t}(x,t) = \gamma^c(x,t)U^c(x,t), \quad x \in \Re, t \in (0,T). \tag{89}$$

We then consider the effects of a deterministic time change and show that a necessary and sufficient condition for (89) to be transformable to the heat equation under these maps is that $\gamma^c(x,t)$ must be a quadratic function of x with arbitrary time-dependent coefficients. Finally, we use this result in order to derive a restriction on the coefficients $a(S,t)$, $b(S,t)$, and $c(S,t)$.

Let:

$$U(x,t) \equiv V(S,t) \text{ where } x(S,t) \equiv \int_{S_0}^{S} \frac{1}{a(Z,t)} dZ \tag{90}$$

is our new independent variable. Differentiating V in (90) w.r.t. S implies:

$$\frac{\partial V(S,t)}{\partial S} = \frac{\partial U(x,t)}{\partial x}\frac{1}{a(S,t)}.$$

Differentiating w.r.t. S again implies:

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{\partial^2 U(x,t)}{\partial x^2}\frac{1}{a^2(S,t)} - \frac{\partial U(x,t)}{\partial x}\frac{1}{a^2(S,t)}\frac{\partial a(S,t)}{\partial S}.$$

Differentiating V in (90) w.r.t. t implies:

$$\frac{\partial V(S,t)}{\partial t} = -\frac{\partial U(x,t)}{\partial x}\int_{S_0}^{S} \frac{1}{a^2(Z,t)}\frac{\partial a(Z,t)}{\partial t} dZ + \frac{\partial U(x,t)}{\partial t}. \tag{91}$$

Substituting (90) to (91) in (88) yields:

$$\frac{1}{2}\frac{\partial^2 U}{\partial x^2}(x,t) + \beta_1(x,t)\frac{\partial U(x,t)}{\partial x} + \frac{\partial U}{\partial t}(x,t) = \gamma(x,t)U(x,t), \quad x \in \Re, t \in (0,T), \tag{92}$$

where $\gamma(x,t) \equiv c(S,t)$ is the proportional cost of carrying the overlying as a function of x, and:

$$\beta_1(x,t) \equiv -\int_{S_0}^{S} \frac{1}{a^2(Z,t)}\frac{\partial a(Z,t)}{\partial t} dZ + \frac{b(S,t)}{a(S,t)} - \frac{1}{2}\frac{\partial a(S,t)}{\partial S}. \tag{93}$$

Consider the new dependent variable:

$$U^c(x,t) \equiv e^{\int_0^x \beta_1(y,t)dy} U(x,t). \tag{94}$$

Then:

$$U(x,t) = e^{-\int_0^x \beta_1(y,t)dy} U^c(x,t). \tag{95}$$

Differentiating (95) w.r.t. x implies:

$$\frac{\partial U(x,t)}{\partial x} = e^{-\int_0^x \beta_1(y,t)dy}\left[-\beta_1(x,t)U^c(x,t) + \frac{\partial U^c(x,t)}{\partial x}\right].$$

Differentiating w.r.t. x again:

$$\frac{\partial^2 U(x,t)}{\partial x^2} = e^{-\int_0^x \beta_1(y,t)dy}\left[\beta_1^2(x,t)U^c(x,t) - \frac{\partial \beta_1(x,t)}{\partial x}U^c(x,t) - 2\beta_1(x,t)\frac{\partial U^c(x,t)}{\partial x} + \frac{\partial^2 U^c(x,t)}{\partial x^2}\right].$$

Differentiating (95) w.r.t. t implies:

$$\frac{\partial U(x,t)}{\partial t} = e^{-\int_0^x \beta_1(y,t)dy}\left[-\int_0^x \frac{\partial \beta_1(y,t)}{\partial t}dy U^c(x,t) + \frac{\partial U^c(x,t)}{\partial t}\right]. \tag{96}$$

Substituting (95) to (96) in (92) and dividing out the exponential term yields:

$$\frac{1}{2}\frac{\partial^2 U^c}{\partial x^2}(x,t) + \frac{\partial U^c}{\partial t}(x,t) = \gamma^c(x,t)U^c(x,t), \quad x \in \Re, t \in (0,T), \tag{97}$$

where:

$$\gamma^c(x,t) \equiv \gamma(x,t) + \frac{1}{2}\frac{\partial \beta_1(x,t)}{\partial x} + \int_0^x \frac{\partial \beta_1(y,t)}{\partial t}dy + \frac{\beta_1^2(x,t)}{2}. \tag{98}$$

Finally, consider the new dependent and independent variables:

$$u(w,\tau) \equiv e^{F_0(t)+\int_0^x \beta_2(z,t)dz} U^c(x,t), \tag{99}$$

where:

$$w \equiv w(x,t) \text{ and } \tau \equiv \tau(x,t).$$

Then:

$$U^c(x,t) = e^{-F_0(t)-\int_0^x \beta_2(z,t)dz} u(w,\tau). \tag{100}$$

Differentiating (100) w.r.t. x implies:

$$\frac{\partial U^c(x,t)}{\partial x} = e^{-F_0(t)-\int_0^x \beta_2(z,t)dz}\left[-\beta_2(x,t)u(w,\tau) + \frac{\partial u(w,\tau)}{\partial w}\frac{\partial w(x,t)}{\partial x} + \frac{\partial u(w,\tau)}{\partial \tau}\frac{\partial \tau(x,t)}{\partial x}\right].$$

Differentiating w.r.t. x again:

$$\begin{aligned}\frac{\partial^2 U^c(x,t)}{\partial x^2} \quad = \quad & e^{-F_0(t)-\int_0^x \beta_2(z,t)dz}\left[\beta_2^2(x,t)u(w,\tau) - \frac{\partial \beta_2(x,t)}{\partial x}u(w,\tau) + 2\frac{\partial^2 u(w,\tau)}{\partial w \partial \tau}\frac{\partial w(x,t)}{\partial x}\frac{\partial \tau(x,t)}{\partial x}\right. \\ & -2\beta_2(x,t)\frac{\partial u(w,\tau)}{\partial w}\frac{\partial w(x,t)}{\partial x} + \frac{\partial^2 u(w,\tau)}{\partial w^2}\left(\frac{\partial w(x,t)}{\partial x}\right)^2 + \frac{\partial u(w,\tau)}{\partial w}\frac{\partial^2 w(x,t)}{\partial x^2} \\ & \left. -2\beta_2(x,t)\frac{\partial u(w,\tau)}{\partial \tau}\frac{\partial \tau(x,t)}{\partial x} + \frac{\partial^2 u(w,\tau)}{\partial \tau^2}\left(\frac{\partial \tau(x,t)}{\partial x}\right)^2 + \frac{\partial u(w,\tau)}{\partial \tau}\frac{\partial^2 \tau(x,t)}{\partial x^2}\right].\end{aligned}$$

