A Survey of Preference-Free Option Valuation with Stochastic Volatility

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Part I

Introduction

Overview

Fischer Black once wrote:

> Suppose we use the standard deviation ... of possible future returns on a stock ... as a measure of its volatility. Is it reasonable to take that volatility as constant over time? I think not.

- I review 6 approaches for handling stochastic volatility in a preference-free manner:
  1. instantaneous volatility is a specified function of stock price and time
  2. instantaneous volatility is an autonomous diffusion process
  3. bounding quadratic variation
  4. deterministic local volatility surface
  5. stochastic forward local volatility surface
  6. stochastic implied volatility surface
- This list is not exhaustive!
- Much of what I review can be downloaded from www.math.nyu.edu/research/carrp/papers

Part II

Instantaneous Volatility is a Specified Function of Stock Price and Time

Introduction

- An early approach to option pricing with stochastic volatility assumes that the risk-neutral process is:

$$dS_t = rS_t dt + a(S_t, t)dW_t, \quad t \in [0, T],$$

  where the absolute volatility is some specified function of the stock price and time.

- Some specifications which lead to closed form solutions for option prices are:
  1. constant $a(S, t) = a$ (Cox/Ross (1976))
  2. square root $a(S, t) = a\sqrt{S}$ (Cox/Ross (1976))
  3. proportional $a(S, t) = aS$ (Black/Scholes (1973))
- These 3 can be embedded in the CEV process where $a(S, t) = aS^p, p \in [0, 1]$, due to Cox (1975).
- Cox's solution technique is to transform the CEV process into a square root process. Goldenberg (1991) presents option pricing formulas when $a(S, t) = a(t)\sqrt{S}$ and $a(S, t) = a\sqrt{c_1 e^{-rt} S_t + c_2}$. He argues that generalized CEV processes can be mapped into these.

A Different Approach

- When the stock is regarded as a call on the firm's assets (Black-Scholes (1973), Merton (1974), Geske (1979)), then the instantaneous volatility of the stock depends on the firm's assets.
- Since there is also a one to one map between the GBM describing firm value and the SBM driving the system, the instantaneous volatility of the stock also depends on the driving SBM.
- Since there is a one to one map between the stock and the driving SBM, the stock's instantaneous volatility can be said to depend on the stock price.
- When the relationship between the stock price and the driving SBM is explicitly invertible, closed form option pricing formulas result.

Path-Independence Approach

- Assuming that the risk-neutral process is:

$$dS_t = (r-q)S_t dt + a(S_t, t)dW_t, \quad t \in [0, T],$$

  Carr, Tari, and Zariphopolou (1999) characterize the entire class of volatility functions which permit the stock price to be transformed into standard Brownian motion by scale changes alone:

$$\frac{a^2(S,t)}{2}\frac{\partial^2 a(S,t)}{\partial S^2} + (r-q)S\frac{\partial a}{\partial S}(S,t) + \frac{\partial a}{\partial t}(S,t) = (r-q)a(S,t).$$

  for $S > 0, t \in [0, T]$.

- They present the general solution to this nonlinear PDE and present three new examples of explicitly invertible relationships between the stock price and the driving SBM.
- They thus derive 3 new vol specifications and 3 new closed form option pricing formulas.
- I next cover the first of these 3.

Example 1: Stock Price is Hyperbolic Sine

- Suppose that the risk-neutral stock price process is

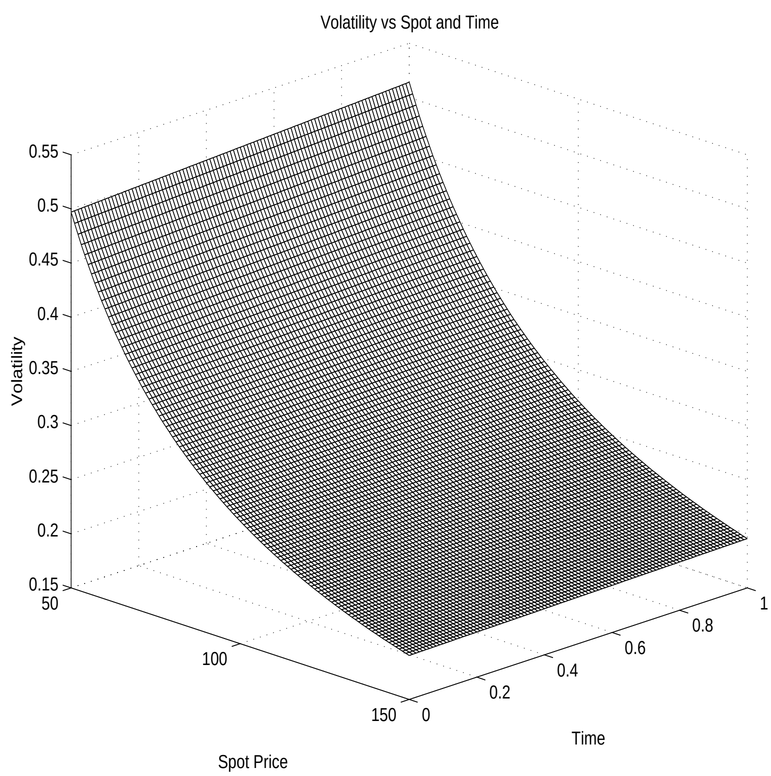
$$S_t = \beta e^{-\mu(T-t)} \sinh[\alpha(W_t - L)], \quad t \in [0,T], W_t > L,$$

where $\beta \equiv S_0 e^{\mu T} \text{csch}(-\alpha L), \quad \mu \equiv r - q - \alpha^2/2$, and L and α are free parameters.

- Then the "relative" instantaneous volatility is:

$$\sigma(S,t) \equiv \frac{a(S,t)}{S} = \alpha\sqrt{1 + \left(\frac{S_0}{Se^{-\mu t)}}\right)^2}, \qquad S > 0, t \in [0, \tau).$$

- For high S, the process behaves locally like GBM, while for low S, it behaves locally like OU.

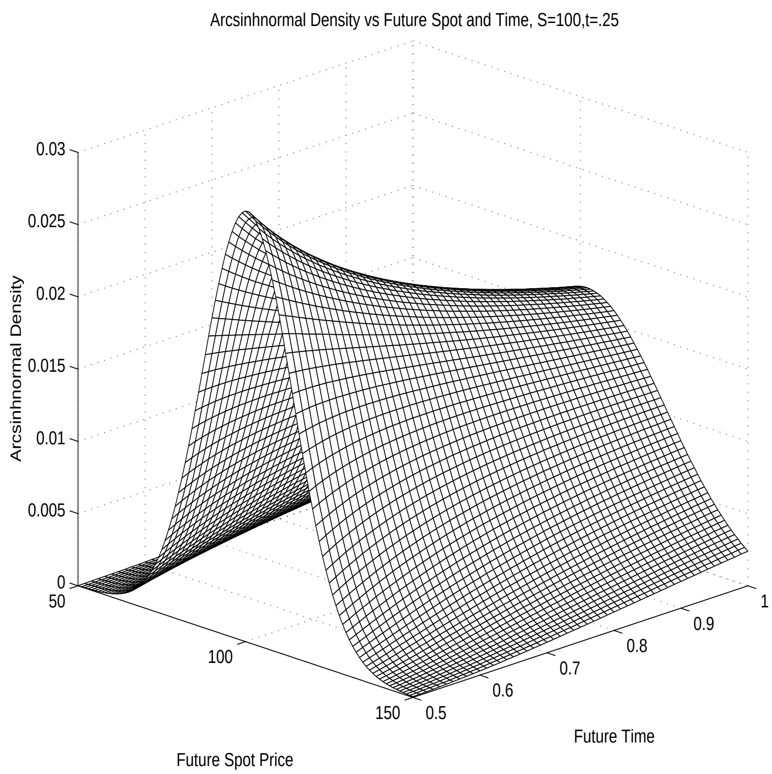


Stock Price is Hyperbolic Sine (con'd)

- Using the standard change of variables formula, the risk-neutral stock pricing density is:

$$q(Z,M;S,t) = \frac{1}{\sqrt{2\pi(M-t)}}\frac{1}{\alpha\sqrt{Z^2+\beta^2 e^{-2\mu(T-M)}}}$$
$$\left\{\exp\left\{-\frac{1}{2}\left[\frac{\sinh^{-1}\left(\frac{Z}{\beta}e^{\mu(T-M)}\right)-\sinh^{-1}\left(\frac{S}{\beta}e^{\mu(T-t)}\right)}{\alpha\sqrt{M-t}}\right]^2\right\}\right.$$
$$\left.-\exp\left\{-\frac{1}{2}\left[\frac{\sinh^{-1}\left(\frac{Z}{\beta}e^{\mu(T-M)}\right)+\sinh^{-1}\left(\frac{S}{\beta}e^{\mu(T-t)}\right)}{\alpha\sqrt{M-t}}\right]^2\right\}\right\},$$

for $S>0, t\in[0, M\wedge\tau)$ and where $\beta = S_0 e^{\mu T}\text{csch}(-\alpha L)$.



