

A Discrete Time Synthesis of Derivative Security Valuation Using a Term Structure of Futures Prices

Abstract

This paper synthesizes the existing discrete-time option pricing literature. We posit a binomial process on a term structure of futures prices and derive existing discrete-time results as special cases of our general framework.

by

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Comments are welcome.

I Introduction

Because options are specialized and relatively unimportant financial securities, the amount of time and space devoted to the development of a pricing theory might be questioned. One justification is that . . .

The above quote is taken from Robert C. Merton's classic article[45] "Theory of Rational Option Pricing". Apparently, the justification given by the author was persuasive; as of April 1994, this paper had 543 citations. The classic paper by Black and Scholes[4] "The Pricing of Options and Corporate Liabilities" had 955 citations. In his entry entitled "Finance" in the New Palgrave Dictionary of Economics and Finance, Ross[51] writes:

when judged by its ability to explain the empirical data, option-pricing theory is the most successful theory not only in finance, but in all of economics. It is now widely employed by the financial industry and its impact on economics has been far-ranging.

Although Ross eloquently surveys the entire field of finance in about 15 pages, we feel that an attempt to survey the voluminous futures and option pricing literature would be impossible. We have instead opted to provide an analytic synthesis of the literature in the text, and to briefly describe several surveys, texts, anthologies, and journals on the subject in this introduction. Since we may have omitted several important references in our brief review, we would like to issue an apology to our slighted readers.

The progression of academic thought on the subject of futures and option pricing theory is easily discerned by scanning the large number of surveys on the subject. An early survey by Smith[55] stresses the analytic point of view prevalent in the early to mid 70's. A later survey[56] by the same author emphasizes applications in corporate finance, as does the even later survey by Mason and Merton[43]. More recent introductory surveys are contained in Merton[47], Rubinstein[52], and Van Hulle[59]. Entries relating to Option Pricing Theory in the New Palgrave Dictionary of Economics include those by Babbs and Selby[1], Finnerty[20], Ingersoll[35], Jarrow[36], Merton[48], and Ross[51]. Empirical validation of option pricing models is ably summarized in Galai[21] and Geske and Trautmann[24]. Alternative stochastic processes for the price of the underlying asset are emphasized in Cox and Rubinstein[11] and Jarrow and Wiggins[40].

Similarly, alternative contractual structures are valued in a review of exotic options by Rubinstein and Reiner[50]. The survey by Cox and Huang[10] is excellent for understanding the early literature in the context of the modern martingale technology. Finally, the surveys by Jarrow[37], Jarrow and Turnbull[39] and by Clewlow, Hodges, Selby, Strickland and Xu[9] discuss term structure models and serve as excellent companion readings for the current paper.

Textbooks on the subject of futures and option pricing theory are usually aimed at either master's students, practitioners, or doctoral students. The early texts by Cox and Rubinstein[12] and Jarrow and Rudd[38] were aimed at master's level students and practitioners. Several later books appeal to both undergraduates and master's students including those by Chance[7], Figlewski, Silber, and Subrahmanyam[19], Hull[31], Ritchken[52], and Stoll and Whaley[58]. Books by Bookstaber[5], Gastineau[23], Hull[32], and McMillan[44], appeal to practitioners. Books primarily oriented towards futures contracts include those by Duffie[15] and Kolb[41]. Several doctoral level books contain useful chapters on option pricing theory, including those by Duffie[16], Dothan[14], Huang and Litzenberger[30], Ingersoll[33], Shimko[54], and Wilmott et. al[61]. Anthologies which wholly or mainly consist of articles on option pricing include those edited by Brenner[6], Bhattacharya and Constantinides[2], Chance and Trippi[8], Fabozzi[17], Field, Jaycolbs, and Tompkins[18], Hodges[29], Kolb[42], Merton[47], Smith and Smithson[57], and Whaley[60]. Two forthcoming books intended to be of general interest are Carr, Reiner, and Rubinstein on exotic options and Heath and Jarrow on interest rate derivatives.

Most of the widely cited articles on option pricing theory have appeared in journals of general interest to finance academics. However, several journals exist which are either exclusively oriented towards derivative securities or have an emphasis on derivatives. These include Journal of Applied Mathematical Finance, Journal of Derivatives, Review of Derivatives Research, Journal of Financial Engineering, Journal of Fixed Income, Journal of Futures Markets, Review of Futures Markets, and Mathematical Finance.

In the remainder of the paper, we present an analytic synthesis of the discrete-time option pricing literature. Our motivation for focussing on discrete-time models lies in the inherent simplicity of this approach. In the next section, we present a general model which posits a stochastic process on a term structure of futures prices. In this model, we show the equivalence between the absence of arbitrage and the

existence of an equivalent martingale measure. Since we work in a complete market, the results of Harrison and Pliska [26] imply that this measure is unique. We illustrate the martingale measure methodology for both European and American-style options. In the third section, we review existing discrete-time option models and show they may be considered as special cases of our general framework. The fourth section discusses the implementation of our futures term structure model. The final section concludes.

II The General Framework

This section presents the general framework for valuing derivative securities. Every existing arbitrage-free derivatives pricing model will be shown to be a special case of this approach.

II-A The Model

Consider a frictionless economy with discrete trading dates $t \in [0, 1, 2, \dots, \tau]$, where for simplicity, the trading horizon is finite ($\tau < \infty$). At each trading date, investors can borrow or lend risklessly and trade in a term structure of futures contracts. At date $t \in [0, \tau]$, the (gross) spot riskless rate is denoted by $r_t \geq 1$ and the futures price is denoted by $F_{t,T}$, where $T \in [t, \tau]$ is the the delivery date. We assume marking-to-market occurs once¹ per period for simplicity. In other words, an investor going long one futures contract with maturity T at time t will receive $F_{t+1,T} - F_{t,T}$ in cash at time $t + 1$.

For simplicity, we use a binomial tree to describe the evolution of the entire futures price curve and spot interest rate over time. The tree starts out at time 0 with the $(\tau + 2)$ vector $[S_0, F_{0,1}, F_{0,2}, \dots, F_{0,\tau}; r_0]$.

It evolves over the first period as given below:

¹Under *deterministic* interest rates, it is well-known that the futures price for any marking-to-market frequency is equal to the forward price. Consequently, the frequency of marking-to-market would have no effect on the futures price.

$$\begin{array}{c}
\left(\begin{array}{c} S_0 \\ F_{0,1} \\ F_{0,2} \\ \vdots \\ F_{0,\tau} \\ \dots \\ r_0 \end{array} \right) \\
\begin{array}{l} \nearrow p_0 \\ \searrow 1 - p_0 \end{array} \\
\left(\begin{array}{c} S_1^u \equiv F_{0,1} \cdot u_{0,1} \\ F_{1,2}^u \equiv F_{0,2} \cdot u_{0,2} \\ \vdots \\ F_{1,\tau}^u \equiv F_{0,\tau} \cdot u_{0,\tau} \\ \dots\dots\dots \\ r_1^u \end{array} \right) \\
\left(\begin{array}{c} S_1^d \equiv F_{0,1} \cdot d_{0,1} \\ F_{1,2}^d \equiv F_{0,2} \cdot d_{0,2} \\ \vdots \\ F_{1,\tau}^d \equiv F_{0,\tau} \cdot d_{0,\tau} \\ \dots\dots\dots \\ r_1^d \end{array} \right)
\end{array}$$

where $u_{0,T} > d_{0,T} > 0$ for all $T \in \{0, 1, 2, \dots, \tau\}$. The entire curve moves “up” with probability $p_0 \in (0, 1)$ or “down” with probability $1 - p_0$. As the curve evolves, there is one less futures price available as $F_{0,1}$ becomes the new spot price S_1 with value $S_1^u \equiv F_{0,1} \cdot u_{0,1}$ if the process moves up and value $S_1^d \equiv F_{0,1} \cdot d_{0,1}$ if the process moves down. The first element S_0 in the vector disappears. Each futures price is allowed to have its own up and down price relative. For example, $u_{0,1}$ can differ from $u_{0,2}$.