Differentiating (100) w.r.t. t implies:

$$\frac{\partial U^c(x,t)}{\partial t} = e^{-F_0(t)-\int_0^x \beta_2(z,t)dz}\left[-F_0'(t) - \int_0^x \frac{\partial \beta_2(z,t)}{\partial t}dz u(w,\tau) + \frac{\partial u(w,\tau)}{\partial w}\frac{\partial w(x,t)}{\partial t} + \frac{\partial u(w,\tau)}{\partial \tau}\frac{\partial \tau(x,t)}{\partial t}.\right] \tag{101}$$

Substituting (100) to (101) in (97) and dividing out $e^{-F_0(t)-\int_0^x \beta_2(z,t)dz}$ yields:

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial w(x,t)}{\partial x}\right)^2 \frac{\partial^2 u(w,\tau)}{\partial w^2} + \frac{\partial w(x,t)}{\partial x}\frac{\partial \tau(x,t)}{\partial x}\frac{\partial^2 u(w,\tau)}{\partial w \partial \tau} + \frac{1}{2}\left(\frac{\partial \tau(x,t)}{\partial x}\right)^2 \frac{\partial^2 u(w,\tau)}{\partial \tau^2} \\
&\quad + \left[\mathcal{L}w(x,t) - \beta_2(x,t)\frac{\partial w(x,t)}{\partial x}\right]\frac{\partial u(w,\tau)}{\partial w} + \left[\mathcal{L}\tau(x,t) - \beta_2(x,t)\frac{\partial \tau(x,t)}{\partial x}\right]\frac{\partial u(w,\tau)}{\partial \tau} \\
= &\left[\gamma^c(x,t) + F_0'(t) + \frac{1}{2}\frac{\partial \beta_2(x,t)}{\partial x} + \int_0^x \frac{\partial \beta_2(z,t)}{\partial t}dz - \frac{\beta_2^2(x,t)}{2}\right]u(w,\tau),
\end{aligned} \tag{102}$$

where recall $\mathcal{L} \equiv \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}$.

In order to obtain the backward diffusion equation (6) from (102), we must have:

$$\frac{\partial \tau(x,t)}{\partial x} = 0, \tag{103}$$

which implies that the time change is deterministic:

$$\tau(x,t) \equiv \tau(t). \tag{104}$$

We must also have that:

$$\left(\frac{\partial w(x,t)}{\partial x}\right)^2 = \mathcal{L}\tau(x,t) = \tau'(t), \tag{105}$$

$$\mathcal{L}w(x,t) - \beta_2(x,t)\frac{\partial w(x,t)}{\partial x} = 0, \tag{106}$$

and:

$$\gamma^c(x,t) = -F_0(t) - \frac{1}{2}\frac{\partial \beta_2(x,t)}{\partial x} - \int_0^x \frac{\partial \beta_2(z,t)}{\partial t}dz + \frac{\beta_2^2(x,t)}{2}. \tag{107}$$

Equation (105) implies that:

$$\frac{\partial w(x,t)}{\partial x} = \sqrt{\tau'(t)} = e^{\int_0^t F_2(s)ds}, \tag{108}$$

where:

$$F_2(t) \equiv \frac{\tau''(t)}{2\tau'(t)}. \tag{109}$$

Integrating (108) implies that the new spatial variable is linear in x:

$$w(x,t) = w_0(t) + e^{\int_0^t F_2(s)ds}x, \tag{110}$$

where $w_0(t)$ is an arbitrary function of time. Let $F_3(t) \equiv e^{-\int_0^t F_2(s)ds} = \frac{1}{\sqrt{\tau'(t)}}$, so that (108) becomes:

$$\frac{\partial w(x,t)}{\partial x} = \frac{1}{F_3(t)}. \tag{111}$$

Substituting (110) and (111) in (106) implies that:

$$\beta_2(x,t) \equiv F_3(t)\mathcal{L}\left[w_0(t) + e^{\int_0^t F_2(s)ds}x\right] = F_3(t)w_0'(t) + F_3(t)F_2(t)e^{\int_0^t F_2(s)ds}x = F_1(t) + F_2(t)x, \tag{112}$$

where:

$$F_1(t) \equiv F_3(t)w_0'(t). \tag{113}$$

Substituting (112) in (107) and simplifying implies that $\gamma^c(x,t)$ is quadratic in x:

$$\gamma^c(x,t) = \left[F_2^2(t) - F_2'(t)\right]\frac{x^2}{2} + \left[F_2(t)F_1(t) - F_1'(t)\right]x + \left[-\frac{F_2(t)}{2} + \frac{F_1^2(t)}{2} - F_0'(t)\right]. \tag{114}$$

Suppose $\gamma^c(x,t)$ is specified as:

$$\gamma^c(x,t) = q_2(t)\frac{x^2}{2} + q_1(t)x + q_0(t), \tag{115}$$

where $q_2(\cdot), q_1(\cdot)$, and $q_0(\cdot)$ are arbitrary functions of time. Then equating (114) with (115) and comparing the coefficients of x^2 implies that the function $F_2(t)$ can be determined by solving the Riccati ODE:

$$F_2'(t) - F_2^2(t) + q_2(t) = 0, \quad t \in [0,T]. \tag{116}$$

Equivalently, $F_3(t) \equiv e^{-\int_0^t F_2(s)ds}$ solves the second order linear ODE:

$$F_3''(t) - q_2(t)F_3(t) = 0, \quad t \in [0,T]. \tag{117}$$

Given a solution for $F_3(t)$, its definition implies that $F_2(t)$ is given by:

$$F_2(t) = -\frac{d\ln F_3(t)}{dt}. \tag{118}$$

Comparing the coefficients of x in (114) and (115) implies that the function $F_1(t)$ solves the first order linear ODE:

$$F_1'(t) + \frac{d\ln F_3(t)}{dt}F_1(t) + q_1(t) = 0, \quad t \in [0,T], \tag{119}$$

whose general solution is:

$$F_1(t) = \frac{F_1(0) - \int_0^t F_3(s)q_1(s)ds}{F_3(t)}, \quad t \in [0,T], \tag{120}$$

where $F_1(0)$ is an arbitrary constant. Finally, comparing the constant terms in (114) and (115) implies:

$$F_0(t) = \int_0^t \left[-\frac{F_2(s)}{2} + \frac{F_1^2(s)}{2} - q_0(s) \right] ds, \quad t \in [0,T]. \tag{121}$$

Equating the RHS's of (98) and (18) gives:

$$\gamma(x,t) + \frac{1}{2}\frac{\partial \beta_1(x,t)}{\partial x} + \int_0^x \frac{\partial \beta_1(y,t)}{\partial t} dy + \frac{\beta_1^2(x,t)}{2} = q_0(t) + q_1(t)x + q_2(t)\frac{x^2}{2}. \tag{122}$$

