

Time-Changed Lévy Processes and Option Pricing

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Overview

- Apply **stochastic time change to Lévy processes**:
 - Lévy processes can generate non-normal return innovations.
 - Stochastic time changes generate stochastic volatility.
 - Correlation between the two captures the “leverage effect”.
- ⇒ Our framework encompasses almost all extant option pricing models and points to new directions for designing new models.
- What we do:
 - Derive the *generalized* characteristic function (CF) of the time-changed Lévy process.
 - Propose FFT algorithms to price European-style options via this generalized CF.
 - Specification analysis (model design, examples).

Related Literature

- Affine jump-diffusion stochastic volatility models of Duffie, Pan, Singleton (2000):
 - Finite-activity compound Poisson jumps: Jumps are regarded as rare events.
Evidence: Asset prices display many small jumps on a finite time scale:
⇒ Infinite-activity jumps may perform better.
 - Affine volatility dynamics:
A linear-quadratic structure is more flexible for incorporating correlations.
- Time-changed Lévy processes:
 - The Lévy process can accommodate both low and high frequency jumps.
 - Stochastic time change can accommodate both affine and quadratic volatility dynamics.
 - Stochastic volatility can be driven by stochastic diffusion variance or stochastic jump arrival rate, or both.
 - Flexible correlations between return and volatility are possible.

Lévy Processes and the Lévy-Khintchine Formula

- X : a d -dimensional Lévy process: rcll (right continuous with left limits), stationary independent increments, stochastic continuity.
- Defined on a probability space (Ω, \mathcal{F}, P) endowed with a standard complete filtration $\mathbf{F} = \{\mathcal{F}_t | t \geq 0\}$.

- The Lévy-Khintchine formula for the characteristic function of X_t :

$$\phi_{X_t}(\theta) \equiv E \left[e^{i\theta^\top X_t} \right] = e^{-t\Psi_x(\theta)}, \quad t \geq 0, \theta \in \mathbb{R}^d$$

where the *characteristic exponent* $\Psi_x(\theta)$, $\theta \in \mathbb{R}^d$, is given:

$$\Psi_x(\theta) \equiv -i\mu^\top \theta + \frac{1}{2}\theta^\top \Sigma \theta - \int_{\mathbb{R}^d - \{0\}} \left(e^{i\theta^\top x} - 1 - i\theta^\top x 1_{|x|<1} \right) \Pi(dx).$$

- *Lévy characteristics*: (μ, Σ, Π) , with μ a d -vector, Σ a semi-definite symmetric $d \times d$ matrix, and $\Pi : \mathbb{R}^d - \{0\} \mapsto \mathbb{R}^+$ a measure with some integrability properties.
- The *generalized* Fourier transform (or CF): $\theta \in \mathcal{D} \subseteq \mathbb{C}^d$.

Stochastic Time Change

- Let $t \rightarrow T_t, t \geq 0$ be an increasing rcll stochastic process such that for each fixed t , the random variable T_t is a stopping time with respect to \mathbf{F}_t .
- T_t is finite P -a.s. for all $t \geq 0$ and $T_t \rightarrow \infty$ as $t \rightarrow \infty$.
- The family of stopping times $\{T_t; t \geq 0\}$ defines a **stochastic time change**.
- Define Y by evaluating X at T , i.e.

$$Y_t \equiv X_{T_t}, \quad t \geq 0.$$

- We use the *time changed Lévy process*, Y , as the source of all uncertainty in the economy.

Lévy Subordinators

- The random time T_t can be modeled as a nondecreasing semimartingale:

$$T_t = \alpha_t + \int_0^t \int_0^\infty z \mu(dz, ds)$$

- Example: Lévy subordinators as random time changes:

$$T_t = \int_0^t \int_0^\infty z \Pi(dz) ds$$

- Bertoin (1999): A Lévy process time changed by a Lévy subordinator yields a new Lévy process.
- We can always suppress the subordinator by directly specifying the appropriate Lévy characteristics for X .

Business Activity Rates

- We focus on locally predictable time changes:

$$T_t = \alpha_t = \int_0^t v(s_-) ds.$$

- We call $v(t)$ the **instantaneous (business) activity rate**.
- Economic Interpretations:
 - t — calendar time; T — business time.
 - $v(t)$ captures the intensity of the business activity at calendar time t .
- If X is SBM, then $v(t)$ is the instantaneous variance rate of $Y_t \equiv X_{T_t}$.
- if X is a pure jump Lévy process, then $v(t)\Pi(dy)$ is the arrival rate of a jump of size y in Y .
- Although T_t is assumed to be continuous, $v(t)$ can jump.