Stock Price is Hyperbolic Sine (con'd)

- Integrating the call's payoff against this density yields the following pricing formula:

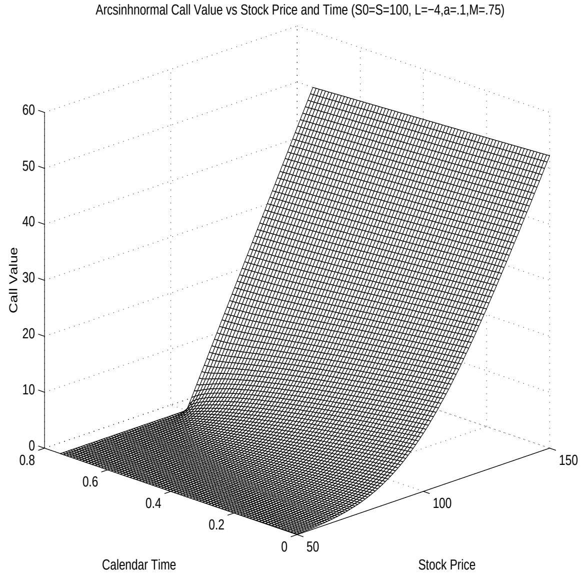
$$\begin{aligned} C(S,t) &= \frac{e^{-q(M-t)}}{2}\left(S+\sqrt{S^2+\beta^2 e^{-2\mu(T-t)}}\right)\cdot \\ &\quad [N(d_+ + \alpha\sqrt{M-t}) + N(d_- - \alpha\sqrt{M-t})] \\ &- \frac{\beta^2 e^{-q(M-t)}}{2e^{2\mu(T-t)}}\frac{1}{S+\sqrt{S^2+\beta^2 e^{-2\mu(T-t)}}}\cdot \\ &\quad [N(d_+ - \alpha\sqrt{M-t}) + N(d_- + \alpha\sqrt{M-t})] \\ &- Ke^{-r(M-t)}[N(d_+) - N(d_-)], \end{aligned}$$

where:

$$d_\pm \equiv \frac{\pm\sinh^{-1}\left(\frac{S}{\beta}e^{\mu(T-t)}\right) - \sinh^{-1}\left(\frac{K}{\beta}e^{\mu(T-t)}\right)}{\alpha\sqrt{M-t}},$$

and where:

$$\beta = S_0 e^{\mu T}\operatorname{csch}(-\alpha L).$$



Bibliography

[1] Adjaoute, K., 1994, "Les Modeles D'Evaluation D'Option a Elasticite de Variance Constante," HEC Lausanne working paper.

[2] Ang J. and D. Peterson, 1984, "Empirical Properties of the Elasticity Coefficient in the Constant Elasticity of Variance Model," *Financial Review*, Vol. 19, No. 4, 372-80.

[3] Beckers, S., 1980 "The Constant Elasticity of Variance Model and its Implications for Option Pricing," *Journal of Finance*, **35**, 661-73.

[4] Bensoussan A., M. Crouhy, and D. Galai, 1994, Stochastic Equity Volatility and the Capital Structure of the Firm, Philosophical Transactions of the Royal Society of London, Series A, **347**, 449-598.

[5] Bergman, Y., B. Grundy, and Z. Wiener, 1996, General Properties of Option Prices, *Journal of Finance*, **51**, 5, 1573-1610.

[6] Black, F., 1976, Studies in Stock Price Volatility Change, in Proceedings of the 1976 Business and Economics Statistics Selection, American Statistical Association, 177–181.

[7] Boyle P. and Y. Tian, 199, "Pricing Lookback and Barrier Options under the CEV Process", *Journal of Financial and Quantitative Analyis*, 1999, forthcoming.

[8] Carr, P., 1988, "European Option Pricing When Dividends and Interest Rates are Unknown," UCLA working paper.

[9] Carr, P., M. Tari, and T. Zariphopolou, "Closed Form Option Pricing with Smiles", Banc of America Securities working paper.

[10] Choi, J. and F. Longstaff, 1985, "Pricing Options on Agricultural Futures: An Application of the Constant Elasticity of Variance Option Pricing Model," *Journal of Futures Markets*, **5**, Spring, 247-58.

[11] Christie, A., 1982, "The Stochastic Behavior of Common Stock Variances," *Journal of Financial Economics*, **10**, 407-32.

[12] Christie, A., 1993, "Constant Elasticity of Variance Models and Portfolio Formation," University of Rochester working paper.

[13] Cox, J., 1975, "Notes on Option Pricing I: Constant Elasticity of Variance Diffusions," Stanford University working paper.

[14] Cox, J. and S. Ross, 1976, "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics*, **3**, 145-166.

[15] Emanuel, D. and J. MacBeth, 1982, "Further Results on the Constant Elasticity of Variance Call Option Pricing Model," *Journal of Financial and Quantitative Analysis*, **17**, Nov., 533-54.

[16] Feller, W., 1951, "Two Singular Diffusions," *Annals of Mathematics*, 54, **1**, July, 173-82.

[17] Geman, H. and M. Yor, 1992, "Bessel Processes, Asian Options and Perpetuities, Université Pierre et Marie Curie working paper.

[18] Gibbons, M. and C. Jacklin, 1990, "CEV Diffusion Estimation," Stanford University working paper

[19] Goldenberg, D., 1991, "A Unified Method for Pricing Options on Diffusion Processes," *Journal of Financial Economics*, **29** Mar., 3-34.

[20] Hauser, S., and C. Bagley, 1986, "Estimation and Empirical Evidence on Foreign Currency Call Options: Constant Elasticity of Variance Model," Temple University working paper.

[21] Linetsky, V. and D. Davidov, 1999, "Pricing Options on One Dimensional Diffusions: A Unified Approach", University of Michigan working paper.

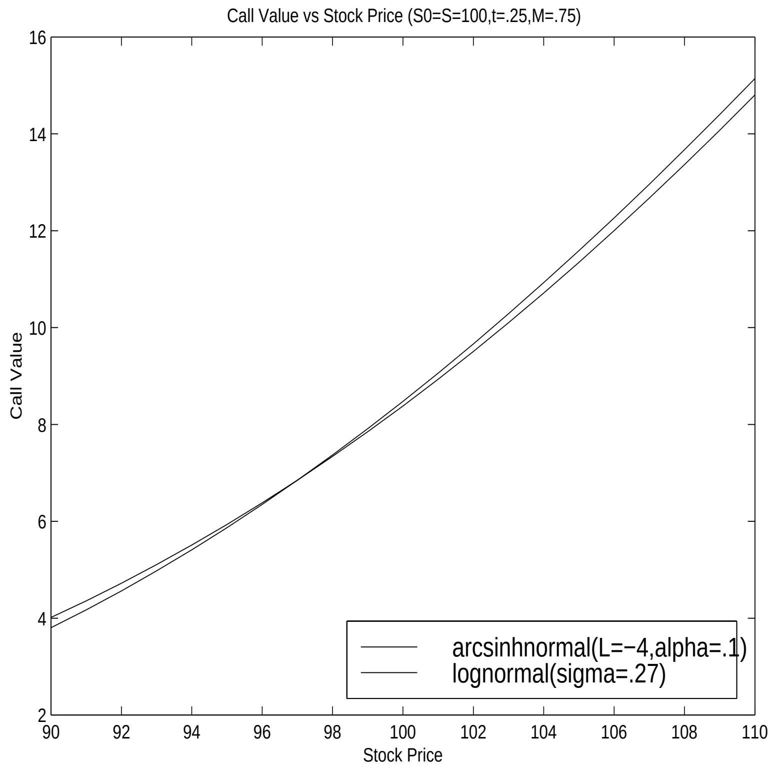
[22] MacBeth J., and L. Merville, "Tests of the Black-Scholes and Cox Call Option Pricing Models," *Journal of Finance*, 35, **2**, 285-300

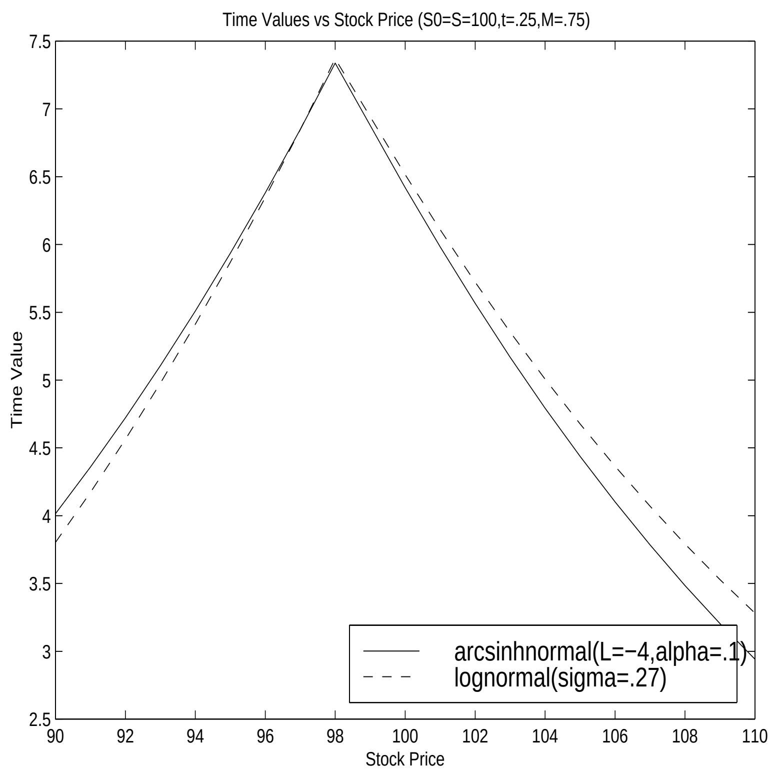
[23] Milonas, N., 1986, "A Note on Agricultural Options and the Variance of Futures Prices," *The Journal of Futures Markets*, Vol. 4, No. 4, 671-76.