The evolution of the spot interest rate is appended onto each vector. The initial spot rate r_0 changes to r_1^u with probability p_0 and to r_1^d with probability $1 - p_0$. The implicit assumption here is that the futures price curve and spot interest rate are driven by the same factor. This assumption can be easily relaxed by adding additional branches to the tree. Three branches emanating from each node would imply a two factor model; four branches would imply a three factor model, and so on. Our restriction of two branches is easily relaxed; the generalization is left to the reader.

At an arbitrary step t to $t + 1$, the tree looks like:

$$\begin{array}{c}
\left(\begin{array}{c} S_t \\ F_{t,t+1} \\ F_{t,t+2} \\ \vdots \\ F_{t,\tau} \\ \dots \\ r_t \end{array} \right) \\
\begin{array}{l} \nearrow p_t \\ \searrow 1-p_t \end{array}
\end{array}
\begin{array}{c}
\left(\begin{array}{c} S_{t+1}^u \equiv F_{t,t+1} \cdot u_{t,t+1} \\ F_{t+1,t+2}^u \equiv F_{t,t+2} \cdot u_{t,t+2} \\ \vdots \\ F_{t+1,\tau}^u \equiv F_{t,\tau} \cdot u_{t,\tau} \\ \dots \\ r_{t+1}^u \end{array} \right) \\
\left(\begin{array}{c} S_{t+1}^d \equiv F_{t,t+1} \cdot d_{t,t+1} \\ F_{t+1,t+2}^d \equiv F_{t,t+2} \cdot d_{t,t+2} \\ \vdots \\ F_{t+1,\tau}^d \equiv F_{t,\tau} \cdot d_{t,\tau} \\ \dots \\ r_{t+1}^d \end{array} \right)
\end{array}$$

where $p_t \in (0, 1)$ and $u_{t,T} > d_{t,T} > 0$ for all $T \in [t + 1, \dots, \tau]$.

At the last step from $\tau - 1$ to τ , the process is:

$$\begin{array}{c}
\left(\begin{array}{c} S_{\tau-1} \\ F_{\tau-1,\tau} \\ \dots \\ r_{\tau-1} \end{array} \right) \\
\begin{array}{l} \nearrow p_{\tau-1} \\ \searrow 1-p_{\tau-1} \end{array}
\end{array}
\begin{array}{c}
\left(\begin{array}{c} S_{\tau}^u \equiv F_{\tau-1,\tau} \cdot u_{\tau-1,\tau} \\ \dots \\ r_{\tau}^u \end{array} \right) \\
\left(\begin{array}{c} S_{\tau}^d \equiv F_{\tau-1,\tau} \cdot d_{\tau-1,\tau} \\ \dots \\ r_{\tau}^d \end{array} \right)
\end{array}$$

where $p_{\tau} \in (0, 1)$ and $u_{\tau-1,\tau} > d_{\tau-1,\tau} > 0$. The process ends here. By definition, $S_{\tau}^u \equiv F_{\tau-1,\tau} \cdot u_{\tau-1,\tau}$ is the spot price at τ if the process moves up over the last period and $S_{\tau}^d \equiv F_{\tau-1,\tau} \cdot d_{\tau-1,\tau}$ is the spot price at τ if the process moves down.

We have assumed that the probabilities p_t and $1 - p_t$ and the price relatives $u_{t,T}$ and $d_{t,T}$ are strictly positive. If either probability is ever zero, then the process is locally riskless, which is economically uninteresting. If the down price relative is ever zero, then the process absorbs at the origin the first time that such a down jump occurs. For simplicity of the notation and presentation, we have assumed that these probabilities and price relatives do not depend on the state (i.e. node in the tree) or path. The generalization to state or path-dependent probabilities or price relatives is straightforward.

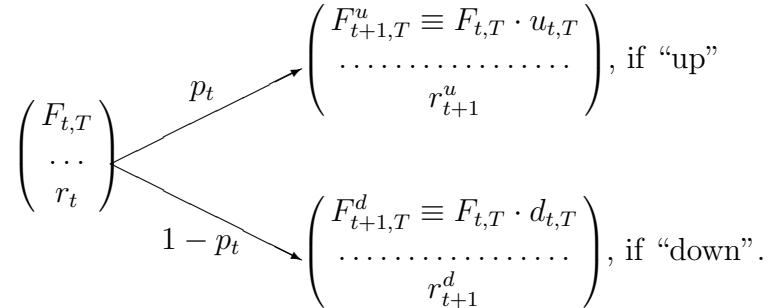
The futures contracts of different maturities are close substitutes, having the same underlying asset

with only differing delivery dates. Consequently, when exogenously specifying the stochastic process for the futures price curve, we must be concerned that the process specified admits no arbitrage opportunities. This is the topic of the next subsection.

II-B Arbitrage Opportunities, Market Completeness, and Martingale Probabilities

Given the stochastic process for the evolution of the futures price curve, we next determine the restrictions required on the up and down price relatives $u_{t,T}$ and $d_{t,T}$ across maturities $T \in [t, \tau], t \in [0, \tau]$ such that there are no arbitrage opportunities.

First, consider the futures price of a fixed maturity T . Over the interval $[t, t + 1]$, the evolution of this futures price and the spot interest rate is:



We claim that there are no arbitrage opportunities between these two securities only if²

$$u_{t,T} > 1 > d_{t,T} \text{ for all } t \in [0, \tau] \text{ and for all } T \in [t, \tau]. \quad (1)$$

Otherwise, for example, if $1 \leq d_{t,T} < u_{t,T}$, then an arbitrage opportunity is generated by going long the futures at t and receiving at least $F_{t,T}(d_{t,T} - 1) \geq 0$ at the next marking-to-market at $t + 1$. The arbitrageur then reverses the long position costlessly. Conversely, if $1 \geq u_{t,T} > d_{t,T}$, then an arbitrage opportunity is generated by going short the futures at t and receiving at least $F_{t,T}(1 - u_{t,T}) > 0$ at $t + 1$. These contradictions yield the result.

²Condition (1) is also sufficient for no arbitrage. A sketch of the proof is that (4) is equivalent to (1), and (4) implies $\frac{V_t}{B_t}$ in (8) is a martingale. An arbitrage opportunity would contradict that $\frac{V_t}{B_t}$ is a martingale (for the arbitrage opportunity). This completes the argument.

Expression (1) holds if and only if there exists a *unique* number $\pi_{t,T}$ satisfying $0 < \pi_{t,T} < 1$ such that:

$$\pi_{t,T} \cdot u_{t,T} + (1 - \pi_{t,T}) \cdot d_{t,T} = 1. \quad (2)$$

The unique number $\pi_{t,T}$ is called a *pseudo-probability* or a *martingale probability*. To see this latter interpretation, multiply expression (2) by $F_{t,T}$:

$$\pi_{t,T} F_{t,T} \cdot u_{t,T} + (1 - \pi_{t,T}) F_{t,T} \cdot d_{t,T} = F_{t,T}. \quad (3)$$

or

$$E_{t,T} F_{t+1,T} = F_{t,T}, \quad (4)$$

for all $T \in [t + 1, \tau]$. To prove $E_{t,T} F_{t+i,T} = F_{t,T}$ for all $i = 1, \dots, \tau - t$ and $T \in [t + i, \tau]$, use induction and the law of iterated expectations. Thus, $F_{t,T}$ is a martingale using the pseudo-probability $\pi_{t,T}$. We can solve for $\pi_{t,T}$ explicitly using (2):

$$\pi_{t,T} \equiv \frac{1 - d_{t,T}}{u_{t,T} - d_{t,T}}, \quad (5)$$

for all $T \in [t, \tau]$.

In summary, for each delivery date T , no arbitrage in a futures contract of that maturity is equivalent to:

$$u_{t,T} > 1 > d_{t,T}. \quad (6)$$

or equivalently there exists a unique $\pi_{t,T}$ for all $t \in [0, \tau]$ and for all $T \in [t, \tau]$ such that:

$$0 < \pi_{t,T} < 1 \text{ and } E_{t,T} F_{t+1,T} = F_{t,T}, \quad (7)$$

where $E_{t,T}$ represents expectations with respect to $\pi_{t,T}$ at time t .

But what about arbitrage opportunities across the futures contracts and riskless lending or borrowing opportunities? To guarantee that these don't exist, we need to consider the construction of synthetic futures contracts. This involves the concept of *market completeness*, to which we now turn.