Encompassing Extant Models

- Heston (1993): $X_t = W_t$, $v(t)$ follows a mean-reverting square-root process.
- Hull and White (1987): $X_t = W_t$, $v(t)$ follows an independent log-normal process.
- Affine jump-diffusion of Duffie, Pan, Singleton (2000):
 X_t is diffusion plus compound Poisson jumps; $v(t)$ follows affine dynamics.

Independent Time Change

- $v(t)$ evolves independently of X_t , e.g. Hull and White (1987), no leverage effect.
- The characteristic function of the time changed Lévy process $Y_t = X_{T_t}$ is

$$\begin{aligned}\phi_y(\theta) &\equiv E e^{i\theta^\top X_{T_t}} = E \left[E \left[e^{i\theta^\top X_u} \middle| \mathcal{F}_t^v \right] \right] = E \left[E \left[e^{i\theta^\top X_u} \middle| T_t = u \right] \right] \\ &= E e^{-T_t \Psi_x(\theta)} = \mathcal{L}_T(\Psi_x(\theta))\end{aligned}$$

which is the Laplace transform of the stochastic time $T_t = \int_0^t v(s-) ds$, evaluated at the characteristic exponent of X .

- To obtain the Laplace transform of T in closed form, consider its specification in terms of the activity rate:

$$\mathcal{L}_T(\lambda) \equiv E \left[\exp \left(-\lambda \int_0^t v(s-) ds \right) \right]$$

analogous to the bond pricing formula if we regard $v(t)$ as analogous to the instantaneous spot interest rate. Hence, we can “borrow” closed form solutions for zero coupon bonds that arise under affine, quadratic term structure models, etc.

The General Case of Correlated Time Changes

- More generally, the generalized CF of $Y_t \equiv X_{T_t}$ under measure P can be represented as the “Laplace transform” of T_t under a new **complex-valued measure** $Q(\theta)$, evaluated at the characteristic exponent $\Psi_x(\theta)$ of X_t ,

$$\phi_{Y_t}(\theta) \equiv E \left[e^{i\theta^\top Y_t} \right] = E^\theta \left[e^{-T_t \Psi_x(\theta)} \right] \equiv \mathcal{L}_{T_t}^\theta (\Psi_x(\theta)). \quad (1)$$

- For each $\theta \in \mathcal{D}$, $Q(\theta)$ is absolutely continuous with respect to P and is defined by

$$E \frac{dQ(\theta)}{dP} \Big|_{\mathcal{F}_{T_t}} \equiv M_t(\theta) \equiv \exp \left(i\theta^\top Y_t + T_t \Psi_x(\theta) \right), \quad \theta \in \mathcal{D}, \quad (2)$$

which is a *complex valued* P -martingale with respect to $\{\mathcal{F}_{T_t} | t \geq 0\}$, for each $\theta \in \mathcal{D}$.

Intuition and Theorem Proof

- Why is $M_t(\theta) \equiv E \frac{dQ(\theta)}{dP} |_{\mathcal{F}_{T_t}} = \exp(i\theta^\top Y_t + T_t \Psi_x(\theta))$, $\theta \in \mathcal{D}$ a P martingale?
 - Recall the familiar Wald martingale defined on a Lévy process
$$Z_t(\theta) \equiv \exp(i\theta^\top X_t + t\Psi_x(\theta)).$$
 - Time change (i.e. replacing t by T_t) preserves the martingality.

- Theorem proof:

$$\begin{aligned} E \left[e^{i\theta^\top Y_t} \right] &= E \left[e^{i\theta^\top Y_t + T_t \Psi_x(\theta) - T_t \Psi_x(\theta)} \right] \\ &= E \left[M_t(\theta) e^{-T_t \Psi_x(\theta)} \right] = E^\theta \left[e^{-T_t \Psi_x(\theta)} \right] \equiv \mathcal{L}_{T_t}^\theta (\Psi_x(\theta)). \end{aligned}$$

- The complex-valued measure loses its probabilistic interpretation, but the mathematical operation remains valid.
- Under measure $Q(\theta)$, we can take “expectations” as if there is no correlation \Rightarrow **leverage-neutral measure**.

