

Static Hedges for Barrier and Lookback Options

or Knocking Out the Barriers to Understanding Exotics

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Introduction

- We show how to value and hedge certain types of exotic options without knowledge of advanced mathematics.
- The hedging strategies are intuitive and simple. The simplicity arises from using standard options in the replicating portfolio along with the underlying asset.
- As a result, analytic results and intuition developed for standard options can be easily transferred to exotic options.
- Furthermore, in contrast to standard approaches, our hedges are static; there is no need to dynamically rebalance the portfolio, saving the hedger both transactions costs and headaches.

Barrier and Lookback Options

- We consider the valuation and hedging of barrier options and lookback options.
- Barrier options come in several flavors:
 - up vs. down
 - in vs. out
 - call vs. put
- Similarly, lookback options come in several flavors:
 - extrema is underlying vs. extrema is strike
 - call vs. put.
- We do not explicitly deal with barrier or lookback options which are:
 - American
 - partial
 - multi-factor

However, some of our insights extend to these types.

Overview of the Presentation

- Throughout the presentation, we assume frictionless markets and no arbitrage.
- Under these and other assumptions, exact valuation and hedging of barrier and lookback options arises whenever the asset underlying the option has no carrying cost.
- Thus, our analysis applies without restriction to options written on forward or futures prices.
- For concreteness, we focus on options on the spot price of a stock.
- For these options, the assumption of no carrying cost implies that the dividend yield equals the interest rate over the life of the option.
- We first use the zero carry assumption in order to develop exact hedges for options on the spot.
- We then relax the zero carry assumption and develop tight bounds on the values of exotic options on the spot.

Barrier Options under Zero Carrying Costs

- We initially focus on barrier options with no rebates.
- Consequently, it suffices to discuss knockins, since knockouts are valued by subtracting off our knockin value from that of a standard option.
- For concreteness, we examine the valuation and hedging of down-and-in calls; up options and puts are treated analogously.
- For now, assume frictionless markets, no arbitrage, and zero carry (for options on the spot).

Down-and-In Call When Barrier Equals Strike

- The easiest case to deal with arises when the barrier of the down-and-in call is equal to the strike.
- In this case, the sale of a down-and-in call is hedged by going long a standard put with the same underlying, maturity, and strike as the knockin.
- If the underlying stays above the barrier (see path B in Figure 1), then both the down-and-in call and the put will die worthless.

Figure 1

Figure 1

- On the other hand, if the underlying is ever at the barrier (see path C in Figure 1), then at such times, put-call parity implies that the standard put has the same value as the standard call.
- At the first time the underlying touches the barrier, the hedger can sell off his standard put and buy a standard call, without incurring any out-of-pocket expense.
- Consequently, buying a standard put at initiation is an exact hedge for writing a down-and-in call.
- Thus, prior to hitting the barrier, the value of a down-and-in call with strike K and barrier H , denoted $DIC(K, H)$, is the same as that of a standard put struck at K , denoted $P(K)$, when $H = K$, i.e.:

$$DIC(K, H) = P(K), \quad H = K. \quad (1)$$

- For example, if the strike and barrier of a down-and-in call are both set at 4, then whenever the spot is above 4, this down-an-in call has the same value as a standard put struck at 4.

Assumptions

- To deal with the more difficult cases where the barrier differs from the strike, we now assume the standard Black Scholes model:
 - frictionless markets,
 - no arbitrage,
 - constant interest rate r
 - constant dividend yield δ ,
 - the stock price, S , obeys geometric Brownian motion with a constant volatility rate, σ .
- When dealing with options on the spot, we continue to assume that the dividend yield equals the interest rates, i.e. $\delta = r$.

Put-Call Symmetry

- When the barrier of the call is below the strike, the previous hedging strategy does not work.
- In this case, the emergent call is out-of-the-money when the underlying touches the barrier.
- For this reason, a put and a call with equal strikes no longer have equal value at this point.
- To hedge a down-and-in call in this case, we use a result known as “Put Call Symmetry”.
- The symmetry is depicted in Figure 2. Figure 2 graphs the lognormal distribution describing the spot at expiration, as well as the payoff from a call struck at 4 and from *two* puts struck at 1. When the spot is at 2, the out-of-the-money call has the same value as the two out-of-the-money puts.