[24] Schroder, M., 1989, "Computing the Constant Elasticity of Variance Option Pricing Formula," *Journal of Finance*, 44, Mar., 211-219.

[25] Tucker, A., D. Peterson and E. Scott, 1988, "Tests of the Black-Scholes and Constant Elasticity of Variance Currency Call option Valuation Models," *Journal of Financial Research*, Vol. 11, No. 3, 201-14.

[26] Zühlsdorff, C., 1999, "The Pricing of Derivatives on Assets with Quadratic Volatility", Bonn University working paper.


Call Value vs Stock Price (S0=S=100,t=.25,M=.75)
Call Value
Stock Price
arcsinhnormal(L=−4,alpha=.1)
lognormal(sigma=.27)


Time Values vs Stock Price (S0=S=100,t=.25,M=.75)
Time Value
Stock Price
arcsinhnormal(L=−4,alpha=.1)
lognormal(sigma=.27)

Part III

Instantaneous Volatility is an Autonomous Diffusion Process

Introduction

- Suppose once again that the risk-neutral process is:

$$dS_t = rS_t dt + a(S_t, t)dW_t, \quad t \in [0, T],$$

- By Itô's lemma, the stock volatility $a_t \equiv a(S_t, t)$ follows a diffusion process:

$$da_t = \mu(S_t, t)dt + \omega(S_t, t)dW_t, \quad t \in [0, T],$$

  for some functions $\mu(\cdot)$ and $\omega(\cdot)$.

- If we suppose further that the map between a and S_t is invertible, then the stock volatility a_t follows an *autonomous* diffusion process:

$$da_t = m(a_t, t)dt + v(a_t, t)dW_t, \quad t \in [0, T],$$

- In the late 1980's and early 90's, several researchers extended this approach to:

$$da_t = m(a_t, t)dt + v(a_t, t)dB_t, \quad t \in [0, T],$$

  where B_t is a second Brownian motion, which may or may not be correlated with W_t.

- For example, Stein and Stein (1991) consider the case where the lognormal volatility $\sigma_t \equiv \frac{a_t}{S_t}$ is an OU process:

$$d\sigma_t = \delta(\theta - \sigma_t)dt + kdB_t, \quad t \in [0, T].$$

A Brief Survey of this Literature

- Let $V_t \equiv a_t^2$ denote the absolute variance process.
- Closed form formulas for option prices were given by:
  - Hull and White (1987):

$$dV_t = \alpha V_t dt + \xi V_t dB_t, \quad t \in [0, T].$$

  - Heston (1993)

$$dV_t = (\omega - \theta V_t) dt + \xi \sqrt{V_t} dB_t, \quad t \in [0, T].$$

- A review article by Ball and Roma (1994) surveys these approaches.
- Ritchken and Trevor (1998) showed that the Hull-White model emerges as a continuous time limit of a GARCH 1-1 process.
- A superb new book called "Option Valuation under Stochastic Volatility" by Alan Lewis extends the Hull White model to correlated Brownian motions, derives the Heston model, and introduces a "3/2" model:

$$dV_t = (\omega V - \tilde{\theta} V_t^2) dt + \xi V_t^{3/2} dB_t, \quad t \in [0, T],$$

  where $\tilde{\theta}$ is a certain function of the parameters.
- The publisher of this book is "Finance Press" at "www.financepress.com".

Market Completeness when Volatility Follows an Autonomous Diffusion

- When volatility follows an autonomous diffusion, the "market price of volatility risk" is usually specified to take some tractable form.
- The emergence of this concept has lead many (respectable) researchers to mistakenly conclude that markets are not complete when volatility follows an autonomous diffusion. They then assume some equilibrium model to justify their analysis.
- In fact, markets are complete so long as one can trade continuously in another option on the same stock. Thus, standard arbitrage pricing arguments can be used to develop "preference-free" option pricing models.
- The key to avoiding preference restrictions is to relate option prices to either other option prices or to known functions of option prices such as "forward local volatility" or Black Scholes implied volatility.
- To illustrate these points, I now show how the market price of volatility risk drops out when an option price is related to another option price.

Elimination of the Market Price of Volatility Risk

- Using standard arguments, one can derive the following PDE governing the function $C^{(1)}(t,S,Y)$ relating the price of an option to time t, the price of the underlying stock S, and the price of a state variable governing volatility Y:
$$\frac{\partial C^{(1)}}{\partial t} + r\left[S\frac{\partial C^{(1)}}{\partial S} - C^{(1)}\right] + [\alpha(m-Y) - \lambda\beta]\frac{\partial C^{(1)}}{\partial Y}$$
$$+\frac{f^2(Y)}{2}\frac{\partial^2 C^{(1)}}{\partial S^2} + \rho f(Y)\beta\frac{\partial^2 C^{(1)}}{\partial S\partial Y} + \frac{\beta^2}{2}\frac{\partial^2 C^{(1)}}{\partial Y^2} = 0,$$
where λ is the market price of volatility risk.
- The above PDE also holds for the price of a second option:
$$\frac{\partial C^{(2)}}{\partial t} + r\left[S\frac{\partial C^{(2)}}{\partial S} - C^{(2)}\right] + [\alpha(m-Y) - \lambda\beta]\frac{\partial C^{(2)}}{\partial Y}$$
$$+\frac{f^2(Y)}{2}\frac{\partial^2 C^{(2)}}{\partial S^2} + \rho f(Y)\beta\frac{\partial^2 C^{(2)}}{\partial S\partial Y} + \frac{\beta^2}{2}\frac{\partial^2 C^{(2)}}{\partial Y^2} = 0.$$
- Let $\gamma(t,S,C^{(2)})$ be the function relating the price of the first option to time t, the underlying stock price S, and the price of the second option:
$$\gamma(t,S,C^{(2)}) \equiv C^{(1)}(t,S,Y),$$
where $C^{(2)}$ solves the above PDE.
- The appendix proves:
$$\gamma_1 + rS\gamma_2 + rC^{(2)}\gamma_3$$
$$+\frac{1}{2}\frac{\mathrm{Var}(dS)}{dt}\gamma_{22} + \frac{\mathrm{Cov}(dC^{(2)},dS)}{dt}\gamma_{23} + \frac{1}{2}\frac{\mathrm{Var}(dC^{(2)})}{dt}\gamma_{33} = r\gamma.$$
- If one can exogenously model the volatility structure of $C^{(2)}$ without reference to λ, then one does not need to specify λ. There are many ways to do this.

Bibliography

[1] Amin, K., and V. Ng, 1993, Option Valuation with Systematic Stochastic Volatility, *Journal of Finance*, **48**, 3, 881-910.

[2] Amin, K., and V. Ng, 1994, A Comparison of Predictable Volatility Models Using Option Data, *Preprint*, Research Department, International Monetary Fund.

[3] Ball, C., and A. Roma, 1994, Stochastic Volatility Option Pricing, *Journal of Financial and Quantitative Analysis*, **29**, 4, 589–607.

[4] Bollerslev, T., R. Chou and K. Kroner, 1992, ARCH Modeling in Finance. A Review of the Theory and Empirical Evidence, *Journal of Econometrices*, bf 52, 5-59.

[5] Bookstaber R. and J. McDonald, 1987, A General Distribution for Describing Security Price Returns, *Journal of Business*, **60**, 3, 401–424.

[6] Bookstaber R. and J. McDonald, 1988, Option Pricing for Generalized Distributions, Brigham Young University working paper.

[7] Cherian, J., and R. Jarrow, 1997, "Options Markets, Self-fulfilling Prophecies and Implied Volatilities", Boston University working paper, forthcoming in *Review of Derivatives Research*.

[8] Chesney, M., and L. Scott, Pricing European Options: A Comparison of the Modified Black Scholes model and a Random Variance

Model, *Journal of Financial and Quantitative Analysis* **24** 267–284.

[9] Chou, R., 1988, Volatility Persistence and Stock Valuations: Some Empirical Evidence Using GARCH, *Journal of Applied Econometrices*, **3**, 279-294.