Asset Pricing under Time-Changed Lévy Processes

- Let S_t be the time- t price of a limited liability asset under statistical measure:

$$S_t \equiv S_0 e^{\vartheta^\top Y_t}, t \geq 0, \text{ for given } S_0 > 0, \text{ where recall } Y_t \equiv X_{T_t}.$$

- The generalized CF of the log return $s_t \equiv \ln(S_t/S_0)$ is

$$\phi_s(\theta) \equiv E [e^{i\theta s_t}] = E [e^{i\theta\vartheta^\top Y_t}] = \mathcal{L}_T^{\theta\vartheta}(\Psi_x(\theta\vartheta)), \quad t \geq 0, \theta\vartheta \in \mathcal{D} \subseteq \mathbb{C}^d.$$

- Let $F_t(M)$ be the M maturity forward price at time $t \in [0, M]$. To value most European-style claims maturing at M , specify F as a positive martingale

$$F_t(M) \equiv F_0(M) e^{\vartheta^\top Y_t + T_t^\top \Psi_x(-i\vartheta)}, \quad t \in [0, M]$$

under an M -forward measure. T_t is now a vector of stochastic clocks so that $e^{\vartheta^\top Y_t + T_t^\top \Psi_x(-i\vartheta)}$ is an exponential martingale.

- The generalized CF of the terminal log return $f_M \equiv \ln(F_M(M)/F_0(M))$ is

$$\phi_f(\theta) \equiv E [e^{i\theta f_M}] = \mathcal{L}_T^{\theta\vartheta}(\Psi_x(\theta\vartheta) - i\theta\Psi_x(-i\vartheta)).$$

Specification Analysis Ia

Lévy processes and Characteristic Exponents:

Lévy Components	Lévy Density $\Pi(dx)/dx$	Characteristic Exponent $\Psi(\theta)$
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Pure Continuous Lévy component

$\mu t + \sigma W_t$	—	$-i\mu\theta + \frac{1}{2}\sigma^2\theta^2$
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Finite Activity Pure Jump Lévy components

Merton (76)	$\lambda \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left(-\frac{(x-\alpha)^2}{2\sigma_j^2}\right)$	$\lambda \left(1 - e^{i\theta\alpha - \frac{1}{2}\sigma_j^2\theta^2}\right)$
Kou (99)	$\lambda \frac{1}{2\eta} \exp\left(-\frac{ x-k }{\eta}\right)$	$\lambda \left(1 - e^{i\theta k \frac{1-\eta^2}{1+\theta^2\eta^2}}\right)$
Eraker (2001)	$\lambda \frac{1}{\eta} \exp\left(-\frac{x}{\eta}\right)$	$\lambda \left(1 - \frac{1}{1-i\theta\eta}\right)$

Specification Analysis Ib

Lévy processes and Characteristic Exponents: *Infinite Activity Jumps*

Lévy Components	Lévy Density $\Pi(dx)/dx$	Characteristic Exponent $\Psi(\theta)$
NIG	$e^{\beta x} \frac{\delta \alpha}{\pi x } K_1(\alpha x)$	$-\delta \left[\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + i\theta)^2} \right]$
Hyperbolic	$\frac{e^{\beta x}}{ x } \left[\int_0^\infty \frac{e^{-\sqrt{2y+\alpha^2} x }}{\pi^2 y (J_{ \lambda }^2(\delta\sqrt{2y}) + Y_{ \lambda }^2(\delta\sqrt{2y}))} dy \right. \\ \left. + 1_{\lambda \geq 0} \lambda e^{-\alpha x } \right]$	$-\ln \left[\frac{\sqrt{\alpha^2 - \beta^2}}{\sqrt{\alpha^2 - (\beta + i\theta)^2}} \right]^\lambda \left[\frac{K_\lambda(\delta\sqrt{\alpha^2 - (\beta + i\theta)^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})} \right]$
CGMY	$\begin{cases} C e^{-G x } x ^{-Y-1}, & x < 0, \\ C e^{-M x } x ^{-Y-1}, & x > 0 \end{cases}$	$C \Gamma(-Y) [M^Y - (M - i\theta)^Y + G - (G + i\theta)^Y]$
VG	$\frac{\mu_\pm^2 \exp\left(-\frac{\mu_\pm}{v_\pm} x \right)}{v_\pm x }$	$\lambda \ln \left(1 - iu\alpha + \frac{1}{2} \sigma_j^2 u^2 \right)$
LS	$(\mu_\pm = \sqrt{\frac{\alpha^2}{4\lambda^2} + \frac{\sigma_j^2}{2}} \pm \frac{\alpha}{2\lambda}, v_\pm = \mu_\pm^2 / \lambda)$ $c x ^{-\alpha-1}, \quad x < 0$	$-c \Gamma(-\alpha) (i\theta)^\alpha$