Differentiating (122) w.r.t. x implies that β_1 solves Burger's equation with an arbitrary forcing term:

$$\frac{1}{2}\frac{\partial^2 \beta_1(x,t)}{\partial x^2} + \frac{\partial \beta_1(x,t)}{\partial t} + \beta_1(x,t)\frac{\partial \beta_1(x,t)}{\partial x} = q_1(t) + q_2(t)x - \frac{\partial \gamma(x,t)}{\partial x}. \tag{123}$$

Recall from (93) that $\beta_1(x,t)$ is given by:

$$\beta_1(x,t) \equiv -\int_{S_0}^S \frac{1}{a^2(Z,t)}\frac{\partial a(Z,t)}{\partial t} dZ + \frac{b(S,t)}{a(S,t)} - \frac{1}{2}\frac{\partial a(S,t)}{\partial S}, \tag{124}$$

where $S = s(x,t)$ is the inverse map of $x(S,t)$ defined in (90). Noting that $\frac{\partial S}{\partial x} = \frac{1}{\frac{\partial x}{\partial S}} = a(S,t)$ from (90), differentiating (124) w.r.t. x implies:

$$\begin{aligned}
\frac{\partial \beta_1(x,t)}{\partial x} &= \left[-\frac{1}{a^2(S,t)}\frac{\partial a(S,t)}{\partial t} + \frac{1}{a(S,t)}\frac{\partial b(S,t)}{\partial S} - \frac{b(S,t)}{a^2(S,t)}\frac{\partial a(S,t)}{\partial S} - \frac{1}{2}\frac{\partial^2 a(S,t)}{\partial S^2} \right] a(S,t) \\
&= -\frac{\partial \ln a(S,t)}{\partial t} + \frac{\partial b(S,t)}{\partial S} - b(S,t)\frac{\partial \ln a(S,t)}{\partial S} - \frac{a(S,t)}{2}\frac{\partial^2 a(S,t)}{\partial S^2}.
\end{aligned} \tag{125}$$

Differentiating (124) w.r.t. t implies:

$$\begin{aligned}
\frac{\partial \beta_1(x,t)}{\partial t} &= \left[-\frac{1}{a^2(S,t)}\frac{\partial a(S,t)}{\partial t} + \frac{1}{a(S,t)}\frac{\partial b(S,t)}{\partial S} - \frac{b(S,t)}{a^2(S,t)}\frac{\partial a(S,t)}{\partial S} - \frac{1}{2}\frac{\partial^2 a(S,t)}{\partial S^2} \right] \frac{\partial S(x,t)}{\partial t} \\
&\quad + \int_{S_0}^S \frac{2}{a^3(Z,t)} \left(\frac{\partial a(Z,t)}{\partial t} \right)^2 dZ - \int_{S_0}^S \frac{1}{a^2(Z,t)}\frac{\partial^2 a(Z,t)}{\partial t^2} dZ \\
&\quad + \frac{1}{a(S,t)}\frac{\partial b(S,t)}{\partial t} - \frac{b(S,t)}{a^2(S,t)}\frac{\partial a(S,t)}{\partial t} - \frac{1}{2}\frac{\partial^2 a(S,t)}{\partial S \partial t} \\
&= \frac{\partial \beta_1(x,t)}{\partial x}\frac{\frac{\partial S(x,t)}{\partial t}}{a(S,t)} + \ell(S,t),
\end{aligned} \tag{126}$$

where $S(x,t)$ is the inverse of the map $x(S,t)$ defined in (90) and:

$$\ell(S,t) \equiv \int_{S_0}^{S} \frac{2}{a^3(Z,t)} \left(\frac{\partial a(Z,t)}{\partial t} \right)^2 dZ - \int_{S_0}^{S} \frac{1}{a^2(Z,t)}\frac{\partial^2 a(Z,t)}{\partial t^2} dZ + \frac{1}{a(S,t)}\frac{\partial b(S,t)}{\partial t} - \frac{b(S,t)}{a^2(S,t)}\frac{\partial a(S,t)}{\partial t} - \frac{1}{2}\frac{\partial^2 a(S,t)}{\partial S \partial t}. \tag{127}$$

Hence, from (124) and (126), $\frac{\partial \beta_1(x,t)}{\partial t} + \beta_1(x,t)\frac{\partial \beta_1(x,t)}{\partial x} =$

$$\frac{\partial \beta_1(x,t)}{\partial x}\left[\frac{\frac{\partial S(x,t)}{\partial t}}{a(S,t)} - \int_{S_0}^{S} \frac{1}{a^2(Z,t)}\frac{\partial a(Z,t)}{\partial t}dZ + \frac{b(S,t)}{a(S,t)} - \frac{1}{2}\frac{\partial a(S,t)}{\partial S}\right] + \ell(S,t). \tag{128}$$

From (7), $S(x,t)$ is defined implicitly by:

$$x = \int_{S_0}^{S(x,t)} \frac{1}{a(Z,t)}dZ. \tag{129}$$

Differentiating w.r.t. t while holding x constant gives:

$$0 = \frac{1}{a(S(x,t),t)}\frac{\partial S(x,t)}{\partial t} - \int_{S_0}^{S(x,t)} \frac{1}{a^2(Z,t)}\frac{\partial a(Z,t)}{\partial t}dZ. \tag{130}$$

Hence, as $S = S(x,t)$:

$$\frac{\frac{\partial S(x,t)}{\partial t}}{a(S,t)} = \int_{S_0}^{S} \frac{1}{a^2(Z,t)}\frac{\partial a(Z,t)}{\partial t}dZ. \tag{131}$$

Substituting (131) in (128) implies $\frac{\partial \beta_1(x,t)}{\partial t} + \beta_1(x,t)\frac{\partial \beta_1(x,t)}{\partial x}$:

$$\begin{aligned} &= \frac{\partial \beta_1(x,t)}{\partial x}\left[\frac{b(S,t)}{a(S,t)} - \frac{1}{2}\frac{\partial a(S,t)}{\partial S}\right] + \ell(S,t) \qquad (132)\\ &= \left[-\frac{\partial \ln a(S,t)}{\partial t} + \frac{\partial b(S,t)}{\partial S} - b(S,t)\frac{\partial \ln a(S,t)}{\partial S} - \frac{a(S,t)}{2}\frac{\partial^2 a(S,t)}{\partial S^2}\right]\left[\frac{b(S,t)}{a(S,t)} - \frac{1}{2}\frac{\partial a(S,t)}{\partial S}\right] + \ell(S,t), \end{aligned}$$

from (125). Differentiating (125) w.r.t. x implies:

$$\begin{aligned} \frac{\partial^2 \beta_1(x,t)}{\partial x^2} = &\left\{-\frac{\partial^2 \ln a(S,t)}{\partial S \partial t} + \frac{\partial^2 b(S,t)}{\partial S^2} - \frac{\partial b(S,t)}{\partial S}\frac{\partial \ln a(S,t)}{\partial S} - b(S,t)\frac{\partial^2 \ln a(S,t)}{\partial S^2}\right.\\ &\left. -\frac{1}{2}\frac{\partial a(S,t)}{\partial S}\frac{\partial^2 a(S,t)}{\partial S^2} - \frac{a(S,t)}{2}\frac{\partial^3 a(S,t)}{\partial S^3}\right\} a(S,t). \end{aligned} \tag{133}$$