[10] Clewlow L., and X. Xu, 1992, A Review of Option Pricing with Stochastic Volatility, University of Warwick working paper.

[11] Di Masi G. B. , Y. Kabanov, and W.J. Runggaldier, 1996, Mean-variance hedging of options on stocks with Markov volatilities, Universita di Padova working paper.

[12] Di Masi G. B. , Y. Kabanov, and W.J. Runggaldier, 1996, Hedging of options under discrete observations on assets with stochastic volatility, Universita di Padova working paper.

[13] Duan, J. C., 1991, The GARCH Option Pricing Model, McGill University working paper.

[14] Duan, J. C., 1995, Fitting the "Smile Family" - A GARCH Approach, McGill University working paper.

[15] Duan, J. C., 1996, The GARCH Option Pricing Model, *Mathematical Finance*, **5**, 13-32.

[16] Duan, J. C., 1996, Augmented GARCH(p,q) Process and its Diffusion Limit, McGill University working paper.

[17] Duan, J. C., 1996, A Unified Theory of Option Pricing under Stochastic Volatility - from GARCH to Diffusion, *Preprint*, Hong University of Science and Technology.

[18] Engle, R., Autoregressive Conditional Heteroscedasticity with Estimates of the variance of United Kingdom Inflation, *Econometrica*, **50**, 987-1007.

[19] Engle, R., and Rosenberg J., 1995, GARCH Gamma, *The Journal of Derivatives*, 47-59.

[20] Fleming, J., B. Ostdiek, and R. Whaley, 1993, "Predicting Stock Market Volatility: A New Measure", Duke University working paper.

[21] Frey, R., 1996, Derivative Asset Analysis in Models with Level-Dependent and Stochastic Volatility, ETH Zurich working paper.

[22] Ghysels E., C. Gourieroux and J. Jasiak, 1996, Market Time and Asset Price Movements, Theory and Estimation, CIRANO working paper.

[23] Ghysels, E., C. Harvey, and E. Renault, 1997 Stochastic Volatility, in Handbook of Statistics, **14**, Statistical Methods in Finance, North Holland.

[24] Grunbichler A., and F. Longstaff, 1993, "Valuing Options on Volatility", UCLA working paper.

[25] A. Harvey, E. Ruiz, and N Shephard, 1994, Multivariate stochastic variance models, *Review of Economic Studies*, **61**, 247–264.

[26] Heath, D., and E. Platen, 1997, Quantitative Methods for a class of Stochastic Volatility Models, Australian National University working paper.

[27] Heston, Stephen, 1993, Closed-Form Solution for Options with Stochastic Volatility, with Application to Bond and Currency Options, *Review of Financial Studies*, **6**, 327–343.

[28] Heynen, R., A. Kemna, and T. Vorst, 1994, Analysis of the Term Structure of Implied Volatilities, *Journal of Financial and Quantitative Analysis*, **29**, 31–56.

[29] Hilliard, J. and A. Schwartz, 1994, Binomial Option Pricing Under Stochastic Volatility and Correlated State Variables, University of Georgia working paper.

[30] Hofmann N., E. Platen, M. Schweizer, 1992, Option Pricing under incompleteness and stochastic volatility, *Mathematical Finance*, **2**, 153–187.

[31] Hull J. and A. White, 1987, The Pricing of Options with Stochastic Volatility, *Journal of Finance*, **42**, 2, 281–300.

[32] Hull J. and A. White, 1988, An Analysis of the Bias in Option Pricing Caused by a Stochastic Volatility, Advances in Futures and Options Research **3**, 29–61.

[33] Hurst S. and E. Platen, 1997, On the Dynamics of Stochastic Volatility. Australian National University working paper.

[34] Jagannathan R., 1984, "Call Options and the Risk of Underlying Securities", *Journal of Financial Economics*, **13**, 3, 425–434.

[35] Jarrow, R. and A. Rudd, 1982, Approximate Option Valuation for Arbitrary Stochastic Processes, *Journal of Financial Economics*, **10**, 347–369.

[36] Johnson, H. and D. Shanno, 1987, Option Pricing when the Variance is Changing, *Journal of Financial and Quantitative Analysis*, **22**, 2, 143–151.

[37] Knoch, H., 1991, The Pricing of Foreign Currency Options with Stochastic Volatility, Ph.D. dissertation, Yale University.

[38] Lazrak, A., 1997, General equilibrium foundation of the stochastic volaitlity model: A theoretical investigation and an example, GREMAQ, Universite des Sciences Sociales working paper.

[39] Lewis A., 1996, A New Closed Form Solution for Options Under Stochastic Volatility, Analytic TSA working paper.

[40] Liesen D., 1997, Stock evolution under stochastic volatility: A discrete time approach, CREST working paper.

[41] Liu M. and H. Zhang, 1997, Specification tests in the efficient method of moments framework with application to the stochastic volatility models, Chinese University of Hong Kong working paper.

[42] Madan, D., and F. Milne, 1991, Option Pricing with V.G. Martingale Components, University of Maryland working paper.

[43] Melino, A. and S. Turnbull, 1990, Pricing Foreign Currency Options with Stochastic Volatility, *Journal of Econometrics*, **45**, 239–265.

[44] Nelson, D., ARCH Models as Diffusion Approximations, *Journal of Econometrices*, **45**, 7-38.

[45] Neuberger, A. and S. Hodges, 1996, Equilibrium and the Role of Options in an Economy with Stochastic Volatility, University of Warwick working paper.

[46] Oztukel, A., and P Wilmott, 1999, Uncertain Parameters, an Empirical Stochastic Volatility Model and Confidence Limits, Mathematical Institute, Oxford working paper.

[47] Pham, H., and N. Touzi, 1996, Equlibrium Prices in a Stochastic Volatility Model, *Mathematical Finance*, **6**, 2, 215-236.

[48] Platen E., and M. Schweitzer, 1994, On Smile and Skewness, University of Bonn working paper.

[49] Platen E., 1997, Constructing Consistent Market Models, Universtiy of Sydney working paper.

[50] Renault E., and N. Touzi, Option Hedging and Implied Volatilities in a Stochastic Volatility Model, *Mathematical Finance*, **6**, 279–302.

[51] Ritchken, P., and R. Trevor, 1999, Pricing Options under generalized GARCH and stochastic volatility processes, *Journal of Finance*.

[52] Romano, M., and N. Touzi, 1997, Contingent Claims and Market Completeness in a Stochastic Volatility Model, *Mathematical Finance*, **7**, 4, 399-412.

[53] Scott, L., 1987, Option Pricing When the Variance Changes Randomly: Theory, Estimation, and an Application, *Journal of Financial and Quantitative Analysis*, **22** 4, 419–438.

[54] Scott, L., 1997, Pricing Stock Options in a Jump-Diffusion Model with Stochastic Volatility and Interest Rates: Applications of Fourier Inversion Methods, *Mathematical Finance*, **7**, 3, 345-358.

[55] Shastri, K., and K. Wethyavivorn, 1987, The Valuation of Currency Options for Alternate Stochastic Processes, *Journal of Financial Research*, **X**, 2, 283–293.

[56] Sin, C., 1996, Complications with Stochastic Volatility Models, *Journal of Applied Probability*, forthcoming.

[57] Sin C., 1996, Strictly Local Martingales and Hedge Ratios in Stochastic Volatility Models, Cornell University dissertation.

[58] Stein, Jeremy, 1989, Overreactions in the Options Market, *Journal of Finance*, **44** 1011–1023.

[59] Stein, Elias, and Jeremy Stein, 1991, Stock Price Distributions with Stochastic Volatility: An Analytic Approach, *Review of Financial Studies*, **4**, 4, 727–752.

[60] Taylor S., Modelling Stochastic Volatility, A Review and Comparative Study, *Mathematical Finance*, **4**, 183–204.

[61] Taylor S. and X. Xu, 1993, The Magnitude of Implied Volatility Smiles, University of Warwick working paper.

[62] Taylor S., B. Blair, and S. Poon, 1997, Modelling S&P-100 volatility: The information content of stock returns.

[63] Walsh D. and G. Tsou, 1997, Forecasting index volatility: Sampling interval and non-trading effects, University of Western Australia working paper.

[64] Wiggins, J., 1987, Option Values Under Stochastic Volatility: Theory and Empirical Estimates, *Journal of Financial Economics*, **19**, 351–372.

[65] Willner, R., 1990, A Jump Process Stock Variance Call Option Model, Citibank working paper.

[66] Xu, X., and S.J. Taylor, 1994, The Term Structure of Volatility Implied by Foreign Exchange Options, *Journal of Financial and Quantitative Analysis*, **29**, 57–74.