Specification Analysis IIa

Activity Rate Dynamics and the Laplace Transform :

Affine: Duffie, Pan, Singleton (2000)

Activity Rate Specification $v(t)$	Laplace Transform $\mathcal{L}_{T_t}(\lambda) \equiv E [e^{-\lambda T_t}]$
$v(t) = \mathbf{b}_v^\top Z_t + c_v,$ $\mu(Z_t) = a - \kappa Z_t,$ $\left[\sigma(Z_t) \sigma(Z_t)^\top \right]_{ii} = \alpha_i + \beta_i^\top Z_t,$ $\left[\sigma(Z_t) \sigma(Z_t)^\top \right]_{ij} = 0, \quad i \neq j,$ $\gamma(Z_t) = a_\gamma + \mathbf{b}_\gamma^\top Z_t.$	$\exp(-\mathbf{b}(t)^\top z_0 - c(t)),$ $\mathbf{b}'(t) = \lambda \mathbf{b}_v - \kappa^\top \mathbf{b}(t) - \frac{1}{2} \beta \mathbf{b}(t)^2$ $- \mathbf{b}_\gamma (\mathcal{L}_q(\mathbf{b}(t)) - 1),$ $c'(t) = \lambda c_v + \mathbf{b}(t)^\top a - \frac{1}{2} \mathbf{b}(t)^\top \alpha \mathbf{b}(t)$ $- a_\gamma (\mathcal{L}_q(\mathbf{b}(t)) - 1),$ $\mathbf{b}(0) = 0, c(0) = 0.$

Specification Analysis IIb

Activity Rate Dynamics and the Laplace Transform :
Generalized Affine: Filipovic (2001)

Activity Rate Specification $v(t)$

$$\begin{aligned} \mathcal{A}f(x) &= \frac{1}{2}\sigma^2 x f''(x) + (a' - \kappa x) f'(x) \\ &\quad + \int_{\mathbb{R}_0^+} (f(x+y) - f(x) + f'(x)(1 \wedge y)) \\ &\quad (m(dy) + x\mu(dy)), \\ a' &= a + \int_{\mathbb{R}_0^+} (1 \wedge y) m(dy), \\ \int_{\mathbb{R}_0^+} [(1 \wedge y) m(dy) + (1 \wedge y^2) \mu(dy)] &< \infty. \end{aligned}$$

Laplace Transform $\mathcal{L}_{T_t}(\lambda) \equiv E [e^{-\lambda T_t}]$

$$\begin{aligned} &\exp(-b(t)v_0 - c(t)), \\ b'(t) &= \lambda - \kappa b(t) - \frac{1}{2}\sigma^2 b(t)^2 \\ &\quad + \int_{\mathbb{R}_0^+} (1 - e^{-yb(t)} - b(t)(1 \wedge y)) \mu(dy), \\ c'(t) &= ab(t) + \int_{\mathbb{R}_0^+} (1 - e^{-yb(t)}) m(dy), \\ b(0) &= c(0) = 0. \end{aligned}$$

Specification Analysis IIc

Activity Rate Dynamics and the Laplace Transform

Quadratic: Leippold and Wu (2002)