Figure 2

Figure 2

- We can imagine that a mirror has been placed at the spot level of 2. The reflection in this mirror of the payoff from the call struck at 4 is the payoff from the two puts struck at 1.
- More generally, the reflection of the payoff from a call struck at K when the spot is at H is the payoff from $\frac{K}{H}$ puts struck at $\frac{H^2}{K}$.
- The put strikes are chosen so that the geometric mean of the call and put strike is the barrier, eg. the geometric mean of 4 and 1 is $\sqrt{4 \times 1} = 2$.
- The number of puts chosen is the ratio of distances to the respective strikes when the spot is at the barrier, eg. when the spot is at 2, the distance to the call strike is $4 - 2 = 2$, while the distance to the put strike is $2 - 1 = 1$, so the number of puts purchased is $\frac{2}{1} = 2$.
- This ratio ensures that the puts have the same value as a standard call *when-ever* the spot is at the barrier.

Down-and-In Calls when Barrier Below Strike

- From Put-Call Symmetry, we can see that the sale of a down-and-in call with strike K and barrier $H \leq K$ is hedged by going long $\frac{K}{H}$ puts, each with strike $\frac{H^2}{K}$.
- For example, the sale of a down-and-in call with strike 4 and barrier 2 is hedged by going long 2 puts, each with strike 1.
- Note that when $H = K$, we obtain the same hedge as before.
- Moving forward through time, the hedge works in the same manner as before.
 - When the call strike is above the barrier as assumed, the put strike must be below the barrier, in order that their geometric mean is equal to the barrier. Consequently, if the spot never touches the barrier of 2, the hedger is assured that both the hedged item (call struck at 4) and the hedging instruments (2 puts struck at 1) expire worthless.
 - If the spot touches the barrier, then at that moment, Put Call Symmetry implies that the 2 puts struck at 1 have the same value as 1 call struck at 4. Consequently, the hedger can sell off both puts and buy a call without incurring any out-of-pocket expense.
- Since buying 2 standard puts is an exact hedge for writing this down-and-in call, it follows that prior to hitting the barrier, a down-and-in call has the same value and behavior as two standard puts, or more generally:

$$DIC(K, H) = \frac{K}{H} P\left(\frac{H^2}{K}\right), \quad H \leq K. \quad (2)$$

Hedging Intrinsic Value

- So far, we have dealt with a down-and-in call, where the call, once knocked in, had zero intrinsic value at the barrier.
- Thus, we have really been hedging the *time value* of a down-and-in call.
- We now develop a static hedge for the *intrinsic value* of the down-and-in call at the barrier, so that we can tackle the knockin problem when the knocked in call has positive intrinsic value at the barrier.
- Define a down-and-in *bond* as a security which pays \$1 at expiration if and only if the underlying touches the barrier prior to expiration.
- Then a portfolio of $H - K$ down-and-in bonds has the same value at the barrier as the intrinsic value of the emergent call.
- It turns out that a down-and-in bond can be synthesized using standard and *binary* puts. By definition, a binary put pays out \$1 if the underlying is less than a fixed strike at expiration. A binary put struck at K , denoted $BP(K)$, can be valued and hedged using standard puts as follows:

$$BP(K) = \lim_{n \uparrow \infty} \frac{n}{2} \left[P \left(K + \frac{1}{n} \right) - P \left(K - \frac{1}{n} \right) \right], \quad (3)$$

In words, the investor buys a large number of standard puts struck just above the binary put strike and writes an equal number of standard puts struck just below the binary put strike.

- Clearly, the problem in implementing (3) in practice is the requirement to trade an infinite number of options to obtain exact hedging. We soon determine how many options are required to obtain reasonable convergence. For now, note that the above centered difference approximation converges faster than either a forward or backward difference.