First, for $t \in [0, \tau)$, consider forming a portfolio consisting of n_t futures contracts with maturity $T_1 \in (t, \tau]$ and lending³ β_t risklessly. Its time t value is:

$$V_t \equiv n_t \Phi_{t,T_1} + \beta_t, \quad (8)$$

³If $\beta_t < 0$, then the investor is borrowing $|\beta_t|$ risklessly.

where $\Phi_{t,T_1} = 0$ is a placeholder to remind us of our costless investment in futures. The portfolio's time $t + 1$ value, including cash flows, is:

$$\begin{aligned} V_{t+1}^u &\equiv n_t F_{t,T_1}(u_{t,T_1} - 1) + \beta_t r_t, \text{ if "up", or} \\ V_{t+1}^d &\equiv n_t F_{t,T_1}(d_{t,T_1} - 1) + \beta_t r_t, \text{ if "down".} \end{aligned} \quad (9)$$

Since the matrix $\begin{pmatrix} u_{t,T_1} - 1 & 1 \\ d_{t,T_1} - 1 & 1 \end{pmatrix}$ is invertible for all $t \in [0, \tau)$ and for all $T_1 \in (t, \tau]$, by selecting (n_t, β_t) appropriately, the above portfolio can generate any desired payoff vector at time $t + 1$. For example, suppose we wish to obtain the time $t + 1$ payoff $[X^u, X^d]$. This can be done by choosing:

$$\begin{aligned} n_t^* &= \frac{X^u - X^d}{F_{t,T_1}(u_{t,T_1} - d_{t,T_1})} \text{ and} \\ \beta_t^* &= \frac{X^d - n_t^* F_{t,T_1}(d_{t,T_1}) - 1}{r_t} \\ &= \frac{X^d + (X^u - X^d)\pi_{t,T_1}}{r_t} \\ &= r_t^{-1} E_{t,T_1} X_{t+1}. \end{aligned} \quad (10)$$

Note that the value of the portfolio at time t is then:

$$V_t^* \equiv n_t^* \Phi_{t,T_1} + \beta_t^* = r_t^{-1} E_{t,T_1} X_{t+1}.$$

Because this is true for all times $t \in [0, \tau)$, the market is said to be *dynamically complete*.

To understand these results, consider a graph of the desired payoff X_{t+1} against the futures price change $F_{t+1,T_1} - F_{t,T_1}$. In a binomial framework, the desired payoff, X_{t+1} , may be represented by a straight line with slope $m_x \equiv \frac{X_{t+1}^u - X_{t+1}^d}{F_{t+1,T}^u - F_{t+1,T}^d}$ and vertical intercept $X_{t+1}^u - m_x(F_{t+1,T_1}^u - F_{t,T_1})$. The time $t + 1$ marking-to-market proceeds from going long n_t futures contracts, $F_{t+1,T_1} - F_{t,T_1}$, is represented by a straight line going through the origin with slope n_t . The time $t + 1$ future value of lending β_t at t , $\beta_t r_t$, is given by a horizontal line with vertical intercept $\beta_t r_t$. The desired number of futures contracts, n_t^* , rotates the slope of the time $t + 1$ portfolio value, V_{t+1} , to match that of the desired payoff, X_{t+1} . The desired amount of lending, β_t^* , then moves the vertical intercept of this portfolio to match that of the desired payoff. This vertical intercept simultaneously represents the expected payoff, $E_{t,T_1} X_{t+1}$, the expected replicating portfolio value, $E_{t,T+1} V_{t+1}$, and the future market value of the riskless component of this replicating portfolio, $\beta_t^* r_t$.

Given these insights, we can now determine the additional restrictions on the price relatives $u_{t,T}$ and $d_{t,T}$ needed to guarantee that there is no arbitrage among all futures contracts and riskless lending and borrowing opportunities. To obtain this restriction, consider forming a portfolio of the futures contract with maturity T_1 and riskless lending, as above, to duplicate the marking-to-market proceeds of a different futures contract with maturity $T_2 \geq t + 1$. That is, choose:

$$\begin{aligned} X^u &= F_{t,T_2}(u_{t,T_2} - 1) \quad \text{and} \\ X^d &= F_{t,T_2}(d_{t,T_2} - 1) \end{aligned}$$

in expression (10). Consequently, the number of futures contracts held and the amount lent are:

$$\begin{aligned} n_t^* &= \frac{F_{t,T_2}(u_{t,T_2} - d_{t,T_2})}{F_{t,T_1}(u_{t,T_1} - d_{t,T_1})} \text{ and} \\ \beta_t^* &= r_t^{-1} E_{t,T_1}(F_{t+1,T_2} - F_{t,T_2}), \end{aligned} \tag{11}$$

while the time t value of this replicating portfolio is:

$$V_t^* = r_t^{-1} E_{t,T_1}(F_{t+1,T_2} - F_{t,T_2}).$$

There is no arbitrage among these securities if and only if the time t value of this portfolio equals the time t value of the futures contract with maturity T_2 , i.e.,

$$r_t^{-1} E_{t,T_1}(F_{t+1,T_2} - F_{t,T_2}) = \Phi_{t,T_2} = 0. \tag{12}$$

Equivalently, tomorrow's expected futures price is just today's under an equivalent martingale measure, i.e. if and only if:

$$E_{t,T_1} F_{t+1,T_2} = F_{t,T_2}. \tag{13}$$

However by (4):

$$E_{t,T_2} F_{t+1,T_2} = F_{t,T_2}, \tag{14}$$

and thus, by the uniqueness of π_{t,T_2} :

$$\pi_{t,T_1} = \pi_{t,T_2}. \tag{15}$$

Condition (15) states that the martingale probabilities are independent of the maturity of the particular futures contract selected. In summary, therefore, we have the following proposition:

Proposition 1: (No Arbitrage Opportunities)

There are no arbitrage opportunities in the above economy if and only if:

1) there exists a unique π_t for $t \in [0, \dots, \tau)$ such that:

$$E_t F_{t+1, T} = F_{t, T} \text{ for all } T \in (t, \dots, \tau] \quad (16)$$

where $E_t(\cdot)$ represents expectations with respect to $(\pi_t, \pi_{t+1}, \dots, \pi_{\tau-1})$ at time t , and

2) the unique π_t is given by:

$$\pi_{t, T} = \frac{1 - d_{t, T_1}}{u_{t, T_1} - d_{t, T_1}} = \frac{1 - d_{t, T_2}}{u_{t, T_2} - d_{t, T_2}} \text{ for all } t \in [0, \tau) \text{ and for all } T_1, T_2 \in (t, \tau]. \quad (17)$$

Condition (17) provides the necessary and sufficient restrictions required on the up and down price relatives $u_{t, T}$ and $d_{t, T}$ across maturities T , such that there are no arbitrage opportunities in the economy. We assume for the remainder of this paper that these restrictions are satisfied.

II-C Contingent Claim Valuation and Delta Hedging

Given the traded term structure of futures prices, riskless lending and borrowing opportunities, and the arbitrage-free restrictions, we can now discuss the pricing of options and contingent claims. Contingent claims are divided into two styles: European or American. A *European-style* contingent claim is defined to be any time sequence of cash flows which are dependent on the history of the evolution of the term structure of futures prices. An *American-style* contingent claim is defined to be a *set of* sequences of cash flows dependent upon the history of the evolution of the term structure of futures prices *and* an optimal stopping time⁴ which determines which sequence in the set of cash flow sequences is received.

For example, a European-style call option on the spot price of an asset with maturity date $\bar{t} \leq \tau$ and strike price $K \geq 0$ has a cash flow at time \bar{t} equal to $\max[S_{\bar{t}} - K, 0]$. In contrast, the corresponding American-style call option has the set of cash flows, $\max[S_t - K, 0]$ for $t \in [0, \bar{t}]$ possible, where the stopping time $\tilde{t} \in [0, \bar{t}]$ determines which of these is received. The stopping time \tilde{t} depends on (at most) the history of the evolution of the term structure of futures prices up to the time it is stopped. Given a stopping time \tilde{t} , the cash flow received is $\max[S_{\tilde{t}} - K, 0]$.