Substituting (132) and (133) in (123) implies:

$$\begin{aligned} &\frac{1}{2}\left\{-\frac{\partial^2 \ln a(S,t)}{\partial S \partial t} + \frac{\partial^2 b(S,t)}{\partial S^2} - \frac{\partial b(S,t)}{\partial S}\frac{\partial \ln a(S,t)}{\partial S} - b(S,t)\frac{\partial^2 \ln a(S,t)}{\partial S^2}\right.\\ &\left.-\frac{1}{2}\frac{\partial a(S,t)}{\partial S}\frac{\partial^2 a(S,t)}{\partial S^2} - \frac{a(S,t)}{2}\frac{\partial^3 a(S,t)}{\partial S^3}\right\} a(S,t)\\ &+\left[-\frac{\partial \ln a(S,t)}{\partial t} + \frac{\partial b(S,t)}{\partial S} - b(S,t)\frac{\partial \ln a(S,t)}{\partial S} - \frac{a(S,t)}{2}\frac{\partial^2 a(S,t)}{\partial S^2}\right]\left[\frac{b(S,t)}{a(S,t)} - \frac{1}{2}\frac{\partial a(S,t)}{\partial S}\right]\\ &+\ell(S,t) = q_1(t) + q_2(t)x - \frac{\partial \gamma(x,t)}{\partial x}. \end{aligned} \tag{134}$$

Now:

$$\frac{\partial^2 \ln a(S,t)}{\partial S \partial t} = \frac{\partial}{\partial S} \frac{1}{a(S,t)} \frac{\partial a(S,t)}{\partial t} = -\frac{1}{a^2(S,t)} \frac{\partial a(S,t)}{\partial S} \frac{\partial a(S,t)}{\partial t} + \frac{1}{a(S,t)} \frac{\partial^2 a(S,t)}{\partial S \partial t}. \tag{135}$$

Also:

$$\frac{\partial^2 \ln a(S,t)}{\partial S^2} = \frac{\partial}{\partial S} \frac{1}{a(S,t)} \frac{\partial a(S,t)}{\partial S} = -\frac{1}{a^2(S,t)} \left(\frac{\partial a(S,t)}{\partial S} \right)^2 + \frac{1}{a(S,t)} \frac{\partial^2 a(S,t)}{\partial S^2}. \tag{136}$$

Substituting (127), (135), and (136) in (134) implies:

$$\begin{aligned}
&\frac{1}{2a(S,t)} \frac{\partial a(S,t)}{\partial S} \frac{\partial a(S,t)}{\partial t} - \frac{1}{2} \frac{\partial^2 a(S,t)}{\partial S \partial t} + \frac{a(S,t)}{2} \frac{\partial^2 b(S,t)}{\partial S^2} - \frac{1}{2} \frac{\partial b(S,t)}{\partial S} \frac{\partial a(S,t)}{\partial S} \\
&+ \frac{b(S,t)}{2a(S,t)} \left(\frac{\partial a(S,t)}{\partial S} \right)^2 - \frac{b(S,t)}{2} \frac{\partial^2 a(S,t)}{\partial S^2} - \frac{a(S,t)}{4} \frac{\partial a(S,t)}{\partial S} \frac{\partial^2 a(S,t)}{\partial S^2} - \frac{a^2(S,t)}{4} \frac{\partial^3 a(S,t)}{\partial S^3} \\
&- \frac{b(S,t)}{a^2(S,t)} \frac{\partial a(S,t)}{\partial t} + \frac{b(S,t)}{a(S,t)} \frac{\partial b(S,t)}{\partial S} - \frac{b^2(S,t)}{a^2(S,t)} \frac{\partial a(S,t)}{\partial S} - \frac{b(S,t)}{2} \frac{\partial^2 a(S,t)}{\partial S^2} \\
&+ \frac{1}{2a(S,t)} \frac{\partial a(S,t)}{\partial S} \frac{\partial a(S,t)}{\partial t} - \frac{1}{2} \frac{\partial a(S,t)}{\partial S} \frac{\partial b(S,t)}{\partial S} + \frac{b(S,t)}{2a(S,t)} \left(\frac{\partial a(S,t)}{\partial S} \right)^2 + \frac{a(S,t)}{4} \frac{\partial a(S,t)}{\partial S} \frac{\partial^2 a(S,t)}{\partial S^2} \\
&+ \int_{S_0}^{S} \frac{2}{a^3(Z,t)} \left(\frac{\partial a(Z,t)}{\partial t} \right)^2 dZ - \int_{S_0}^{S} \frac{1}{a^2(Z,t)} \frac{\partial^2 a(Z,t)}{\partial t^2} dZ + \frac{1}{a(S,t)} \frac{\partial b(S,t)}{\partial t} - \frac{b(S,t)}{a^2(S,t)} \frac{\partial a(S,t)}{\partial t} - \frac{1}{2} \frac{\partial^2 a(S,t)}{\partial S \partial t} \\
&= q_1(t) + q_2(t)x - \frac{\partial \gamma(x,t)}{\partial x}.
\end{aligned} \tag{137}$$

Recalling from (12) that $\gamma(x,t) \equiv c(S(x,t),t)$, differentiating w.r.t. x implies:

$$\frac{\partial \gamma(x,t)}{\partial x} = \frac{\partial c(S,t)}{\partial S} a(S,t). \tag{138}$$