Part IV

Bounding Quadratic Variation

Introduction

- Rather than exactly specifying the volatility process and then attempting exact pricing several authors have instead placed bounds on either the instantaneous volatility process or the quadratic variation process, and then found corresponding bounds on option prices.
- Suppose we assume that:
$$\frac{dS_t}{S_t} = (r - q)dt + \sigma_t dW_t, \quad t \in [0, T],$$
where the volatility σ_t is a process.
- Consider a claim with a convex payoff such as a European call.
- If the variance rate σ_t^2 is bounded above by a constant $\bar{\sigma}^2$, then an upper bound on any such claim is given by the Black Scholes value with $\sigma^2 = \bar{\sigma}^2$.
- Similarly, if the variance rate is bounded below by a constant $\underline{\sigma}^2$, then a lower bound on any such claim is given by the Black Scholes value with $\sigma^2 = \underline{\sigma}^2$.
- Clearly, the inequalities reverse for concave claims.
- In all of these analyses, the claim should be delta-hedged using the Black-Scholes delta with volatility given by the applicable bound.
- Avellaneda and Buff[1] analyze barrier options which need not be globally convex or concave.

Two Generalizations

- Mykland[11] generalizes the analysis in two ways:
- First his bound is on the total quadratic variation $\langle \ln S \rangle_T \equiv \int_0^T \sigma_t^2 dt$, rather than on the variance rate σ_t^2. This implies that jumps in the stock are possible.
- Second, he shows that if under the statistical probability measure P:

  $$P\{\int_0^T \sigma_t^2 dt > Q\} = p,$$

  then the probability of loss can also be made to be p.
- To accomplish this, the sale of a convex claim should be delta-hedged using the Black-Scholes delta with $\sigma^2(T-t)$ replaced with $Q - \langle \ln S \rangle_t$.

Bibliography

[1] Avellaneda, M., A. Levy, and R. Buff, 1997, "Combinatorial Implications of Nonlinear Uncertain Volatility Models: the Case of Barrier Options", Courant working paper.

[2] Avellaneda, M., A. Levy, and A. Paras, 1995, "Pricing and Hedging Derivative Securities in Markets with Uncertain Volatilities", *Applied Mathematical Finance*, **2**, 73–88.

[3] Avellaneda, M., A. Levy, and A. Paras, 1996, "Managing the Volatility Risk of Portfolios of Derivative Securities: The Lagrangian Uncertain Volatility Model", *Applied Mathematical Finance*, **3**, 21–52.

[4] El Karoui N., M. Jeanblanc Picque, and S. Shreve, Robustness of the Black and Scholes Formula, *Mathematical Finance*, forthcoming.

[5] Frey G., and C. Sin, 1997, Bounds on European Option Prices Under Stochastic Volatility, ETH Zurich working paper.

[6] Hobson, D., 1997, Stochastic Volatility, University of Bath working paper.

[7] Hobson, D., 1999, Volatility Misspecification, Option Pricing and SuperReplication via Coupling, *Annals of Applied Probability***8**, 1, 193-205.

[8] Lyons, T., 1995, "Uncertain Volatility and the Risk-free Synthesis of Derivatives", *Applied Mathematical Finance*, **2**, 117-33.

[9] Lyons, T., 1999, “Capital Adequacy for portfolios of Derivatives in the face of volatility risk”,

[10] Lyons T., and A. Smith, 1999, “Uncertain Volatility, Price Risk, and Path Dependent Derivatives”, Imperial College working paper.

[11] Mykland, P., 1998, Conservative Delta Hedging, University of Chicago working paper.

Part V

Deterministic Local Volatility Surface

Introduction

- Once again assuming that

$$dS_t = (r-q)S_t dt + a(S_t,t)dW_t, \quad t \in [0,T],$$

  Dupire (1994) uncovered a forward PDE which governs standard European option prices. On the domain $K > 0, T > t$, the PDE is:

$$\frac{a^2(K,T)}{2}\frac{\partial^2 V(K,T)}{\partial K^2} - (r-q)K\frac{\partial V(K,T)}{\partial K} - qV(K,T) = \frac{\partial V(K,T)}{\partial T},$$

  where the initial stock price S and the initial time t are held fixed.

- In contrast to the (Black Scholes) implied volatility, the local volatility function $a(K,T)$ can be explicitly represented in terms of option prices, assuming that one has all strikes and maturities.
- However, again in contrast to the Black Scholes model, the option price cannot be explicitly represented in terms of the local volatility surface. This difference manifests itself if the local volatility surface is assumed to be stochastic. The determination of the risk-neutral drift in this case is difficult although solvable.
- In this theory, the local volatility surface can be backed out one day and used to price options on every later date.
- In contrast to the autonomous diffusion approach, the hedge for an option still involves only the stock. In fact, even volatility swaps can be hedged using only the stock in this model.

Related Work

- To reduce the input requirement (i.e. option prices with a double continuum of strikes and maturities), Derman and Kani (1994) implement the same idea in a binomial setting. This reduces the number of options required to the number of nodes on the tree.
- Rubinstein (1994) also works in a binomial setting, and reduces the input requirement even further by requiring only a single strike structure of options. To use less options as inputs, he assumes path-independence, which in continuous time amounts to $S_t = s(t, W_t), t \in [0, T]$ for some $C^{1,2}$ function $s(\cdot, \cdot)$.
- The Dupire approach was empirically tested by Dumas, Fleming, and Whaley (1998).
- They conclude that a more parsimonious model works better both in sample and out of sample.
- In their concluding section, they advocate examining relating the volatility surface to past changes in the index level. We explore a model for this in the next part.

Bibliography

[1] Bouchouev, I. and V. Isakov, 1999, "Uniqueness, stability and numerical methods for the inverse problem that arises in financial markets", *Inverse Problems*, **15**, forthcoming.

[2] Dumas B., J. Fleming, and R. Whaley, 1998 Implied Volatility Functions: Empirical Tests, *Journal of Finance*, **53**, 2059-2106.

[3] Dupire, B. 1994, Pricing With a Smile, *Risk*, **7**, 1, 18–20.

[4] Rubinstein, M. 1994, Implied Binomial Trees, *Journal of Finance*, **49**, 771–818.

Part VI

Stochastic Forward Local Volatility Surface

Introduction

- Suppose that the price process is continuous and that the instantaneous volatility follows an arbitrary stochastic process:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma_t dW_t, \quad t \in [0, T].$$

- Bick (1990) showed that the error from delta-hedging a short claim as if volatility is constant at σ_h is:

$$\int_0^T e^{r(T-t)}(\sigma_h^2 - \sigma_t^2)\frac{S_t^2}{2}\frac{\partial^2}{\partial S^2}V(S_t, t; \sigma_h)dt,$$

  where $V(S_t, t; \sigma_h)$ is the Black Scholes value function.

- Neuberger (1990) noticed that the gamma of the log contract is $\frac{\partial^2}{\partial S^2}V(S_t, t; \sigma_h) = -\frac{e^{-r(T-t)}}{S_t^2}$, so that delta-hedging this contract at constant vol creates (half) a variance swap.
- Dupire (1993) considers a calendar spread of log contracts to lock in the variance between two dates. Mimmicking HJM, he models the evolution of a term structure of these forward variances.
- Carr and Jarrow (1990) explore the "stop-loss start gain" strategy for hedging a call when the stock price is a continuous semimartingale. They conclude that its error is the local time (a.k.a. variance along a strike).

The Local Volatility Surface as an Autonomous Diffusion

- Combining these ideas, Dupire (1996) shows that a calendar spread of European calls is a probe for random local volatility.
- By financing the purchase of the calendar spread with butterfly spreads, a local variance swap is synthesized. The fixed rate on this local variance swap is termed the "forward local variance" $\phi_t^2(T,K)$. Under zero interest rates and dividends:

$$\phi_t^2(T,K) = \frac{\frac{\partial C_t(T,K)}{\partial T}}{\frac{K^2}{2}\frac{\partial^2 C_t(T,K)}{\partial K^2}}, \quad t \in [0,\tau], T \in [t,\tau], K > 0.$$

- Dupire imposes an autonomous diffusion process on the local volatility surface:

$$\frac{dS_t}{S_t} = s_t' dW_t, \text{ where } s_t' s_t = \phi_t^2(t, S_t)$$
$$\frac{d\phi_t^2(T,K)}{\phi_t^2(T,K)} = \alpha_t(T,K)dt + \omega_t(T,K)' dW_t, \quad t \in [0,T].$$

  The restriction on the risk-neutral drift $\alpha_t(T,K)$ is very complicated.
- Derman, Kani, and Kamal (1997) and Britten-Jones and Neuberger (2000) explore these ideas in a binomial context.

Local Volatility Surface Depending on Stock Price History

- Kind, Lipster, and Runggaldier (1991) assume that the stock price process has dependence on the past which occurs only through the quadratic variation process.
- Similarly, Hobson and Rogers (1998) propose that the instantaneous volatility of a stock depend on exponentially weighted moments of historic log-price.
- Markets are complete in these models, and so once again option prices can be determined in a preference-free manner.
- These models are Markov in a larger number of state variables. As Hobson and Rogers note, they can also be combined with the standard assumption that the instantaneous volatility is a function of the contemporaneous stock price.
- Combining these ideas, here's a simple model which captures the spirit of GARCH(1,1) in continuous time:

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma(S_t, t, H_t)dW_t, \quad t \in [0,T],$$

where H_t is the historical variance using exponential weighting:

$$H_t = \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} d\langle \ln S \rangle_s, \quad t \in [0,T],$$

and where κ is a positive constant.