Down-and-in Bonds and Binary Puts

- To synthesize a down-and-in bond with barrier H , an investor should buy *two* binary puts struck at H and write $\frac{1}{H}$ standard puts struck at H .
- If the spot stays above the barrier, then the down-and-in bond and the hedging instruments all expire worthless.
- If the spot touches the barrier, then at that moment, the (risk-neutral) probabilities of finishing above and below the barrier are *roughly* equal. If the distribution were symmetric, then when the spot is at the barrier, the two at-the-money binary puts could be liquidated for proceeds sufficient to buy a bond paying \$1 at expiration.
- However, the assumed lognormal distribution is left-skewed. As a result, when the spot is at the barrier, the (risk-neutral) probability of finishing above the barrier is less than the probability of finishing below. Consequently, in order to purchase a bond paying \$1 at expiration the hedger needs less money than that generated by the sale of the two at-the-money binary puts. By writing $\frac{1}{H}$ of a standard put initially and covering when the spot touches the barrier, the hedger ensures that the transition at the barrier is self-financing.
- As a result, the down-and-in bond value is given by:

$$DIB(H) = 2BP(H) - \frac{1}{H}P(H).$$

Down-and-In Bonds and Standard Puts

- Recall that the down-and-in bond value is given by:

$$DIB(H) = 2BP(H) - \frac{1}{H}P(H).$$

- Also recall that the binary put is replicated using standard puts by:

$$BP(K) = \lim_{n \uparrow \infty} \frac{n}{2} \left[P\left(K + \frac{1}{n}\right) - P\left(K - \frac{1}{n}\right) \right], \quad (4)$$

- Since the binary put is more efficiently replicated using standard puts struck just above and below H , less strikes are traded when synthesizing the down-and-in bond if we replace $P(H)$ with an average of puts struck just above and below H :

$$DIB(H) = 2BP(H) - \frac{1}{H} \lim_{n \uparrow \infty} \frac{P\left(H + \frac{1}{n}\right) + P\left(H - \frac{1}{n}\right)}{2}. \quad (5)$$

- Substituting (4) in (5) implies that the down-and-in bond can be valued in terms of standard puts alone:

$$DIB(H) = \lim_{n \uparrow \infty} \left(n - \frac{1}{2H} \right) P\left(H + \frac{1}{n}\right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2H} \right) P\left(H - \frac{1}{n}\right). \quad (6)$$

Down-and-in Rebates

- An immediate bonus of this analysis is that *rebates* associated with knockin options can be hedged and therefore valued.
- The rebate attached to a down-and-in call is paid at expiration if the underlying stays above the barrier.
- Clearly, a rebate of $\$R$ attached to a down-and-in call is synthesized by going long R standard bonds paying $\$1$ at expiration and shorting R down-and-in bonds.

$$DIR(H) = Re^{-rT} - R \cdot DIB(H). \quad (7)$$

- Recall that the down-and-in bond can be valued in terms of standard puts:

$$DIB(H) = \lim_{n \uparrow \infty} \left(n - \frac{1}{2H} \right) P \left(H + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2H} \right) P \left(H - \frac{1}{n} \right). \quad (8)$$

- Substituting (8) in (7) implies that the down-and-in rebate can be valued in terms of standard puts alone:

$$DIR(H) = R \left[e^{-rT} - \lim_{n \uparrow \infty} \left(n - \frac{1}{2H} \right) P \left(H + \frac{1}{n} \right) + \lim_{n \uparrow \infty} \left(n + \frac{1}{2H} \right) P \left(H - \frac{1}{n} \right) \right]. \quad (9)$$