Combining and cancelling terms in (137) and substituting (90) and (138) in it yields the equation (44) restricting $a(S,t)$, $b(S,t)$, and $c(S,t)$:

$$\begin{aligned}
&\frac{\partial a(S,t)}{\partial S} \frac{\partial \ln a(S,t)}{\partial t} - \frac{\partial^2 a(S,t)}{\partial S \partial t} - \frac{\partial a(S,t)}{\partial S} \frac{\partial b(S,t)}{\partial S} + \frac{b(S,t)}{a(S,t)} \left(\frac{\partial a(S,t)}{\partial S} \right)^2 - \frac{a^2(S,t)}{4} \frac{\partial^3 a(S,t)}{\partial S^3} \\
&+ \frac{1}{a(S,t)} \left[\frac{a^2(S,t)}{2} \frac{\partial^2 b(S,t)}{\partial S^2} + b(S,t) \frac{\partial b(S,t)}{\partial S} + \frac{\partial b(S,t)}{\partial t} \right] \\
&- \frac{2b(S,t)}{a^2(S,t)} \left[\frac{a^2(S,t)}{2} \frac{\partial^2 a(S,t)}{\partial S^2} + \frac{\partial a(S,t)}{\partial t} + \frac{b(S,t)}{2} \frac{\partial a(S,t)}{\partial S} \right] \\
&+ \int_{S_0}^{S} \frac{2}{a^3(Z,t)} \left(\frac{\partial a(Z,t)}{\partial t} \right)^2 dZ - \int_{S_0}^{S} \frac{1}{a^2(Z,t)} \frac{\partial^2 a(Z,t)}{\partial t^2} dZ = q_1(t) + q_2(t) \int_{S_0}^{S} \frac{1}{a(Z,t)} dZ - \frac{\partial c(S,t)}{\partial S} a(S,t),
\end{aligned} \tag{139}$$

for $S \geq 0, t \in [0,T]$ and for some arbitrary functions $q_1(t)$ and $q_2(t)$. As discussed in section 3, this condition is also sufficient. In other words, if $a(S,t)$, $b(S,t)$, and $c(S,t)$ solve (139), then the valuation PDE (3) can be mapped to the backward diffusion equation (6). The next appendix derives the map.

Appendix 2: Mapping to the Backward Diffusion Equation

Given that $\gamma^c(x,t)$ is quadratic in x, this appendix finds the map taking the general linear parabolic PDE (3) to the backward diffusion equation (6). Given some specification of the three functions $q_i(t), i = 0,1,2$ in (18), we assume that the ODE (117) has been solved for $F_3(t)$, so that the corresponding functions $F_i(t), i = 0,1,2$, are determined by (118) to (121). We also assume that we have identified a triplet $(a(S,t), b(s,t), c(s,t))$ solving (139). Substituting (94) in (99) implies:

$$u(w,\tau) \equiv e^{F_0(t)+\int_0^x [\beta_1(z,t)+\beta_2(z,t)]dz} U(x,t), \tag{140}$$

where $\beta_1(z,t)$ and $\beta_2(z,t)$ are defined in (93) and (112) respectively, and from (110) and (113):

$$w = \frac{x}{F_3(t)} + \int_0^t \frac{F_1(s)}{F_3(s)} ds + c \equiv w(x,t), \tag{141}$$

where c is an arbitrary constant, while from (108) and (21):

$$\tau = \int_0^t \frac{1}{F_3^2(s)} ds \equiv \tau(t). \tag{142}$$

Let $Y = S(z,t)$ be a change of variable in the integral in (140), where recall that for each t, $S(\cdot,t)$ is the inverse map of $x(S,t)$ defined in (90). Then $dY = \frac{\partial S(z,t)}{\partial z} dz = a(Y,t)dz$ from (65). Using the definitions of β_1 and β_2, substituting (90) in (140) implies:

$$u(w,\tau) \equiv e^{F_0(t)+\int_{S_0}^S \left[F_2(t)\int_{S_0}^Y \frac{1}{a(Z,t)}dZ + F_1(t) - \int_{S_0}^Y \frac{1}{a^2(Z,t)}\frac{\partial a(Z,t)}{\partial t}dZ + \frac{b(Y,t)}{a(Y,t)} - \frac{1}{2}\frac{\partial a(Y,t)}{\partial Y}\right]\frac{dY}{a(Y,t)}} V(S,t), \tag{143}$$

where from substituting (90) in (141):

$$w = \frac{1}{F_3(t)} \int_{S_0}^S \frac{1}{a(Z,t)} dZ + \int_0^t \frac{F_1(s)}{F_3(s)} ds + c \equiv W(S,t). \tag{144}$$

Equations (143), (144), and (142) define the map from (3) to (6) when the three coefficients solving (139) are known. The next appendix derives a set of sufficient conditions which permit the construction of these coefficients.

Appendix 3: Quadratic Carrying Costs for the State Variables

In this appendix, we assume that (52) holds, so that:

$$\frac{1}{2}\frac{\partial^2 R}{\partial x^2}(x,t) + \frac{\partial R}{\partial t}(x,t) = \left[q_0^r(t) + q_1^r(t)x + q_2^r(t)\frac{x^2}{2}\right] R(x,t), \quad x \in \Re, t \in (0,T). \tag{145}$$

From section 2.3, the PDE (145) can be transformed to the backward diffusion equation:

$$\frac{1}{2}h_{11}(w^r,\tau^r) + h_2(w^r,\tau^r) = 0, \tag{146}$$

where subscripts denote partial derivatives. The map (35) to (37) implies that the required map is:

$$h(w^r,\tau^r) \equiv e^{F_2^r(t)\frac{x^2}{2} + F_1^r(t)x + F_0^r(t)} R(x,t), \tag{147}$$

where $F_3^r(t)$, $F_2^r(t)$, $F_1^r(t)$, $F_0^r(t)$, $w^r(x,t)$, and $\tau^r(t)$ are respectively defined by (117), (118), (120), (121), (36), and (37) with q_i replaced by q_i^r for $i = 0,1,2$.

Let $\bar{\tau}^r \equiv \tau^r(T)$ denote the final time in the new clock, and let:

$$p(w,\tau;k,\bar{\tau}) \equiv \frac{1}{\sqrt{2\pi[\bar{\tau}-\tau]}} \exp\left\{-\frac{1}{2}\left[\frac{k-w}{\sqrt{\bar{\tau}-\tau}}\right]^2\right\} \tag{148}$$

denote the fundamental solution to the backward diffusion equation (6). To uniquely determine a solution to (146), suppose that the new exchange rate h at the terminal time $\bar{\tau}^r$ is related to the new spatial variable w^r by:

$$h(w^r,\bar{\tau}^r) = \phi^r(w^r), \quad w^r \in \Re, \tag{149}$$

where $\phi^r(\cdot)$ is some function of w^r whose convolution with the fundamental solution (148) is finite. Thus, the solution of (146) subject to (149) is expressible as:

$$h(w^r,\tau^r) = \int_{-\infty}^{\infty} \phi^r(k) p(w^r,\tau^r;k,\bar{\tau}^r)dk, \quad w^r \in \Re, \tau^r \in [0,\bar{\tau}^r].$$

Treating $h(w^r,\tau^r)$ as known, (147) implies that $R(x,t)$ is also known:

$$R(x,t) = e^{-F_2^r(t)\frac{x^2}{2} - F_1^r(t)x - F_0^r(t)} h(w^r(x,t),\tau^r(t)), \quad x \in \Re, t \in [0,T]. \tag{150}$$

By (50), the drift coefficient $\beta_1(x,t)$ in (10) is now also known, and is simply the logarithmic derivative of a solution to (145), i.e. a backward diffusion equation with quadratic potential $q_0^r(t) + q_1^r(t)x + q_2^r(t)\frac{x^2}{2}$.