⁴A *stopping time* $\tilde{t} : \Omega \rightarrow [0, \infty]$ is defined as a function such that $\{\omega \in \Omega : \tilde{t}(\omega) \leq t\}$ is measurable with respect to the σ -algebra generated by the futures price curve as it evolves through time. The set Ω represents the 2^τ possible paths through the binomial tree.

II-D European-style Contingent Claims

To value contingent claims, we first start with the simplest case, and then complicate the analysis as our knowledge increases. First, consider a European-style contingent claim, i.e. a security which pays off a cash flow at a fixed date, or possibly at multiple fixed dates. Given are a time sequence of the possibly *path-dependent* cash flows. By path-dependent, we mean that the cash flows received can depend upon the particular history of the evolution of the term structure of futures prices prior to the node of the tree at which the cash flow is received. In general, there may be multiple payout dates associated with the European-style contingent claim. However, the simplest case is where a cash flow is received at only one date, say time $\bar{t} \in (t, \tau)$.

To value this claim by the arbitrage-pricing methodology, start at time $\bar{t} - 1$. At this date, the term structure of futures prices is:

$$\begin{array}{c}
 \left(\begin{array}{c} S_{\bar{t}-1} \\ F_{\bar{t}-1, \bar{t}} \\ F_{\bar{t}-1, \bar{t}+1} \\ \vdots \\ F_{\bar{t}-1, \tau} \\ \dots\dots\dots \\ r_{\bar{t}-1} \end{array} \right)
 \end{array}
 \begin{array}{c}
 \nearrow \pi_{\bar{t}-1} \\
 \searrow 1 - \pi_{\bar{t}-1}
 \end{array}
 \begin{array}{c}
 \left(\begin{array}{c} S_{\bar{t}}^u \equiv F_{\bar{t}-1, \bar{t}} \cdot u_{\bar{t}-1, \bar{t}} \\ F_{\bar{t}, \bar{t}+1}^u \equiv F_{\bar{t}-1, \bar{t}+1} \cdot u_{\bar{t}-1, \bar{t}+1} \\ \vdots \\ F_{\bar{t}, \tau}^u \equiv F_{\bar{t}-1, \tau} \cdot u_{\bar{t}-1, \tau} \\ \dots\dots\dots \\ r_{\bar{t}-1}^u \end{array} \right) \\
 \\
 \left(\begin{array}{c} S_{\bar{t}}^d \equiv F_{\bar{t}-1, \bar{t}} \cdot d_{\bar{t}-1, \bar{t}} \\ F_{\bar{t}, \bar{t}+1}^d \equiv F_{\bar{t}-1, \bar{t}+1} \cdot d_{\bar{t}-1, \bar{t}+1} \\ \vdots \\ F_{\bar{t}, \tau}^d \equiv F_{\bar{t}-1, \tau} \cdot d_{\bar{t}-1, \tau} \\ \dots\dots\dots \\ r_{\bar{t}-1}^d \end{array} \right)
 \end{array}$$

Let the cash flow to the contingent claim at time \bar{t} be $[X^u, X^d]$ at the two possible states.

Because the market is complete, we know that by mixing a futures contract maturing at $T_1 > \bar{t}$ with riskless lending or borrowing, we can form a portfolio at time $\bar{t} - 1$ as in expressions (8)–(10) which will duplicate this cash flow at time \bar{t} . To avoid arbitrage, the price of this contingent claim at time $\bar{t} - 1$, denoted $C_{\bar{t}-1}$, should equal the value of the portfolio, i.e.,

$$C_{\bar{t}-1} = n_{\bar{t}-1} \Phi_{\bar{t}-1, T_1} + \beta_{\bar{t}-1} = r_{\bar{t}-1}^{-1} E_{\bar{t}-1}(X_{\bar{t}}) = r_{\bar{t}-1}^{-1} [\pi_{\bar{t}-1} X_{\bar{t}}^u + (1 - \pi_{\bar{t}-1}) X_{\bar{t}}^d] \quad (18)$$

where $X_{\bar{t}} \equiv \begin{cases} X_{\bar{t}}^u, & \text{if "up"} \\ X_{\bar{t}}^d, & \text{if "down"}. \end{cases}$

The value of the contingent claim at time $\bar{t} - 1$ is seen to be the discounted expected value of the time \bar{t} cash flows to the contingent claim using the martingale probabilities. This is the same price that would be obtained in a risk-neutral economy with probability beliefs π . Hence, this is called *risk-neutral valuation* and the approach called the *risk-neutrality argument*.

The hedge ratio (or delta) of the contingent claim at time $\bar{t} - 1$ is the quantity $n_{\bar{t}-1}^*$ for the futures contract with maturity T_1 . By going long $n_{\bar{t}-1}^*$ futures and lending the current model value $C_{\bar{t}-1}$ risklessly, the contingent claim is *synthesized*.

Next, to value this contingent claim at time $\bar{t} - 2$, we follow a recursive procedure. At this step, the “cash flows” received at time $\bar{t} - 1$ are given by $[C_{\bar{t}-1}^u, C_{\bar{t}-1}^d]$ where the superscripts u and d correspond to the up and down branches. Consequently, the value of the contingent claim at time $\bar{t} - 2$ is given by expression (18) with $X_{\bar{t}}^u$ replaced with $C_{\bar{t}-1}^u$ and $X_{\bar{t}}^d$ replaced with $C_{\bar{t}-1}^d$:

$$\begin{aligned} C_{\bar{t}-2} &= r_{\bar{t}-2}^{-1} \left[\pi_{\bar{t}-2}^* C_{\bar{t}-1}^u + (1 - \pi_{\bar{t}-2}^*) C_{\bar{t}-1}^d \right] \\ &= r_{\bar{t}-2}^{-1} E_{\bar{t}-2} (C_{\bar{t}-1}). \end{aligned} \tag{19}$$

The hedge ratio over $[\bar{t} - 2, \bar{t} - 1]$ is again obtained from expression (10) with the stated replacements for X^u and X^d .

Substitution of (18) into (19), and the law of iterated expectations yields:

$$C_{\bar{t}-2} = E_{\bar{t}-2} \left(\frac{x}{r_{\bar{t}-2} \cdot r_{\bar{t}-1}} \right). \tag{20}$$

Again, the contingent claim’s value is seen to be the expected value of its cash flows, appropriately discounted, and using the martingale probabilities.

It is now easy to see that by induction:

$$C_0 = E_0 \left(\frac{x}{r_0 \cdot r_1 \cdot \dots \cdot r_{\bar{t}-1}} \right). \tag{21}$$

Recall that the martingale probabilities are given by $\pi_t = \frac{1-d_{t,T}}{u_{t,T}-d_{t,T}}$ for any $T > t$. Expression (21) yields a very general procedure for calculating present values in the above economy. Notice that in the valuation formula, the original probabilities $(p_0, p_1, \dots, p_{\tau-1})$ do not appear. Instead, they are replaced

by the pseudo-probabilities $(\pi_0, \pi_1, \dots, \pi_{\tau-1})$. The pseudo-probabilities can be thought of as the original probabilities, adjusted for risk aversion, so that the risk-neutral valuation procedure is valid.

An example will help to illustrate the above procedure. Consider a European-style call option on the spot price of the asset with maturity date $\bar{t} \leq \tau$ and strike price $K \geq 0$. By expression (21), the call's value at time 0 is given by:

$$C_0 = E_0 \left(\frac{\max[S_{\bar{t}} - K, 0]}{r_0 \cdot r_1 \cdot \dots \cdot r_{\bar{t}-1}} \right). \quad (22)$$

An explicit calculation is possible given particular values for $(u_{t,T}, d_{t,T})$ and $(S_0, F_{0,1}, \dots, F_{0,\tau}; r_0)$.