Local Volatility Surface Depending on Stock Price History (Continued)

- Recall the simple model:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma(S_t, t, H_t)dW_t, \quad t \in [0, T],$$

  where H_t is the historical variance using exponential weighting:

$$H_t = \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} d\langle \ln S \rangle_s, \quad t \in [0, T].$$

- The total derivative of H is:

$$dH_t = \kappa[\sigma^2(t, S_t, H_t) - H_t]dt, \quad t \in [0, T].$$

- Thus, the historical variance rises if the local variance is higher than it, and falls otherwise.
- The bivariate process (S_t, H_t) is Markov. Thus, the same techniques used to value Asian options in the Black Scholes model can be used to value standard options in this model.
- For volatility based products (eg. vol swaps) each point on the forward local volatility surface $\{\phi_t(T, K), t \in [0, \tau], T \in [t, \tau], K > 0\}$ can be posited to depend on a single state variable, $H_t(T, K)$ which is defined as the the (exponentially weighted) historical variance conditional on $S_T = K$.

Bibliography

[1] Bick, A., 1990, "Imperfect Dynamic Replication" University of British Columbia working paper.

[2] Bossaerts, P., E. Ghysels, and C. Gourieroux, Arbitrage-Based Pricing When Volatility is Stochastic, California Institute of Technology working paper.

[3] Brenner, M., and D. Galai, 1989, "New Financial Instruments for Hedging Changes in Volatility", *Financial Analyst's Journal*, July-August 1989, 61–65.

[4] Brenner, M., and D. Galai, 1993, "Hedging Volatility in Foreign Currencies", *The Journal of Derivatives*, Fall 1993, 53–9.

[5] Brenner, M., and D. Galai, 1996, "Options on Volatility", Chapter 13 of Option Embedded Bonds, I. Nelken, ed. 273–286.

[6] Britten-Jones, M., and A. Neuberger, 2000, "Option Prices, Implied Price Processes, and Stochastic Volatility", *Journal of Finance* forthcoming.

[7] Carr P. and R. Jarrow, 1990, "The Stop-Loss Start-Gain Strategy and Option Valuation: A New Decomposition into Intrinsic and Time Value", *Review of Financial Studies*, **3**, 469–492.

[8] Cvitanic J., H. Pham and N. Touzi, 1997, Cost of Dominating Strategies in a Stochastic Volatility model under portfolio constraints, Columbia University working paper.

[9] Derman E., and I. Kani, 1997, Stochastic Implied Trees: Arbitrage Pricing with Stochastic Term and Strike Structure of Volatility, Goldman Sachs Quantitative Strategies Technical Note.

[10] Derman, E., Kamal, M., I. Kani, J. McLure, Cyrus Pirasteh, and J. Zhou., 1996, Investing in Volatility, Goldman Sachs Quantitative Strategies Research Note.

[11] Derman E., I. Kani, and M. Kamal, 1997, "Trading and Hedging Local Volatility" *Journal of Financial Engineering*, **6**, 3, 233-68.

[12] Dupire, B., 1992, Arbitrage Pricing with Stochastic Volatility, Societe Generale Division Options, Paris.

[13] Dupire, B., 1993, Model Art, *Risk*, **6**, 9, 118-120.

[14] Dupire, B., 1996, A Unified Theory of Volatility, Banque Paribas working paper.

[15] Galai, D., 1979, "A Proposal for Indexes for Traded Call Options", *Journal of Finance*, XXXIV, 5, 1157–72.

[16] Gastineau, G., 1977, "An Index of Listed Option Premiums", *Financial Analyst's Journal*, May-June 1977.

[17] Hobson, D., and C. Rogers, 1996, Complete Models with Stochastic Volatility, *Mathematical Finance*, forthcoming

[18] Neuberger, A. 1990, "Volatility Trading", London Business School working paper.

[19] Whaley, R., 1993, "Derivatives on Market Volatility: Hedging Tools Long Overdue", *The Journal of Derivatives*, Fall 1993, 71–84.

Part VII

Stochastic Implied Volatility Surface

Introduction

- Avellaneda and Zhu (1998) model the evolution of a single implied volatility.
- Albanese, Carr, and Madan (1998) consider the arbitrage-free evolution of an implied volatility surface.
- The drift restriction on the implied volatility surface is much simpler to express than the drift restriction on the forward local volatility surface.
- The drift has three terms, each of which has a straightforward interpretation.
- LeDoit and Santa Clara (1998) independently arrive at the same model.

The ACM Model Notation

- Define:
$$\ell(F, \sigma, \tau; K) \equiv FN(d_1) - KN(d_2)$$
as the Black model value for the forward price of a call, where as usual:
$$d_1 \equiv \frac{\ln(F/K) + \sigma^2\tau/2}{\sigma\sqrt{\tau}} \quad d_2 \equiv \frac{\ln(F/K) - \sigma^2\tau/2}{\sigma\sqrt{\tau}}.$$
- Let $C_t(T, K)$ be the market *forward* price at t of a European call maturing at $T \geq t$ and struck at $K \geq 0$. The delivery date for this forward is also T, and so $C_t(T, K)$ is the price agreed upon at t to be paid at T in return for the delivery of a then expiring call.
- The implied volatility $\sigma_t(T, K)$ at time t for strike $K > 0$ and maturity $T \geq t$ is implicitly defined by:
$$\ell(F_t, \sigma_t(T, K), T - t; K) = C_t(T, K).$$

The ACM Model Assumptions

- Let τ denote some fixed distant future time
- Let F_t denote the stock's forward price at $t \in [0, \tau]$ with maturity date τ.
- Let $\sigma_t(T, K)$ be the implied (by Black model) (relative) (annualized) volatility (rate) at t for fixed maturity date $T \geq t$ and strike price $K \geq 0$.
- Under an equivalent martingale measure Q,

$$\frac{dF_t}{F_t} = \mathrm{b}_t' \, d\mathrm{W}_t, \tag{1}$$

and:

$$\frac{d\sigma_t(T, K)}{\sigma_t(T, K)} = m_t(T, K)\, dt + \mathrm{v}_t(T, K)' \, d\mathrm{W}_t, \tag{2}$$

for $t \in [0, \tau], \forall\, T \in [t, \tau], \forall\, K \geq 0$, where:

  - $m_t(T, K)$ is the risk-neutral drift of the implied volatilility,
  - $\mathrm{v}_t(T, K)$ is the vol vol vector, and
  - $\mathrm{b}_t'\mathrm{b}_t \equiv \sigma_t^2(t, F_t)$ is the at-the-money, at-the-moment implied variance.

The ACM Model Drift Restriction

- Recall the assumption that implied volatility follows a diffusion:

$$\frac{d\sigma_t(T,K)}{\sigma_t(T,K)} = m_t(T,K)\,dt + \mathrm{v}_t(T,K)'\,d\mathrm{W}_t,$$

where implied volatility is implicitly defined by:

$$\ell(F_t, \sigma_t(T,K), T-t; K) = C_t(T,K).$$

- To obtain a restriction on the risk-neutral drift m, use Itô's lemma to compute the total derivative of the LHS:

$$\begin{aligned} d\ell &= \ell_1\,dF_t + \ell_2\,d\sigma_t(T,K) - \ell_3\,dt \\ &\quad + \frac{1}{2}\ell_{11} d\langle F, F\rangle_t + \ell_{12} d\langle F, \sigma(T,K)\rangle_t + \frac{1}{2}\ell_{22} d\langle \sigma(T,K), \sigma(T,K)\rangle_t. \end{aligned}$$

- Now recall $\frac{dF_t}{F_t} = \mathrm{b}_t'\,d\mathrm{W}_t$. and by the Black P.D.E., $\ell_3 = \frac{\sigma^2 F^2}{2}\ell_{11}$.
- Substituting in these results and the top equation yields:

$$\begin{aligned} d\ell &= \ell_1 F_t \mathrm{b}_t'\,d\mathrm{W}_t + \ell_2[\sigma_t(T,K) m_t(T,K)\,dt + \sigma_t(T,K)\mathrm{v}_t(T,K)'\,d\mathrm{W}_t] \\ &\quad + \frac{1}{2}\ell_{11} F_t^2 [\sigma_t^2(t, F_t) - \sigma_t^2(T,K)]\,dt \\ &\quad + \ell_{12} F_t \sigma_t(T,K) \mathrm{b}_t' \mathrm{v}_t(T,K)\,dt + \frac{1}{2}\ell_{22}\sigma_t^2(T,K)\mathrm{v}_t(T,K)'\mathrm{v}_t(T,K)\,dt \end{aligned}$$