We next find a sufficient condition which will determine $S(x,t)$ solving (64). Let:

$$\gamma^s(x,t) \equiv \frac{b(S(x,t),t)}{S(x,t)} \tag{151}$$

be the proportional risk-neutral drift of the state variable S, when considered as a function of x and t. Substituting (151) in (64) implies that $S(x,t)$ now solves the linear PDE:

$$\frac{1}{2}\frac{\partial^2 S}{\partial x^2}(x,t) + \beta_1(x,t)\frac{\partial S(x,t)}{\partial x} + \frac{\partial S}{\partial t}(x,t) = \gamma^s(x,t)S(x,t), \quad x \in \Re, t \in (0,T). \tag{152}$$

We now assume that $\gamma^s(x,t)$ is some quadratic function in x:

$$\gamma^s(x,t) = q_0^s(t) + q_1^s(t)x + q_2^s(t)\frac{x^2}{2}, \tag{153}$$

for some functions $q_0^s(t)$, $q_1^s(t)$, and $q_2^s(t)$. Letting $S^c(x,t) = R(x,t)S(x,t)$, the PDE (64) transforms to:

$$\frac{1}{2}\frac{\partial^2 S^c}{\partial x^2}(x,t) + \frac{\partial S^c}{\partial t}(x,t) = \tilde{\gamma}^c(x,t)S^c(x,t), \quad x \in \Re, t \in (0,T), \tag{154}$$

where by analogy with (16):

$$\tilde{\gamma}^c(x,t) \equiv \gamma^s(x,t) + \frac{1}{2}\frac{\partial \beta_1(x,t)}{\partial x} + \int_0^x \frac{\partial \beta_1(y,t)}{\partial t}dy + \frac{\beta_1^2(x,t)}{2}. \tag{155}$$

Substituting (153) in (155) implies that $\tilde{\gamma}^c(x,t)$ is also quadratic in x:

$$\begin{aligned}\tilde{\gamma}^c(x,t) &= q_0^s(t) + q_1^s(t)x + q_2^s(t)\frac{x^2}{2} + q_0^r(t) + q_1^r(t)x + q_2^r(t)\frac{x^2}{2} \\ &= \tilde{q}_0(t) + \tilde{q}_1(t)x + \tilde{q}_2(t)\frac{x^2}{2}.\end{aligned}$$

where $\tilde{q}_i(t) \equiv q_i^s(t) + q_i^r(t), i = 0,1,2$.

Hence, respectively defining $\tilde{F}_0(t)$, $\tilde{F}_1(t)$, $\tilde{F}_2(t)$, $\tilde{w}(x,t)$, and $\tilde{\tau}(t)$ by (121), (120), (118), (36), and (37) with $q_i(t)$ replaced by $\tilde{q}_i(t), i = 0,1,2$, the change of variables:

$$s(\tilde{w},\tilde{\tau}) \equiv e^{\tilde{F}_2(t)\frac{x^2}{2} + \tilde{F}_1(t)x + \tilde{F}_0(t)}S^c(x,t), \tag{156}$$

transforms the PDE (154) to the backward diffusion equation:

$$\frac{1}{2}\frac{\partial^2 s}{\partial \tilde{w}^2}(\tilde{w},\tilde{\tau}) + \frac{\partial s}{\partial \tilde{\tau}}(\tilde{w},\tilde{\tau}) = 0. \tag{157}$$

To uniquely determine a solution to (157), suppose that the converted state variable s at $\bar{\tilde{\tau}} \equiv \tilde{\tau}(T)$ is related to w by:

$$s(\tilde{w},\bar{\tilde{\tau}}) = \phi^s(\tilde{w}), \quad \tilde{w} \in \Re, \tag{158}$$

where $\phi^s(\tilde{w})$ is some function of the spatial variable, whose convolution with the fundamental solution (148) is finite. Thus, the solution of (157) subject to (158) is expressible as:

$$s(\tilde{w},\tilde{\tau}) = \int_{-\infty}^{\infty} \phi^s(k)p(\tilde{w},\tilde{\tau};k,\bar{\tilde{\tau}})dk.$$

Treating $s(\tilde{w}, \tilde{\tau})$ as known, (150) and (156) imply that $S(x,t)$ is also known:

$$S(x,t) = e^{[F_2^r(t) - \tilde{F}_2(t)]\frac{x^2}{2} + [F_1^r(t) - \tilde{F}_1(t)]x + [F_0^r(t) - \tilde{F}_0(t)]} \frac{s(\tilde{w}(x,t), \tilde{\tau}(t))}{h(w^r(x,t), \tau^r(t))}, \quad x \in \Re, t \in [0,T]. \tag{159}$$

Thus, the state variable pricing function $S(x,t)$ is the product of an exponential quadratic in x and the ratio of two solutions of the backward diffusion equation, where the two solutions are each composed with a linear function of x and an increasing function of t. From (65), composing the x derivative of $S(x,t)$ with its spatial inverse results in the absolute volatility:

$$a(S,t) = \frac{\partial S}{\partial x}(x(S,t),t), \quad S > 0, t \in [0,T]. \tag{160}$$

From (151) and (153), the state variable's risk-neutral drift can also be expressed in terms of the spatial inverse:

$$b(S,t) = S\left[q_0^s(t) + q_1^s(t)x(S,t) + q_2^s(t)\frac{x^2(S,t)}{2}\right]. \tag{161}$$

From (12) and (52), the cost of carrying the claim is also expressible in terms of this inverse:

$$c(S,t) = q_0^u(t) + q_1^u(t)x(S,t) + q_2^u(t)\frac{x^2(S,t)}{2}. \tag{162}$$

Equations (160) to (162) represent a family of explicit solutions to the transformation condition (44), generated by the three sufficient conditions (18), (52), and (153).

Substituting (52) in (10) and (153) in (152) implies that the sufficient conditions amount to assuming that $R(x,t)$, $S(x,t)$, and $U(x,t)$, solve:

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 R}{\partial x^2}(x,t) + \frac{\partial R}{\partial t}(x,t) &= \left[q_0^r(t) + q_1^r(t)x + q_2^r(t)\frac{x^2}{2}\right] R(x,t), \\
\frac{1}{2}\frac{\partial^2 S}{\partial x^2}(x,t) + \beta_1(x,t)\frac{\partial S(x,t)}{\partial x} + \frac{\partial S}{\partial t}(x,t) &= \left[q_0^s(t) + q_1^s(t)x + q_2^s(t)\frac{x^2}{2}\right] S(x,t), \\
\frac{1}{2}\frac{\partial^2 U}{\partial x^2}(x,t) + \beta_1(x,t)\frac{\partial U(x,t)}{\partial x} + \frac{\partial U}{\partial t}(x,t) &= \left[q_0^u(t) + q_1^u(t)x + q_2^u(t)\frac{x^2}{2}\right] U(x,t),
\end{aligned} \tag{163}$$

where recall $\beta_1(x,t) = \frac{\frac{\partial R}{\partial x}(x,t)}{R(x,t)}$.