II-D-1 Example: Black's Formula

For example, if $u_{t,\bar{t}}$, $d_{t,\bar{t}}$, and r_t are constant at u , d , and r respectively, then the binomial version of Black's model emerges. To see this, let $F_{0,\bar{t}}$ be the futures/forward price for delivery at the option's expiration. Then convergence assures that the terminal futures price is the spot price, i.e. $F_{\bar{t},\bar{t}} = S_{\bar{t}}$. Let $1_{F_{\bar{t},\bar{t}} > K}$ denote the indicator function of the event that the call finishes in-the-money. Since $\max[S_{\bar{t}} - K, 0] = \max[F_{\bar{t},\bar{t}} - K, 0] = 1_{F_{\bar{t},\bar{t}} > K} F_{\bar{t},\bar{t}} - 1_{F_{\bar{t},\bar{t}} > K} K$, the European call's value at time 0 in the binomial model is:

$$C_0 = r^{-\bar{t}} \left\{ E_0 \left[1_{F_{\bar{t},\bar{t}} > K} F_{\bar{t},\bar{t}} \right] - K E_0 \left[1_{F_{\bar{t},\bar{t}} > K} \right] \right\}. \quad (23)$$

The latter expectation is the probability that the call finishes in-the-money:

$$E_0 \left[1_{F_{\bar{t},\bar{t}} > K} \right] = P_0(F_{\bar{t},\bar{t}} > K) = \sum_j 1_{F_{\bar{t},\bar{t}} > K} \binom{\bar{t}}{j} \pi^j (1 - \pi)^{\bar{t}-j}, \quad (24)$$

where $\pi \equiv \frac{1-d}{u-d}$ and where j is the realized number of up jumps that occur over the \bar{t} periods. Since $F_{\bar{t},\bar{t}} = F_{0,\bar{t}} u^j d^{\bar{t}-j}$, the first expectation evaluates to:

$$\begin{aligned} E_0 \left[1_{F_{\bar{t},\bar{t}} > K} F_{\bar{t},\bar{t}} \right] &= F_{0,\bar{t}} \sum_j 1_{F_{\bar{t},\bar{t}} > K} \binom{\bar{t}}{j} [\pi u]^j [(1 - \pi)d]^{\bar{t}-j} \\ &= F_{0,\bar{t}} \sum_j 1_{F_{\bar{t},\bar{t}} > K} \binom{\bar{t}}{j} \hat{\pi}^j (1 - \hat{\pi})^{\bar{t}-j} \end{aligned}$$

where $\hat{\pi} \equiv \pi u$, and consequently $1 - \hat{\pi} = (1 - \pi)d$, since $\pi u + (1 - \pi)d = 1$. Substituting $\hat{P}_0(F_{\bar{t},\bar{t}} > K) = \sum_j 1_{F_{\bar{t},\bar{t}} > K} \binom{\bar{t}}{j} \hat{\pi}^j (1 - \hat{\pi})^{\bar{t}-j}$ and (24) into (23) gives the European call value as:

$$C_0 = r^{-\bar{t}} \left\{ F_{0,\bar{t}} \hat{P}_0(F_{\bar{t},\bar{t}} > K) - K P_0(F_{\bar{t},\bar{t}} > K) \right\}. \quad (25)$$

In words, the forward/futures price and strike price are each multiplied by a probability of finishing in-the-money, and then their difference is discounted for time. These probabilities can be expressed in terms of binomial distribution functions, which have the advantage of being tabulated and furthermore, are well-approximated by normal distribution functions. Let a be the minimum number of up jumps required to finish in-the-money, i.e. a is the smallest nonnegative integer such that $F_{0,\bar{t}}u^a d^{\bar{t}-a} > K$, or equivalently, $a > \frac{\ln(K/F_{0,\bar{t}}d^{\bar{t}})}{\ln(u/d)}$. Then:

$$P_0(F_{\bar{t},\bar{t}} > K) = \sum_{j=a}^{\bar{t}} \binom{\bar{t}}{j} \pi^j (1-\pi)^{\bar{t}-j} = \sum_{i=0}^{\bar{t}-a} \binom{\bar{t}}{i} (1-\pi)^i \pi^{\bar{t}-i},$$

where $i = \bar{t} - j$. Letting $B(k; n, p) \equiv \sum_{i=0}^k \binom{n}{i} p^i (1-p)^{n-i}$ be the binomial distribution function, we have: $P_0(F_{\bar{t},\bar{t}} > K) = B(\bar{t} - a; \bar{t}, 1 - \pi)$, and similarly, $\hat{P}_0(F_{\bar{t},\bar{t}} > K) = B(\bar{t} - a; \bar{t}, 1 - \hat{\pi})$. Substituting into (25) gives the binomial version of Black's formula:

$$C_0 = r^{-\bar{t}} \{F_{0,\bar{t}} B(\bar{t} - a; \bar{t}, 1 - \hat{\pi}) - K B(\bar{t} - a; \bar{t}, 1 - \pi)\}. \quad (26)$$

This completes the example.

Now consider an alternative European-style claim, where there is a sequence of time and path-contingent cash flows given by $X_1, X_2, \dots, X_{\bar{t}}$. Using expression (21), we can value each time's cash flow separately. The generalized contingent claim's arbitrage-free value, denoted C_t , will be the sum of these, i.e.,

$$C_0 = \sum_{t=1}^{\bar{t}} E_0 \left(\frac{X_t}{r_0 \cdot r_1 \cdot \dots \cdot r_{t-1}} \right). \quad (27)$$

II-E American-Style Claims

Finally, we value American-style claims, i.e. claims where the time at which the cash flow is paid out is selected by the owner, rather than being fixed in advanced. Consider a random payout X_t , where the time the payout is received is determined by selection of a stopping time \tilde{t} . The arbitrage-free value of the claim is given by:

$$C_0 = E_0 \left(\frac{X_{\tilde{t}}}{r_0 \cdot r_1 \cdot \dots \cdot r_{\tilde{t}-1}} \right), \quad (28)$$

where $\tilde{t} \in \{\text{stopping times on } [0, \tau]\}$. The stopping time \tilde{t} is called the *exercise date*. To see why expression (28) holds, consider the situation where the contingent claim has not been exercised by date $\bar{t} - 1$. The

cash flow from the claim at time \bar{t} is $[X_{\bar{t}}^u, X_{\bar{t}}^d]$ depending upon whether “up” or “down” is achieved. There is no early exercise decision to be made at this date.

At time $\bar{t} - 1$, if no exercise is chosen, the contingent claim’s value is given by expression (21), denoted $C_{\bar{t}-1}^n$ where the superscript n stands for not exercised, i.e.,

$$C_{\bar{t}-1}^n = r_{\bar{t}-1}^{-1} E_{\bar{t}-1} (X_{\bar{t}}). \quad (29)$$

Now, if exercise is chosen at time $\bar{t} - 1$, the contingent claim’s value is denoted by $C_{\bar{t}-1}^e$, where the superscript e stands for exercised, and it is given by:

$$C_{\bar{t}-1}^e = X_{\bar{t}-1}. \quad (30)$$

The value of the contingent claim at time $\bar{t} - 1$ is the larger of these two, i.e.,

$$\begin{aligned} C_{\bar{t}-1} &= \max [C_{\bar{t}-1}^e, C_{\bar{t}-1}^n] \\ &= \max [X_{\bar{t}-1}, r_{\bar{t}-1}^{-1} E_{\bar{t}-1} (X_{\bar{t}})]. \end{aligned} \quad (31)$$

Thus, the optimal stopping time is:

$$\tilde{t} = \begin{cases} \bar{t} - 1, & \text{if } X_{\bar{t}-1} > r_{\bar{t}-1}^{-1} E_{\bar{t}-1} (X_{\bar{t}}) \\ \bar{t}, & \text{otherwise.} \end{cases}$$

Next, consider moving back to time $\bar{t} - 2$. The contingent claim’s value at time $\bar{t} - 1$, if not exercised at time $\bar{t} - 2$, will be given by expression (28) in an “up” and “down” state, i.e., $[C_{\bar{t}-1}^u, C_{\bar{t}-1}^d]$. Consequently, we have by expression (21) that the contingent claim’s value at time $\bar{t} - 2$, if not exercised, is:

$$C_{\bar{t}-2}^n = r_{\bar{t}-2}^{-1} E_{\bar{t}-2} (C_{\bar{t}-1}). \quad (32)$$

If exercised, it equals:

$$C_{\bar{t}-2}^e = X_{\bar{t}-2}. \quad (33)$$

The claim’s value at time $\bar{t} - 2$ is the larger of these two, i.e.,

$$\begin{aligned} C_{\bar{t}-2} &= \max [C_{\bar{t}-2}^e, C_{\bar{t}-2}^n] \\ &= \max [X_{\bar{t}-2}, r_{\bar{t}-2}^{-1} E_{\bar{t}-2} (C_{\bar{t}-1})], \end{aligned} \quad (34)$$

where the optimal stopping time is expanded to:

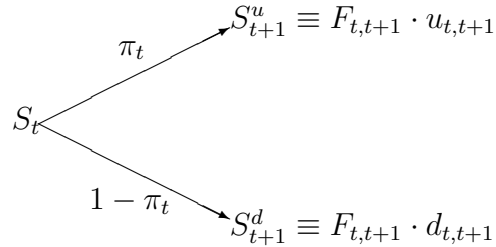
$$\tilde{t} = \begin{cases} \bar{t} - 2, & \text{if } X_{\bar{t}-2} > r_{\bar{t}-2}^{-1} E_{\bar{t}-2}(C_{\bar{t}-1}) \\ \bar{t} - 1, & \text{if } X_{\bar{t}-1} > r_{\bar{t}-2}^{-1} E_{\bar{t}-1}(X_{\bar{t}}) \text{ and not exercised earlier} \\ \bar{t}, & \text{otherwise.} \end{cases}$$

This recursive procedure is easily continued until time 0, at which date the optimal stopping time \tilde{t} will be completely specified, and the value C_0 determined. The resulting value will satisfy expression (28). This is known as Bellman's principle of optimality.

Delta hedges for these American contingent claims can be determined according to expression (10) for any time step $(t, t + 1)$ where the claim is unexercised. When the claim is exercised, delta hedging is no longer required. This then completes the general framework for contingent claims valuation and hedging. Since many contingent claims are written on the spot price of the underlying asset, the next section discusses the relationship between spot and futures prices in our model.

II-F Spot Price Process

At an arbitrary date t , the spot price process under the equivalent martingale measure is given by:



where $u_{t,t+1} > d_{t,t+1} > 0$. Our model has assumed that investors can trade futures contracts of any maturity $T \in [t, \tau]$. However, we have not yet required that investors be able to store the underlying asset from date t to $t + 1$. At date t , if an investor desires possession of the underlying asset at date $t + 1$, he can go long one futures maturing in one period. Similarly, if an investor desires a short position in the underlying asset at date $t + 1$, he can go short one futures at date t . This level of generality is useful when dealing with underlying assets which are not easily held (eg. catastrophe insurance) or stored (eg. eggs)

or easily shorted (eg. illiquid stocks). The analysis is also particularly applicable to real options, when futures contracts trade on the underlying, eg. for agricultural or mineral concerns.

In contrast, traditional option valuation models explicitly assume that the underlying asset can be held over time, where the net benefit from borrowing to buy the asset can be positive eg. for high interest rate currencies, or negative, eg. for low interest rate currencies. For the moment, we continue to assume that the underlying asset need not be storable. Define the *relative basis*, y_t , at date t by:

$$y_t \equiv \frac{r_t S_t - F_{t,t+1}}{S_t} = r_t - \frac{E_t S_{t+1}}{S_t}. \quad (35)$$

Thus, the relative basis at t is the difference between the future value, $r_t S_t$, of the spot price at $t + 1$, and the time t futures price, where this difference is expressed as a proportion of the current spot price. The relative basis can be positive, zero, or negative. Re-arranging the first equality in the above equation implies:

$$F_{t,t+1} = S_t(r_t - y_t). \quad (36)$$

Expression (36) is often referred to as the “cost of carry relation”. This name is justified below. Substituting (36) into the above spot price process implies that it can be reformulated as:

$$\begin{array}{c} \nearrow^{\pi_t} S_{t+1}^u = S_t \cdot U_t \\ S_t \\ \searrow_{1-\pi_t} S_{t+1}^d = S_t \cdot D_t \end{array}$$

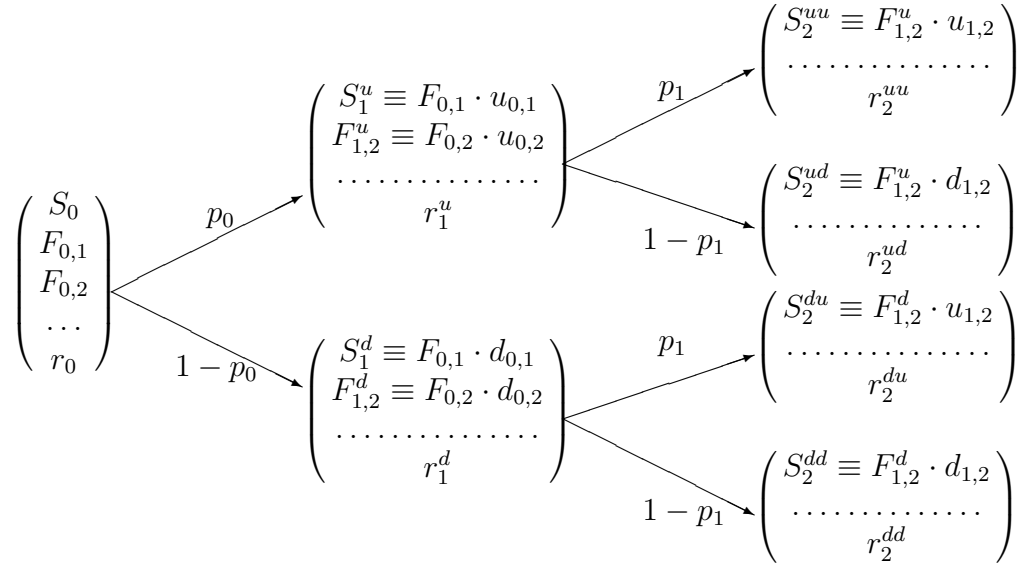
where $U_t \equiv \frac{S_{t+1}^u}{S_t} = (r_t - y_t)u_{t,t+1}$ and $D_t \equiv \frac{S_{t+1}^d}{S_t} = (r_t - y_t)d_{t,t+1}$. At date t , the expected spot price appreciation under the martingale measure is the riskless rate less relative basis:

$$\pi_t U_t + (1 - \pi_t) D_t \equiv \frac{E_t S_{t+1}^u}{S_t} = r_t - y_t,$$

by (35). Consequently, the risk-neutral probabilities are determined by r_t , y_t , and the spot price process parameters:

$$\pi_t = \frac{r_t - y_t - D_t}{U_t - D_t}.$$

A computational issue which arises for derivative securities whose payoff depends directly on the spot price is that in general the spot price tree does not recombine. The tree below illustrates the point by setting $\tau = 2$ for simplicity:



The tree will recombine if

$$F_{1,2}^u d_{1,2} = F_{1,2}^d u_{1,2}, \quad r_2^{ud} = r_2^{du},$$

or equivalently:

$$u_{0,2} d_{1,2} = d_{0,2} u_{1,2}, \quad r_2^{ud} = r_2^{du}.$$

If this equation does not hold for equal length periods, the period lengths can sometimes be chosen so that the tree recombines.

We now assume as in traditional models that **the underlying asset can be stored or short sold without restriction**. Let δ_t denote the “dividend yield” of the underlying asset at date t , achieved by purchasing the asset at date t and storing it to date $t + 1$. This dividend yield is the amount of dollars received at date $t + 1$, expressed as a fraction of the time t spot price. If the underlying asset is a currency, $\delta_t > 0$ is the foreign riskfree rate. If the underlying asset is a non-dividend paying stock, then $\delta_t = 0$. If the underlying asset is a commodity, $\delta_t < 0$ may represent proportional storage costs. Using the standard

cost-of-carry argument⁵, we have:

$$F_{t,t+1} = S_t(r_t - \delta_t). \quad (37)$$

Comparing with (36) implies that the relative basis is this dividend yield:

$$y_t = \delta_t. \quad (38)$$

The analysis easily generalizes to random dividend yields of δ_t^u if the underlying asset jumps up, and of δ_t^d if the underlying jumps down. The generalized expression is $F_{t,t+1} = S_t(r_t - E_t\delta_t)$.