- Collecting coefficients on dt and dW_t implies:

$$\begin{aligned} d\ell &= \Big\{\frac{1}{2}\ell_{11} F_t^2[\sigma_t^2(t,F_t) - \sigma_t^2(T,K)] + \sigma_t(T,K)\Big[\ell_2 m_t(T,K) \\ &\quad + \ell_{12} F_t \mathrm{b}_t' \mathrm{v}_t(T,K) + \frac{1}{2}\ell_{22}\sigma_t(T,K)\mathrm{v}_t(T,K)'\mathrm{v}_t(T,K)\Big]\Big\}\,dt \\ &\quad + \{\ell_1 F_t \mathrm{b}_t' + \ell_2 \sigma_t(T,K)\mathrm{v}_t(T,K)'\}\,d\mathrm{W}_t. \end{aligned}$$

The ACM Model Drift Restriction (Continued)

- Recall:

$$
\begin{aligned}
d\ell \;=\; & \Big\{\frac{1}{2}\ell_{11}F_t^2[\sigma_t^2(t,F_t)-\sigma_t^2(T,K)]+\sigma_t(T,K)\Big[\ell_2 m_t(T,K) \\
& +\ell_{12}F_t \mathrm{b}_t' \mathrm{v}_t(T,K)+\frac{1}{2}\ell_{22}\sigma_t(T,K)\mathrm{v}_t(T,K)'\mathrm{v}_t(T,K)\Big]\Big\}\,dt \\
& +\left\{\ell_1 F_t \mathrm{b}_t' + \ell_2\sigma_t(T,K)\mathrm{v}_t(T,K)'\right\}\,d\mathrm{W}_t.
\end{aligned}
$$

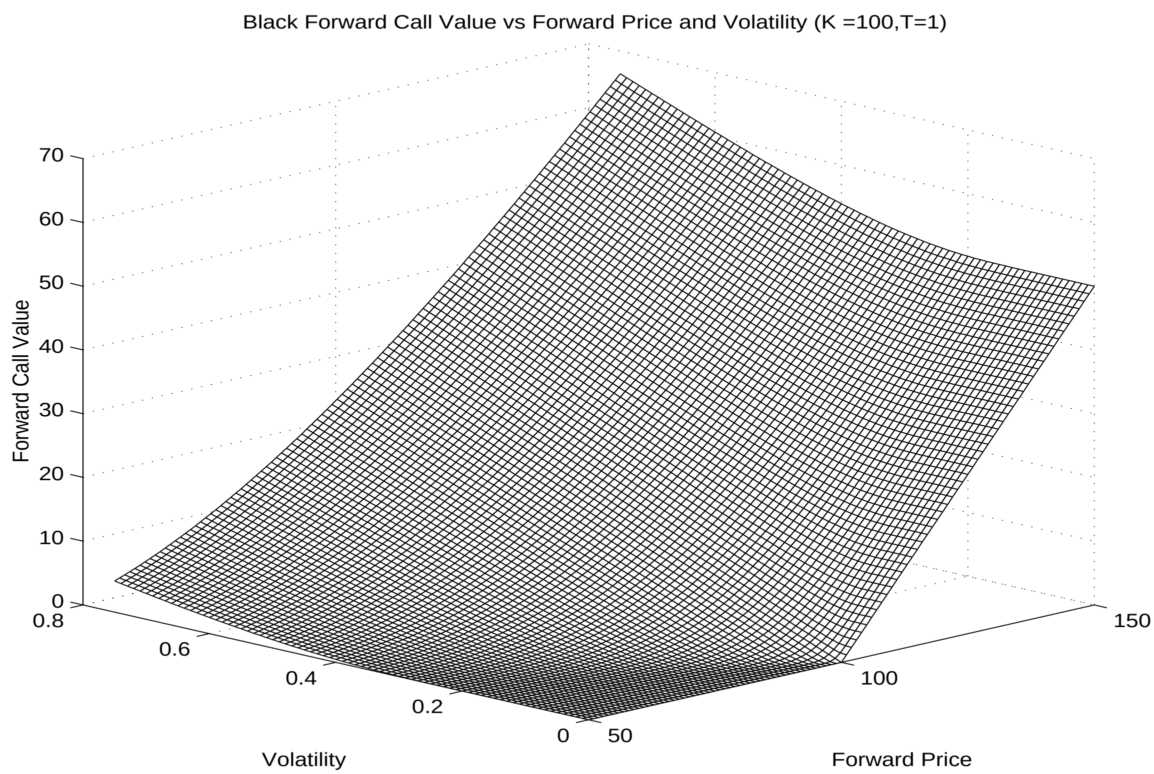
- Setting the drift of the call equal to zero gives our restriction on the drift of the implied:

$$
\begin{aligned}
m_t(T,K) \;=\; & -\frac{\ell_{11}F_t^2}{\ell_2}\frac{\sigma_t^2(t,F_t)-\sigma_t^2(T,K)}{2\sigma_t(T,K)} \\
& -\frac{\ell_{12}}{\ell_2}F_t \mathrm{b}_t' \mathrm{v}_t(T,K) \\
& -\frac{1}{2}\frac{\ell_{22}}{\ell_2}\sigma_t(T,K)\mathrm{v}_t(T,K)'\mathrm{v}_t(T,K).
\end{aligned}
$$

- The drift of the implied must offset the arbitrage profits that would be gained from
  1. The divergence between the gamma trading profit based on the instantaneous variance and the time decay based on the implied.
  2. The Black option value's nonlinearity in F_t and σ_t combined with the covariance between F_t and σ_t.
  3. The Black option value's nonlinearity in implied volatility combined with the randomness in implieds.

Three Sources of Drift in Implied Vol

- Recall that the implied vol drift offsets the arbitrage profits earned from:
  1. The divergence between the gamma trading profit based on the instantaneous variance and the time decay based on the implied.
  2. The Black option value's nonlinearity in F_t and σ_t combined with the covariance between F_t and σ_t.
  3. The Black option value's nonlinearity in implied volatility combined with the randomness in implieds.
- The graph below shows the nonlinear relationship between call value, forward price, and implied:



Simplifying the ACM Model Drift Restriction

- Recall our drift restriction:

$$m_t(T,K) = -\frac{\ell_{11}F_t^2}{\ell_2}\frac{\sigma_t^2(t,F_t)-\sigma_t^2(T,K)}{2\sigma_t(T,K)}$$
$$-\frac{\ell_{12}}{\ell_2}F_t \mathrm{b}_t' \mathrm{v}_t(T,K) - \frac{1}{2}\frac{\ell_{22}}{\ell_2}\sigma_t(T,K)\mathrm{v}_t(T,K)'\mathrm{v}_t(T,K).$$

- Now, straightforward calculations verify that:

$$\ell_1 = N(d_1) \tag{3}$$
$$\ell_2 = K\sqrt{\tau}N'(d_2) = F\sqrt{\tau}N'(d_1)$$
$$\frac{\ell_{11}F_t^2}{\ell_2} = \frac{1}{\sigma\tau}$$
$$\frac{\ell_{12}}{\ell_2}F = -\frac{d_2}{\sigma\sqrt{\tau}}$$
$$\frac{\ell_{22}}{\ell_2}\sigma = d_1 d_2 \tag{4}$$

Letting:

$$d_{1t} \equiv \frac{\ln(F_t/K)+\sigma_t^2(T,K)(T-t)/2}{\sigma_t(T,K)\sqrt{T-t}}, d_{2t} \equiv d_{1t}-\sigma_t(T,K)\sqrt{T-t},$$

substituting (3) to (4) in the top equation implies:

$$m_t(T,K) = \frac{1}{2(T-t)}\left[1-\frac{\sigma_t^2(t,F_t)}{\sigma_t^2(T,K)}\right]$$
$$+\frac{d_{2t}}{\sigma_t(T,K)\sqrt{T-t}}\mathrm{b}_t'\mathrm{v}_t(T,K) - \frac{d_{1t}d_{2t}}{2}\mathrm{v}_t(T,K)'\mathrm{v}_t(T,K).$$

Bibliography

[1] Albanese, Carr, and Madan, 1998, Stochastic Implied Volatility Surfaces, Banc of America Securities working paper.

[2] LeDoit, O., and P. Santa-Clara, 1998, "Relative Pricing of Options with Stochastic Volatility", UCLA working paper.

[3] Zhu, Y., and M. Avellaneda, 1998, A Risk-Neutral Stochastic Volatility Model, *International Journal of Theoretical and Applied Finance*, **1**, 2, 289-310.