In summary, when the three coefficients are not specified *ex ante*, a triplet consistent with the transformation condition (139) can be identified by assuming that their form is consistent with the three PDE's in (163). The PDE for $R(x,t)$ must be solved first, followed by the one for $S(x,t)$. The solutions are given explicitly in (150) and (159). The triplet $(a(S,t), b(s,t), c(s,t))$ consistent with $q_i(t) \equiv q_i^u(t) + q_i^r(t), i = 1,2$, is then given in terms of $S(x,t)$ and its inverse $x(S,t)$ in (160) to (162). The boundary value problem for $U(x,t)$ is also expressed in terms of $S(x,t)$. Finally, the solution to the original PDE for $V(S,t)$ is expressed by composing $U(x,t)$ with $x(S,t)$.

Appendix 4: Specifying Measure Changes

This appendix proposes an alternative approach to generating solutions to the transformation condition (44). We begin by assuming that (54) holds, so that the exchange rate $R(x,t)$ satisfies the backward diffusion equation:

$$\frac{1}{2}\frac{\partial^2 R}{\partial x^2}(x,t) + \frac{\partial R}{\partial t}(x,t) = 0. \tag{164}$$

If we also specify a terminal condition for R, then $R(x,t)$ is known up to quadrature.

To obtain a solution $S(X,t)$ to (65) and to the semi-linear PDE (64), consider the nonlinear change of dependent variable:

$$h(S,t) \equiv \frac{1}{R(x,t)}, \qquad S > 0, t \in [0,T]. \tag{165}$$

where:

$$S = S(x,t) \tag{166}$$

is a change of independent variable. The function $S(x,t)$ is not yet known explicitly, but it satisfies both (64) and (65).

From (165), we have:

$$R(x,t) = \frac{1}{h(S,t)}. \tag{167}$$

Differentiating (167) w.r.t. x implies:

$$\frac{\partial R(x,t)}{\partial x} = -\frac{1}{h^2(S,t)}\frac{\partial h}{\partial S}(S,t)\frac{\partial S}{\partial x}(x,t). \tag{168}$$

Differentiating w.r.t. x again:

$$\begin{aligned}
\frac{\partial^2 R(x,t)}{\partial x^2} &= \frac{2}{h^3(S,t)}\left(\frac{\partial h}{\partial S}(S,t)\frac{\partial S}{\partial x}(x,t)\right)^2 - \frac{1}{h^2(S,t)}\left[\frac{\partial^2 h}{\partial S^2}(S,t)\left(\frac{\partial S}{\partial x}(x,t)\right)^2 + \frac{\partial h}{\partial S}(S,t)\frac{\partial^2 S}{\partial x^2}(x,t)\right] \\
&= -\frac{1}{h^2(S,t)}\left\{\left[\frac{\partial^2 h}{\partial S^2}(S,t) - \frac{2}{h(S,t)}\left(\frac{\partial h}{\partial S}(S,t)\right)^2\right]\left(\frac{\partial S}{\partial x}(x,t)\right)^2 + \frac{\partial h}{\partial S}(S,t)\frac{\partial^2 S}{\partial x^2}(x,t)\right\}.
\end{aligned}$$

Differentiating (167) w.r.t. t implies:

$$\frac{\partial R(x,t)}{\partial t} = -\frac{1}{h^2(S,t)}\left[\frac{\partial h}{\partial S}(Z,t)\frac{\partial S}{\partial t}(x,t) + \frac{\partial h}{\partial t}(S,t)\right]. \tag{169}$$

Hence, (164) implies that $0 = \frac{\partial R}{\partial t} + \frac{1}{2}\frac{\partial^2 R}{\partial x^2}$:

$$= -\frac{1}{h^2(S,t)}\left\{\frac{\partial h}{\partial S}(S,t)\left\{\frac{\partial S}{\partial t}(x,t) + \frac{1}{2}\frac{\partial^2 S}{\partial x^2}(x,t)\right\} + \frac{\partial h}{\partial t}(S,t)\right.$$

$$
\begin{aligned}
&+\frac{1}{2}\left[\frac{\partial^2 h}{\partial S^2}(S,t)-\frac{2}{h(S,t)}\left(\frac{\partial h}{\partial S}(S,t)\right)^2\right]\left(\frac{\partial S}{\partial x}(x,t)\right)^2\Bigg\} \\
= \;& -\frac{1}{h^2(S,t)}\left\{\frac{\partial h}{\partial S}(S,t)\left[b(Z,t)-\frac{\frac{\partial R}{\partial x}(x,t)}{R(x,t)}\frac{\partial S}{\partial x}(x,t)\right]+\frac{\partial h}{\partial t}(S,t)\right. \\
&\left.+\frac{a^2(S,t)}{2}\left[\frac{\partial^2 h}{\partial S^2}(S,t)-\frac{2}{h(S,t)}\left(\frac{\partial h}{\partial S}(S,t)\right)^2\right]\right\}, \qquad (170)
\end{aligned}
$$

from (65) and (64). From (167) and (168), we have:

$$
-\frac{\partial h}{\partial S}(S,t)\frac{\frac{\partial R}{\partial x}(x,t)}{R(x,t)}\frac{\partial S}{\partial x}(x,t)=\frac{\left(\frac{\partial h}{\partial S}(S,t)\right)^2}{h(S,t)}\left(\frac{\partial S}{\partial x}(x,t)\right)^2=\frac{a^2(S,t)}{h(S,t)}\left(\frac{\partial h}{\partial S}(S,t)\right)^2, \qquad (171)
$$

from (65). Substituting (171) in (170) implies:

$$
0=-\frac{1}{h^2(S,t)}\left\{b(S,t)\frac{\partial h}{\partial S}(S,t)+\frac{\partial h}{\partial t}(S,t)+\frac{a^2(S,t)}{2}\frac{\partial^2 h}{\partial S^2}(S,t)\right\}, \qquad (172)
$$

or equivalently:

$$
\frac{a^2(S,t)}{2}\frac{\partial^2 h}{\partial S^2}(S,t)+b(S,t)\frac{\partial h}{\partial S}(S,t)+\frac{\partial h}{\partial t}(S,t)=0, \quad S>0, t\in[0,T]. \qquad (173)
$$

To understand the role of $h(S,t)$, recall that $R(x,t)$ is the exchange rate converting the base currency to the target currency by multiplication. Thus, $h(S,t)$ defined in (165) is clearly the exchange rate which converts the target currency back to the base currency. Put another way, conditional on the information at t, the condition (164) implies that $R_t \equiv R(X_t,t)$ is a martingale in the target economy, while the condition (173) implies that its reciprocal $H_t \equiv h(S_t,t)$ is a martingale in the base economy. As a result, if either exchange rate is normalized so as to have an initial value of one, then the normalized rate can be regarded as a measure change. Hence, the PDE's for $R(x,t)$ and $h(S,t)$ are essentially consequences of the condition that the process for the Radon Nikodym derivative be a martingale.