Activity Rate Specification

$v(t)$

Laplace Transform

$\mathcal{L}_{T_t}(\lambda) \equiv E [e^{-\lambda T_t}]$

$$\begin{aligned}\mu(Z) &= -\kappa Z, \quad \sigma(Z) = I, \\ v(t) &= Z_t^\top A_v Z_t + \mathbf{b}_v^\top Z_t + c_v.\end{aligned}$$

$$\begin{aligned}&\exp \left[-z_0^\top A(t) z_0 - \mathbf{b}(t)^\top z_0 - c(t) \right], \\ A'(t) &= \lambda A_v - A(t) \kappa - \kappa^\top A(t) - 2A(t)^2, \\ \mathbf{b}'(t) &= \lambda \mathbf{b}_v - \kappa \mathbf{b}(t) - 2A(t)^\top \mathbf{b}(t), \\ c'(t) &= \lambda c_v + \text{tr} A(t) - \mathbf{b}(t)^\top \mathbf{b}(t)/2, \\ A(0) &= 0, \mathbf{b}(0) = 0, c(0) = 0.\end{aligned}$$

Specification Analysis IIIa

Correlations and Measure Changes

- Correlation via diffusions: Example

$$X_t = W_t, \quad dv(t) = (a - \kappa v(t))dt + \eta\sqrt{v(t)}dZ_t, \quad dW_t dZ_t = \rho dt.$$

– Measure change:

$$\frac{dQ(\theta)}{dP} \Big|_{\mathcal{F}_t} = \exp \left(i\theta Y_t + \frac{1}{2}\theta^2 \int_0^t v(s)ds \right).$$

– $v(t)$ dynamics under $Q(\theta)$: $dv(t) = (a - (\kappa - i\theta\eta\rho)v(t))dt + \eta\sqrt{v(t)}dZ_t$, which remains affine.

Specification Analysis IIIa

Correlations and Measure Changes: Correlation via jumps:

- Example

$$X_t = L_t^{\alpha, -1}, \quad dv(t) = (a - \kappa v(t)) dt - \beta^{1/\alpha} dL_{T_t}^{\alpha, -1},$$

where $L_t^{\alpha, -1}$ denotes a standard Lévy α -stable motion with tail index $\alpha \in (1, 2]$ and maximum negative skewness.

- Measure change:

$$\frac{dQ(\theta)}{dP} \Big|_{\mathcal{F}_t} = \exp \left(i\theta L_{T_t}^{\alpha, -1} + \Psi_x(\theta) T_t \right), \quad \Psi_x(\theta) = - (i\theta)^\alpha \sec \frac{\pi\alpha}{2}, \quad \text{Im}(\theta) < 0.$$

- Under this new (leverage-neutral) measure, $\Pi^\theta(dx) = e^{i\theta x} \Pi(dx)$, and $v(t)$ satisfies generalized affine:

$$\mathcal{L}_{T_t}^\theta (\Psi_x(\theta)) = \exp(-b(t)v_0 - c(t)). \quad (3)$$

Pricing State Contingent Claims

- Consider a payoff at a given fixed time M which is any linear combination of the following payoffs:

$$\Pi_Y(k; a, b, \vartheta, c) = \left(a + be^{\vartheta^\top Y_M} \right) 1_{c^\top Y_M \leq k}$$

- Examples of claims covered by the above structure include:
 - European call with strike K : $\Pi(\ln(F_0(M)/K); -K, F_0(M), \vartheta, -\vartheta)$,
 - European put with strike K : $\Pi(\ln(K/F_0(M)); K, -F_0(M), \vartheta, \vartheta)$,
 - A protected put: $\max[S_M, K] = \Pi(\ln(F_0(M)/K); 0, F_0(M), \vartheta, -\vartheta) + \Pi(\ln(K/F_0(M)); K, 0, 0, \vartheta)$,
 - A binary call: $\Pi(\ln(F_0(M)/K); 1, 0, 0, -\vartheta)$.

where recall that $F_t(M) = F_0(M)e^{\vartheta^\top Y_t}$ is the M maturity forward price of the underlying asset at time $t \in [0, M]$.

- *State price*: Let $G(k; a, b, \vartheta, c; M)$ denote the price of such a claim. We can compute G with two transform methods using ϕ_Y .