Summary

- We reviewed 6 approaches for handling stochastic volatility
  1. instantaneous volatility is a specified function of stock price and time
  2. instantaneous volatility is an autonomous diffusion process
  3. bounding quadratic variation
  4. deterministic local volatility surface
  5. stochastic forward local volatility surface
  6. stochastic implied volatility surface
- To this list, we should add (at least) regime switching models and discrete time GARCH processes
- Papers by me can be downloaded from www.math.nyu.edu/research/carrp/papers

Appendix 1: Elimination of the Market Price of Volatility Risk

One can derive the following PDE governing the function $C^{(1)}(t,S,Y)$ relating the price of an option to time t, the price of the underlying stock S, and the price of a state variable governing volatility Y:

$$\frac{\partial C^{(1)}}{\partial t}+r\left[S\frac{\partial C^{(1)}}{\partial S}-C^{(1)}\right]+[\alpha(m-Y)-\lambda\beta]\frac{\partial C^{(1)}}{\partial Y}+\frac{f^2(Y)}{2}\frac{\partial^2 C^{(1)}}{\partial S^2}+\rho f(Y)\beta\frac{\partial^2 C^{(1)}}{\partial S\partial Y}+\frac{\beta^2}{2}\frac{\partial^2 C^{(1)}}{\partial Y^2}=0, \quad (5)$$

where λ is the market price of volatility risk. The above PDE also holds for the price of a second option:

$$\frac{\partial C^{(2)}}{\partial t}+r\left[S\frac{\partial C^{(2)}}{\partial S}-C^{(2)}\right]+[\alpha(m-Y)-\lambda\beta]\frac{\partial C^{(2)}}{\partial Y}+\frac{f^2(Y)}{2}\frac{\partial^2 C^{(2)}}{\partial S^2}+\rho f(Y)\beta\frac{\partial^2 C^{(2)}}{\partial S\partial Y}+\frac{\beta^2}{2}\frac{\partial^2 C^{(2)}}{\partial Y^2}=0. \quad (6)$$

Let $\gamma(t,S,C^{(2)})$ be the function relating the price of the first option to time t, the underlying stock price S, and the price of the second option:

$$\gamma(t,S,C^{(2)})\equiv C^{(1)}(t,S,Y),$$

where $C^{(2)}$ solves (6).

Equivalently:

$$C^{(1)}(t,S,Y)=\gamma(t,S,C^{(2)}(t,S,Y)) \tag{7}$$

Differentiating once:

$$\frac{\partial C^{(1)}}{\partial t}=\gamma_1+\gamma_3\frac{\partial C^{(2)}}{\partial t} \tag{8}$$

$$\frac{\partial C^{(1)}}{\partial S}=\gamma_2+\gamma_3\frac{\partial C^{(2)}}{\partial S} \tag{9}$$

$$\frac{\partial C^{(1)}}{\partial Y}=\gamma_3\frac{\partial C^{(2)}}{\partial Y} \tag{10}$$

Differentiating one more time:

$$\frac{\partial^2 C^{(1)}}{\partial S^2}=\gamma_{22}+2\gamma_{23}\frac{\partial C^{(2)}}{\partial S}+\gamma_{33}\left(\frac{\partial C^{(2)}}{\partial S}\right)^2+\gamma_3\frac{\partial^2 C^{(2)}}{\partial S^2} \tag{11}$$

$$\frac{\partial^2 C^{(1)}}{\partial S\partial Y}=\gamma_{23}\frac{\partial C^{(2)}}{\partial Y}+\gamma_{33}\frac{\partial C^{(2)}}{\partial S}\frac{\partial C^{(2)}}{\partial Y}+\gamma_3\frac{\partial^2 C^{(2)}}{\partial S\partial Y} \tag{12}$$

$$\frac{\partial^2 C^{(1)}}{\partial Y^2}=\gamma_{33}\left(\frac{\partial C^{(2)}}{\partial Y}\right)^2+\gamma_3\frac{\partial^2 C^{(2)}}{\partial Y^2} \tag{13}$$

Substituting (8) to (13) in (5):

$$\gamma_1 + \gamma_3 \tfrac{\partial C^{(2)}}{\partial t} + r[S[\gamma_2 + \gamma_3 \tfrac{\partial C^{(2)}}{\partial S}] - \gamma] + [\alpha(m-Y) - \lambda\beta]\gamma_3 \tfrac{\partial C^{(2)}}{\partial Y}$$
$$+\tfrac{f^2(Y)}{2}[\gamma_{22} + 2\gamma_{23}\tfrac{\partial C^{(2)}}{\partial S} + \gamma_{33}\left(\tfrac{\partial C^{(2)}}{\partial S}\right)^2 + \gamma_3 \tfrac{\partial^2 C^{(2)}}{\partial S^2}] + \rho f(Y)\beta[\gamma_{23}\tfrac{\partial C^{(2)}}{\partial Y} + \gamma_{33}\tfrac{\partial C^{(2)}}{\partial S}\tfrac{\partial C^{(2)}}{\partial Y} + \gamma_3 \tfrac{\partial^2 C^{(2)}}{\partial S \partial Y}]$$
$$+\tfrac{\beta^2}{2}[\gamma_{33}\left(\tfrac{\partial C^{(2)}}{\partial Y}\right)^2 + \gamma_3 \tfrac{\partial^2 C^{(2)}}{\partial Y^2}] = 0.$$

Re-arranging terms:

$$\gamma_1 + r[S\gamma_2 - \gamma]$$
$$+\gamma_3\left[\tfrac{\partial C^{(2)}}{\partial t} + rS\tfrac{\partial C^{(2)}}{\partial S} + [\alpha(m-Y) - \lambda\beta]\tfrac{\partial C^{(2)}}{\partial Y} + \tfrac{f^2(Y)}{2}\tfrac{\partial^2 C^{(2)}}{\partial S^2} + \rho f(Y)\beta\tfrac{\partial C^{(2)}}{\partial S}\tfrac{\partial C^{(2)}}{\partial Y} + \tfrac{\beta^2}{2}\tfrac{\partial^2 C^{(2)}}{\partial Y^2}\right]$$
$$+\tfrac{f^2(Y)}{2}\gamma_{22} + \gamma_{23}\left[f^2(Y)\tfrac{\partial C^{(2)}}{\partial S} + \rho f(Y)\beta\tfrac{\partial C^{(2)}}{\partial S}\tfrac{\partial C^{(2)}}{\partial Y}\right] + \gamma_{33}\left[\tfrac{f^2(Y)}{2}\left(\tfrac{\partial C^{(2)}}{\partial S}\right)^2 + \rho f(Y)\beta\tfrac{\partial C^{(2)}}{\partial S}\tfrac{\partial C^{(2)}}{\partial Y} + \tfrac{\beta^2}{2}\left(\tfrac{\partial C^{(2)}}{\partial Y}\right)^2\right] = 0.$$

Substituting (6) into the above simplifies the result to:

$$\gamma_1 + r[S\gamma_2 - \gamma] + rC^{(2)}\gamma_3$$
$$+\tfrac{f^2(Y)}{2}\gamma_{22} + \gamma_{23}\left[f^2(Y)\tfrac{\partial C^{(2)}}{\partial S} + \rho f(Y)\beta\tfrac{\partial C^{(2)}}{\partial S}\tfrac{\partial C^{(2)}}{\partial Y}\right] + \gamma_{33}\left[\tfrac{f^2(Y)}{2}\left(\tfrac{\partial C^{(2)}}{\partial S}\right)^2 + \rho f(Y)\beta\tfrac{\partial C^{(2)}}{\partial S}\tfrac{\partial C^{(2)}}{\partial Y} + \tfrac{\beta^2}{2}\left(\tfrac{\partial C^{(2)}}{\partial Y}\right)^2\right] = 0.$$

Now the risk-neutral dynamics of S are:

$$dS_t = rS_t dt + f(Y_t)dW_{1t}.$$

Note that the variance rate of S is:

$$\text{Var}(dS) = f^2(Y_t)dt.$$

The risk-neutral dynamics of $C^{(2)}$ are given by:

$$dC^{(2)} = rC^{(2)}dt + \frac{\partial C^{(2)}}{\partial S}f(Y_t)dW_{1t} + \frac{\partial C^{(2)}}{\partial Y}\beta dW_{2t},$$

where $dW_{1t}dW_{2t} = \rho dt$. Note that the covariance rate of $C^{(2)}$ with S is:

$$\text{Cov}(dC^{(2)}, dS) = \left[f^2(Y)\frac{\partial C^{(2)}}{\partial S} + \rho f(Y)\beta\frac{\partial C^{(2)}}{\partial S}\frac{\partial C^{(2)}}{\partial Y}\right]dt,$$

while the variance rate of $C^{(2)}$ is:

$$\text{Var}(dC^{(2)}) = \left[f^2(Y)\left(\frac{\partial C^{(2)}}{\partial S}\right)^2 + 2\rho f(Y)\beta\frac{\partial C^{(2)}}{\partial S}\frac{\partial C^{(2)}}{\partial Y} + \beta^2\left(\frac{\partial C^{(2)}}{\partial Y}\right)^2\right]dt.$$

Substituting into the above PDE gives:

$$\gamma_1 + r[S\gamma_2 - \gamma] + rC^{(2)}\gamma_3 + \frac{1}{2}\frac{\text{Var}(dS)}{dt}\gamma_{22} + \frac{\text{Cov}(dC^{(2)}, dS)}{dt}\gamma_{23} + \frac{1}{2}\frac{\text{Var}(dC^{(2)})}{dt}\gamma_{33} = 0. \tag{14}$$

If one can exogenously model the volatility structure of $C^{(2)}$ without reference to λ, then one does not need to specify λ. There are many ways to do this.