Consider any positive function of S and t which is monotonic in S. We can use this function to model one of the (unnormalized) measure changes. Thus, the solution approach outlined in this appendix amounts to modelling both (unnormalized) measure changes as functions of different state variables. Given the drift and volatility of one of these state variables, the drift and volatility of the other is determined. In this paper, the drift and volatility of the state variable X has been chosen to be 0 and 1 respectively (in the target economy). This determines the drift and volatility of the state variable S (in the base economy). For once $h(S,t)$ is specified, substituting (166) in (165) implies:

$$
h(S(x,t),t)=\frac{1}{R(x,t)}. \qquad (174)
$$

Since $h(S,t)$ is monotonic in S, it has a spatial inverse $h^{-1}(H,t)$. Hence, we have a solution to the semi-linear PDE (64):

$$S(x,t) = h^{-1}\left(\frac{1}{R(x,t)}, t\right). \tag{175}$$

The expression (175) for $S(x,t)$ can be used to obtain the desired volatility function $a(S,t)$. Differentiating (174) with respect to x implies:

$$\frac{\partial h}{\partial S}(S,t)\frac{\partial S}{\partial x}(x,t) = -\frac{\frac{\partial R}{\partial x}(x,t)}{R^2(x,t)}. \tag{176}$$

Solving for $\frac{\partial S}{\partial x}(x,t)$:

$$\frac{\partial S}{\partial x}(x,t) = -\frac{h^2(S,t)}{\frac{\partial h}{\partial S}(S,t)}\frac{\partial R}{\partial x}(x,t), \tag{177}$$

from (174). We suppose that $R(x,t)$ determined by (164) is monotonic in x for each t and hence invertible. Let $R^{-1}(\cdot,t)$ be the spatial inverse of $R(x,t)$. Then from (174):

$$x(S,t) = R^{-1}\left(\frac{1}{h(S,t)}, t\right). \tag{178}$$

From (65), (177), and (178), the volatility function $a(S,t)$ is determined explicitly as:

$$a(S,t) = -\frac{h^2(S,t)}{\frac{\partial h}{\partial S}(S,t)}\frac{\partial R}{\partial x}\left(R^{-1}\left(\frac{1}{h(S,t)}, t\right), t\right), \tag{179}$$

where (164) determines $R(x,t)$ and $h(S,t)$ is a positive monotonic function. Once $a(S,t)$ is determined, then $b(S,t)$ is determined by (173).

To determine $c(S,t)$, note that if R solves the backward diffusion equation (164), then $\beta_1(x,t) \equiv \frac{\partial \ln R(x,t)}{\partial x}$ solves Burger's equation:

$$\frac{1}{2}\frac{\partial^2 \beta_1(x,t)}{\partial x^2} + \frac{\partial \beta_1(x,t)}{\partial t} + \beta_1(x,t)\frac{\partial \beta_1(x,t)}{\partial x} = 0. \tag{180}$$

Hence from (47), $c(S,t)$ is given by:

$$c(S,t) = q_0(t) + q_1(t)x(S,t) + q_2(t)\frac{x^2(S,t)}{2}, \tag{181}$$

where again $x(S,t)$ is given by (7).

We now illustrate a way to model h which furthermore captures a pre-specified ratio of the drift to the variance. Suppose that h is independent of t. In this case, (173) reduces to:

$$b(S,t) = g(S)a^2(S,t), \tag{182}$$

where:

$$g(S) \equiv -\frac{h''(S)}{2h'(S)}. \tag{183}$$

Hence, suppose we assume that $b(S,t)$ is related to $a(S,t)$ by:

$$b(S,t) = g(S)a^2(S,t), \tag{184}$$

for some given function $g(S)$. Substituting (184) in (64) and then substituting (65) implies that the PDE governing S is non-linear:

$$S_t + \frac{1}{2}S_{xx} = g(S)S_x^2. \tag{185}$$

To solve this PDE, (184) implies that we must first solve the following linear ODE for $h(S)$:

$$\frac{h''(S)}{h'(S)} = -2g(S). \tag{186}$$

Clearly:

$$\frac{d\ln h'(S)}{dS} = -2g(S).$$

$$\ln h'(S) = -2\int^S g(Y_1)dY_1$$

$$h'(S) = e^{-2\int^S g(Y_1)dY_1}. \tag{187}$$

$$h(S) = \int^S e^{-2\int^{Y_2} g(Y_1)dY_1} dY_2 = \int^S e^{-2\int^{Y_2} \frac{b(Y_1,t)}{a^2(Y_1,t)}dY_1} dY_2, \tag{188}$$

from (184). The solution $h(S)$ is recognized as the scale function of the diffusion process S, which is traditionally defined as in (188), but when the drift function b and volatility function a are independent of time:

$$b(S,t) = b(S) \quad a(S,t) = a(S). \tag{189}$$

Note that $h(S)$ is not uniquely defined until we specify the lower limits of the integrals. We will leave them general at the moment recognizing that (188) defines a 2 parameter family of solutions to the ODE (186). However, any such solution is increasing in S.

From (166),(174), and (188), the solution to the nonlinear PDE (185) for $S(x,t)$ is implicitly given by:

$$\int^{S(x,t)} e^{-2\int^{Y_2} g(Y_1)dY_1} dY_2 = \frac{1}{R(x,t)}. \tag{190}$$

Replacing $S(x,t)$ with S and then inverting this expression for x gives:

$$x(S,t) = R^{-1}\left(\left(\int^S e^{-2\int^{Y_2} g(Y_1)dY_1} dY_2\right)^{-1}, t\right). \tag{191}$$

Hence, when h is independent of time, (179), (187), (188), and (191) imply that the volatility function $a(S,t)$ is:

$$a(S,t) = -\frac{\left[\int^S e^{-2\int^{Y_2} g(Y_1)dY_1} dY_2\right]^2}{e^{-2\int^S g(Y_1)dY_1}} \frac{\partial R}{\partial x}\left(R^{-1}\left(\left(\int^S e^{-2\int^{Y_2} g(Y_1)dY_1} dY_2\right)^{-1}, t\right), t\right). \tag{192}$$

Other choices for the form of $h(S,t)$ should also be explored. For example, one can examine the relationship between b and a that arises when h is a quadratic or exponential function of S.