We have shown how a binomial process on a term structure of futures leads to a binomial process on the spot price of the underlying asset. The traditional model reverses this analysis. It starts with a binomial process on spot prices and derives the resultant binomial process for futures prices of every maturity. Both models imply the existence of unique pseudo-probabilities, as proved next. No arbitrage between trading in the spot market for the underlying asset and riskless borrowing and lending opportunities implies:

$$U_t > r_t - \delta_t > D_t.$$

Consequently, there exists a unique number $\tilde{\pi}_t \in (0, 1)$ such that:

$$\tilde{\pi}_t U_t + (1 - \tilde{\pi}_t) D_t = r_t - \delta_t.$$

Solving for this pseudo-probability gives:

$$\tilde{\pi}_t = \frac{r_t - \delta_t - D_t}{U_t - D_t}. \quad (39)$$

Considering the futures price, $F_{t,T}$, as a “derivative” on S_t , the standard dynamic replication argument gives that at time $T - 1$:

$$0 = [\tilde{\pi}_{T-1}(S_T^u - F_{T-1,T}) + (1 - \tilde{\pi}_{T-1})(S_T^d - F_{T-1,T})]/r_{T-1},$$

or:

$$F_{T-1,T} = \tilde{E}_{T-1}(S_T).$$

⁵The standard cost of carry argument is that buying the asset on the spot market, borrowing to finance the purchase, and storing the asset is equivalent to purchasing the asset forward.

At time $T - 2$,

$$0 = [\tilde{\pi}_{T-2}(F_{T-1,T}^u - F_{T-2,T}) + (1 - \tilde{\pi}_{T-2})(F_{T-1,T}^d - F_{T-2,T})]/r_{T-2},$$

or

$$F_{T-2,T} = \tilde{E}_{T-2}(F_{T-1,T}).$$

Continuing, at time t :

$$F_{t,T} = \tilde{\pi}_t F_{t+1,T}^u + (1 - \tilde{\pi}_t) F_{t+1,T}^d.$$

Substituting $F_{t+1,T}^u = F_{t,T} u_{t,T}$ and $F_{t+1,T}^d = F_{t,T} d_{t,T}$ and algebra gives:

$$\tilde{\pi}_t = \pi_t \text{ for all times } t.$$

Thus, the models can be specified to imply the same prices and hedge ratios of derivative securities. The differences arise in the assumptions. The traditional approach requires that the underlying asset be storable and shortable, whereas our model alternatively assumes that futures contracts of every maturity can be traded. The traditional model characterizes the pseudo-probabilities in terms of the spot price process, whereas our model characterizes these probabilities using the futures price process of any maturity. The next section compares the two models for various categories of underlying assets.

III Existing Models

This section reviews the existing models for pricing options on various commodities and shows how they fit into the above analysis.

III-A Stock Options

All listed stock options are American-style. As a result, the standard model for valuing stock options is the binomial model of Cox, Ross and Rubinstein[13] (henceforth CRR). In the CRR model, the underlying stock is assumed to be storable and shortable without restriction. If short sales restrictions such as the uptick rule are binding, then the CRR model value becomes an upper bound on call prices and a lower bound on put prices. The spot price relatives U and D , the spot rate r , and the dividend yield δ are usually

assumed constant, in order that the tree recombine. The CRR binomial model formula can be obtained from the binomial version of Black's formula (26) using the fact that $F_{0\bar{t}} = S_0 (r - \delta)^{\bar{t}}$. Substitution gives:

$$\begin{aligned} C_0 &= r^{-\bar{t}} \left\{ S_0 (r - \delta)^{\bar{t}} B(\bar{t} - a; \bar{t}, 1 - \hat{\pi}) - KB(\bar{t} - a; \bar{t}, 1 - \pi) \right\} \\ &\approx S_0 (1 - \delta)^{\bar{t}} B(\bar{t} - a; \bar{t}, 1 - \hat{\pi}) - Kr^{-\bar{t}} B(\bar{t} - a; \bar{t}, 1 - \pi), \end{aligned}$$

where $\pi = \frac{1-d}{u-d} = \frac{r-\delta-D}{U-D}$ and where $\hat{\pi} \equiv \pi u = \frac{r-\delta-D}{U-D} \frac{U}{r-\delta}$. When the parameters U , D , r , and δ are time-dependent, unequal time steps can often be used to achieve path-independence. When constant dollar dividends are paid aperiodically (eg. quarterly), path-independence is achieved by escrowing out the dividends and imposing a binomial process on the escrowed stock price. At any exercise date, the exercise value is computed by adding back the present value of the dividends to this escrowed price⁶.

There are no futures contracts trading on individual stocks. Consequently, to implement our model for stock options, forward prices can be used instead of of futures prices, where these forward prices are implied out of (preferably European) put and call stock options. If forward prices are used in our framework, one should also assume that interest rates are deterministic. Since forward prices equal futures prices under deterministic interest rates, our analysis goes through with F interpreted as the forward price. When interest rates are stochastic, any correlation⁷ between spot interest rates and futures prices of stocks leads to theoretical differences between forward and futures prices.

One important difference in the two approaches is that our model does not impose any assumptions on dividends whatsoever. Furthermore, European options can be valued knowing only the process for the forward maturing with the option, while American options require a maturity corresponding to each potential exercise date. Finally, there is no counterpart to the uptick rule for options, forwards, or futures, so this aspect of the frictionless market assumption is more reasonable.

III-B Stock Index Options

The traditional model for valuing index options is also CRR. When compared to valuing options on individual stocks, the implicit assumption in index option models that all of the stocks in the index can

⁶For calls, the cum-dividend stock price is used to compute exercise value, while for puts, the ex-dividend price is used.

⁷In the one factor model presented, there is perfect correlation between spot interest rates and futures prices.

be immediately bought or short-sold without restriction is less reasonable. In contrast to stock options, there is a very liquid market in futures on stock indices. Furthermore, American options on stock index futures trade. For options on the spot price of the index, we suggest using our model to characterize the risk-neutral probabilities, π_t , in terms of the implied spot price parameters, U_t and D_t . For options on stock index futures, these probabilities can be characterized in terms of the assumed futures price parameters $u_{t,T}$ and $d_{t,T}$, where T is any sufficiently long maturity. When the maturity used is the same as that of the underlying futures, our model reduces to a binomial version of Black's model.

III-C Currencies

As in the case of indices, options exist on both the spot and futures exchange rates of currencies. The standard models are Garman and Kohlhagen[22] for European spot options, Black's model for European futures options, and CRR for American options. With the exception of Black's model, these models assume that domestic and foreign interest rates are constant. However, interest rates are about as volatile as the exchange rates themselves. Since our term structure model assumes stochastic domestic interest rates and makes no assumptions on foreign interest rate behavior, it captures this important element of currency option pricing.

III-D Commodities

As in the case of currencies, options exist on both the spot and futures prices of commodities. For commodities such as oil however, the spot market is not well-developed, which makes shorting the physical commodity difficult. Furthermore, convenience yields, which are of significant economic magnitude, are difficult to observe directly, and are even harder to forecast. Nevertheless, commodity option models such as Gibson and Schwartz[25] specify the spot price process directly, requiring the user to input current spot prices and convenience yields.

Commodities such as oil enjoy a rich term structure of futures prices. This term structure can be used in our model to value and hedge derivatives without the need to observe or forecast convenience yields. Since futures are easily shorted, this aspect of the frictionless market ideal is less troubling than for the conventional models described above. Options on futures are easily handled in the specialization

of our model to the binomial version of Black[3]. Furthermore, exotic options, such as options on maturity spreads, are also easily handled in our term structure model.

III-E Interest Rate Options

There exist various types of interest rate options such as caps, floors, collars, etc. Standard models are obtainable as special cases of Heath, Jarrow, and Morton[28]. To get the HJM model, one can use the traditional spot price process approach by letting $P(t)$ be the **vector** of zero coupon bond prices of every maturity. Since the spot rate at each date is determined by the price of the shortest maturity zero coupon bond, this additional restriction is also added to the tree:

$$\begin{array}{c}
 \left(\begin{array}{c} P(t, \tau) \\ P(t, \tau - 1) \\ \vdots \\ P(t, t + 1) \\ \dots\dots\dots \\ r_t = \frac{1}{P(t, t+1)} \end{array} \right) \\
 \swarrow p_t \\
 \left(\begin{array}{c} P^u(t + 1, \tau) \\ P^u(t + 1, \tau - 1) \\ \vdots \\ P^u(t + 1, t + 2) \\ 1 \\ \dots\dots\dots \\ r_{t+1}^u = \frac{1}{P^u(t+1, t+2)} \end{array} \right) \\
 \searrow 1 - p_t \\
 \left(\begin{array}{c} P^d(t + 1, \tau) \\ P^d(t + 1, \tau - 1) \\ \vdots \\ P^d(t + 1, t + 2) \\ 1 \\ \dots\dots\dots \\ r_{t+1}^d = \frac{1}{P^d(t+1, t+2)} \end{array} \right)
 \end{array}$$

The no arbitrage condition (39) is satisfied for each bond maturity. This pseudo-probability must be the same across all bond maturities for the zero-coupon bond price curve to be arbitrage-free. This is the same argument used to get Proposition 1 (see Jarrow[37]).