Transform I

- Let $\mathcal{G}^I(z; a, b, \vartheta, c)$ denote a Fourier transform of state price $G(k; a, b, \vartheta, c)$, defined as

$$\mathcal{G}^I(z; a, b, \vartheta, c) \equiv \int_{-\infty}^{+\infty} e^{izk} dG(k; a, b, \vartheta, c), \quad z \in \mathbb{R}. \quad (4)$$

- $\mathcal{G}^I(z; a, b, \vartheta, c)$ can be written as an affine function of the generalized Fourier transform of Y_M :

$$\mathcal{G}^I(z; a, b, \vartheta, c) = a\phi_Y(zc) + b\phi_Y(zc - i\vartheta).$$

- The price $G(k; a, b, \vartheta, c)$ can then be obtained by inversion:

$$G(k; a, b, \vartheta, c) = \frac{\mathcal{G}_0^I}{2} + \frac{1}{2\pi} \int_0^\infty \frac{e^{izk} \mathcal{G}^I(-z; a, b, \vartheta, c) - e^{-izk} \mathcal{G}^I(z; a, b, \vartheta, c)}{iz} dz,$$

where $\mathcal{G}_0^I = \mathcal{G}^I(0; a, b, \vartheta, c) = a + b\phi_Y(-i\vartheta)$.

Note that this is a *one-dimensional* inversion regardless of the dimensionality of the uncertainty Y_M .

Transform II

- Define a second transform $\mathcal{G}^{II}(z; a, b, \vartheta, c)$:

$$\mathcal{G}^{II}(z; a, b, \vartheta, c) \equiv \int_{-\infty}^{+\infty} e^{izk} G(k; a, b, \vartheta, c) dk, \quad z \in \mathcal{C} \subseteq \mathbb{C}.$$

$$\mathcal{G}^I(z; a, b, \vartheta, c) \equiv \int_{-\infty}^{+\infty} e^{izk} dG(k; a, b, \vartheta, c), \quad z \in \mathbb{R}.$$

Note the two differences between \mathcal{G}^I and \mathcal{G}^{II} .

- \mathcal{G}^{II} , when well-defined, is given by:

$$\mathcal{G}^{II}(z; a, b, \vartheta, c) = \frac{i}{z} (a\phi_Y(zc) + b\phi_Y(zc - i\vartheta)).$$

- Inversion:

$$G(k) = \frac{1}{2\pi} \int_{iz_i - \infty}^{iz_i + \infty} e^{-izk} \mathcal{G}^{II}(z; a, b, \vartheta, c) dz.$$

- This inversion can be performed numerically using FFT or Fractional FFT, generating superior computational efficiency. By vectorizing in maturity, the model's option values at all strikes and maturities can be obtained in one stroke.

Transform II: The Dampening Factor

The choice of $\text{Im } z$ is crucial and depends upon the payoff structure.

Contingent Claim	Generalized transform $-iz\varphi(z)$	Restrictions on $\text{Im } z$
$G(k; a, b, \vartheta, c)$	$a\phi_Y(zc) + b\phi_Y(zc - i\vartheta)$	$(0, \infty)$
$G(-k; a, b, \vartheta, c)$	$a\phi_Y(-zc) + b\phi_Y(-zc - i\vartheta)$	$(-\infty, 0)$
$e^{\alpha k}G(k; a, b, \vartheta, c)$	$a\phi_Y((z - i\alpha)c) + b\phi_Y((z - i\alpha)c - i\vartheta)$	(α, ∞)
$e^{\beta k}G(-k; a, b, \vartheta, c)$	$a\phi_Y(-(z - i\beta)c) + b\phi_Y(-(z - i\beta)c - i\vartheta)$	$(-\infty, \beta)$
$e^{\alpha k}G(k; a_1, b_1, \vartheta_1, c_1)$ $+ e^{\beta k}G(-k; a_2, b_2, \vartheta_2, c_2)$	$a_1\phi_Y((z - i\alpha)c_1) + b_1\phi_Y((z - i\alpha)c_1 - i\vartheta_1)$ $+ a_2\phi_Y(-(z - i\beta)c_2) + b_2\phi_Y(-(z - i\beta)c_2 - i\vartheta_2)$	(α, β)

$(\alpha, \beta, a, b$ are real constants with $\alpha < \beta$.)

Summary

- We provide a powerful tool in generating tractable option pricing models.
 - Apply stochastic time change to Lévy processes
 - Prove a theorem that facilitates the derivation of the CF.
 - Price options via transform methods.
- Applications: Model design and calibration
 - Huang and Wu (JF 2004): Specification analysis for equity index options
 - Carr and Wu (wp, 2004): Stochastic skew in currency options
 - Bakshi, Carr, and Wu (wp, 2004): Stochastic discount factors in international economies.