It is possible to use the futures price curve approach here as well. It entails specifying a term structure of futures prices for *each maturity* of the underlying zero coupon bond. The generalization of the no arbitrage condition in Proposition 1 would be that the pseudo-probabilities implied from the evolution of the futures price curve must be the same across all delivery dates for a particular bond maturity *and* across all possible bond maturities. This approach may be most reasonable for Eurodollar or Treasury bond and

note futures, where a variety of maturities trade actively.

IV Implementation

This section discusses the implementation of our futures term structure model. We assume that there exists a term structure of contracts with current futures prices given by the vector $[S_0, F_{0,1}, F_{0,2}, \dots, F_{0,\tau}]$. Another set of inputs to the model are the current spot rate, r_0 , and all future state contingent spot rates. We refer the reader to Jarrow[37] for guidance on estimating the spot rate process. The final set of inputs to the model are a lower diagonal matrix of up price relatives, $u_{t,T}$:

$$u_{t,T} = \begin{pmatrix} u_{0,1} & 0 & 0 & \dots & 0 \\ u_{0,2} & u_{1,2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{0,\tau} & u_{1,\tau} & u_{2,\tau} & \dots & u_{\tau,\tau} \end{pmatrix}$$

and a lower diagonal matrix of down price relatives, $d_{t,T}$:

$$d_{t,T} = \begin{pmatrix} d_{0,1} & 0 & 0 & \dots & 0 \\ d_{0,2} & d_{1,2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{0,\tau} & d_{1,\tau} & d_{2,\tau} & \dots & d_{\tau,\tau} \end{pmatrix}.$$

Letting Δ be the length of each time step, we reparametrize the price relatives $u_{t,T}$ and $d_{t,T}$ by:

$$u_{t,T} = \exp\left(\mu_{t,T}\Delta + \sigma_{t,T}\sqrt{\Delta}\right).$$

and

$$d_{t,T} = \exp\left(\mu_{t,T}\Delta - \sigma_{t,T}\sqrt{\Delta}\right),$$

where $p_t = \frac{1}{2} + \eta_t\Delta$ for some $\eta_t \in \mathfrak{R}$. p_t are the empirical probabilities. At time $t + 1$, the log price relatives are:

$$\ln(F_{t+1,T}/F_{t,T}) = \begin{cases} \mu_{t,T}\Delta + \sigma_{t,T}\sqrt{\Delta} & \text{if "up"} \\ \mu_{t,T}\Delta - \sigma_{t,T}\sqrt{\Delta} & \text{if "down"} \end{cases}. \quad \text{Simple algebra gives:}$$

$$E^p(\ln(F_{t+1,T}/F_{t,T})) = \mu_{t,T}\Delta + O\left(\Delta^{\frac{3}{2}}\right), \quad \sqrt{\text{Var}^p(\ln(F_{t+1,T}/F_{t,T}))} = \sigma_{t,T}\sqrt{\Delta} + O\left(\Delta^{\frac{3}{2}}\right),$$

where $\lim_{\Delta \downarrow 0} \frac{O\left(\Delta^{\frac{3}{2}}\right)}{\Delta} = 0$. Thus, $\mu_{t,T}$ has the interpretation as the instantaneous expected growth rate in $F_{t,T}$

and $\sigma_{t,T}$ has the interpretation as the instantaneous expected volatility of relative changes in $F_{t,T}$ ⁸.

For valuation, we need to understand the process under the risk-neutral probability $\pi_t \equiv \frac{1-d_{t,T}}{u_{t,T}-d_{t,T}}$. We have:

$$\pi_t = \frac{1 - \exp(\mu_{t,T}\Delta - \sigma_{t,T}\sqrt{\Delta})}{\exp(\mu_{t,T}\Delta + \sigma_{t,T}\sqrt{\Delta}) - \exp(\mu_{t,T}\Delta - \sigma_{t,T}\sqrt{\Delta})}.$$

Since we have a free parameter, without loss of generality, we can set $\pi_t = \frac{1}{2}$ for all t . Then:

$$\exp(-\mu_{t,T}\Delta) = \frac{1}{2} \left[\exp(\sigma_{t,T}\sqrt{\Delta}) + \exp(-\sigma_{t,T}\sqrt{\Delta}) \right].$$

Thus:

$$1 - \mu_{t,T}\Delta + O\left(\Delta^{\frac{3}{2}}\right) = \frac{1}{2} \left[1 + \sigma_{t,T}\sqrt{\Delta} + \frac{\sigma_{t,T}^2}{2}\Delta + 1 - \sigma_{t,T}\sqrt{\Delta} + \frac{\sigma_{t,T}^2}{2}\Delta + O\left(\Delta^{\frac{3}{2}}\right) \right].$$

This gives $\mu_{t,T}\Delta = -\frac{\sigma_{t,T}^2}{2}\Delta + O\left(\Delta^{\frac{3}{2}}\right)$. In summary, the valuation process is approximated by:

$$u_{t,T} = \exp\left(-\frac{\sigma_{t,T}^2}{2}\Delta + \sigma_{t,T}\sqrt{\Delta}\right).$$

and

$$d_{t,T} = \exp\left(-\frac{\sigma_{t,T}^2}{2}\Delta - \sigma_{t,T}\sqrt{\Delta}\right),$$

with $\pi_t = \frac{1}{2}$. This process depends only on the instantaneous volatility of relative changes in futures prices, and is arbitrage-free.

When there is a term structure of futures options trading, these instantaneous volatilities can be obtained from a term structure of implied volatilities. If no futures options trade, one can use a term structure of options on the spot. The implied volatility can be obtained via any of the previously discussed models. If no options are available, historical volatility can be used, if one assumes that $u_{t,T}$ and $d_{t,T}$ depend on their arguments only through their difference $T - t$. In any case, given a vector of volatilities $[\sigma_{01}, \sigma_{02}, \dots, \sigma_{0\tau}]$, then by definition, the *forward volatilities* of futures $\sigma_{t,T}$ satisfy the following relationship:

$$\sigma_{0,t}^2 t + \sigma_{t,T}^2 (T - t) = \sigma_{0,T}^2 T,$$

⁸Given certain technical restrictions on $(\mu_{t,T}, \sigma_{t,T})$, this process will converge weakly, as $\Delta \downarrow 0$, to:

$$\frac{dF_{t,T}}{F_{t,T}} = \left(\mu_{t,T} + \frac{\sigma_{t,T}^2}{2} \right) dt + \sigma_{t,T} dW_t,$$

where $\{W_t, t \in [0, \tau]\}$ is a standard Brownian motion (see He[29]).

i.e.

$$\sigma_{t,T} \equiv \sqrt{\frac{\sigma_{0,T}^2 T - \sigma_{0,t}^2 t}{T - t}}.$$

V Summary

This paper provided an analytic synthesis of the option pricing literature, using a term structure of futures prices approach. Postulating a process for the evolution of the term structure of futures prices, it is shown how to price derivative securities in an arbitrage-free manner. Complete markets are assumed.

This approach generalizes the traditional methodology by relaxing the assumption of a frictionless spot market (or even the existence of a spot market) and that the underlying commodity is storable. Thus, this method is consistent with short sale constraints in the spot market for the underlying commodity. When short sale restrictions are removed, the traditional option pricing models are shown to be obtainable as special cases. This includes the binomial model of CRR, as well as its applications to index options, currency options and commodity options. The new interest rate options models of HJM are also shown to be a subset of this framework. A brief discussion of how to empirically implement the model is also provided. References are given to reviews of the empirical literature and historic surveys of the model development.

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