Down-and-In Call and Down-and-in Bond

- We now address the final and most difficult case where the knockin option has intrinsic value at the barrier.
- The initial sale of a down-and-in call with strike K and barrier $H \geq K$ is hedged by going long $H - K$ down-and-in bonds and long one standard put struck at K .
- Simply speaking, the down-and-in bonds are purchased to provide the intrinsic value of the call at the barrier, while the put is purchased to provide the time value of the call at the barrier.
- To illustrate, the initial sale of a down-and-in call with strike 4 and barrier 6 is hedged by going long $6 - 4 = 2$ down-and-in bonds with barrier 6 and long 1 standard put struck at 4.
- Moving forward through time, the hedge proceeds as follows. If the spot stays above the barrier of 6, then the down-and-in call and the hedging instruments all expire worthless. If the spot touches the barrier of 6, then the 2 down-and-in bonds knock in. By put-call parity, these bonds and the standard put struck at 4 can be sold to finance the purchase of a standard call struck at 4.
- Since buying 2 down-and-in bonds with barrier 6 and buying a standard put struck at 4 is an exact hedge for writing the down-and-in call, it follows that prior to hitting the barrier, a down-and-in call has the same value and behavior as this portfolio, or more generally:

$$DIC(K, H) = (H - K)DIB(H) + P(K), \quad H \geq K. \quad (10)$$

Down-and-In Call and Standard Puts

- Recall that the value of a down-and-in call could be decomposed into the initial cost of hedges for its intrinsic and time value at the barrier:

$$DIC(K, H) = (H - K)DIB(H) + P(K), \quad H \geq K. \quad (11)$$

- Also recall that the down-and-in bond could be replicated using standard puts alone:

$$DIB(H) = \lim_{n \uparrow \infty} \left(n - \frac{1}{2H} \right) P \left(H + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2H} \right) P \left(H - \frac{1}{n} \right). \quad (12)$$

- Substituting (12) in (11) implies that the down-and-in call can be valued in terms of standard puts alone:

$$DIC(K, H) = (H - K) \left[\lim_{n \uparrow \infty} \left(n - \frac{1}{2H} \right) P \left(H + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2H} \right) P \left(H - \frac{1}{n} \right) \right] + P(K).$$

- Table 1 shows the speed of convergence of the hedge as a function of n .
- With $n = 10$, the relative error is $-\frac{5}{8}\%$, which is certainly within the bid-offer spread for these types of options.

Table 1: S=2, K=1.8, H=1.9, $r = \delta = 4\%$, $\sigma = 15\%$, T=1 year

n	$DIC(K, H)$	$=P(K)$	$+(H - K) \left(n - \frac{1}{2H} \right)$	$\times P \left(H + \frac{1}{n} \right)$	$-\left(n + \frac{1}{2H} \right)$	$\times P \left(H - \frac{1}{n} \right)$
10	0.110836796	0.038849078	0.973684211	0.11186	1.026315789	0.039871
100	0.110994179	0.038849078	9.973684211	0.74105	10.02631579	0.668902
1,000	0.110984592	0.038849067	99.97368421	7.08037	100.0263158	7.008238
10,000	0.111051003	0.038849078	999.9736842	70.4786	1000.026316	70.40637

Lookback Options

- A lookback call pays off the difference between the ending and minimum spot over the option's life.
- Goldman, Sosin, and Gatto (JF 79) indicate how to value and hedge a lookback call using a dynamic strategy in straddles when $r = \delta + \frac{\sigma^2}{2}$.
- We show how to value and hedge a lookback call using a static strategy in standard puts when $r = \delta$.
- Furthermore, we explicitly account for the discreteness of the tick size. Lookback puts can be valued and hedged analogously.
- The intuition for our results lies in the ability to rephrase the question, “what was the minimum stock price” as several questions.
- For example, when the underlying starts at 2, we could find the minimum underlying price by asking the following questions:
 1. Did the underlying get as low as $\frac{1}{8}$?
 2. Did the underlying get as low as $\frac{1}{4}$?
 3. Did the underlying get as low as $\frac{3}{8}$? etc.
- The payoff from a down-and-in bond with barrier H can be interpreted as the answer to the logical question “Did the underlying get as low as H ?”
- The payoff is \$1 if the answer is yes and \$0 if the answer is no.
- This intuition suggests a relationship between lookback options and down-and-in bonds which we now explore.

Lookback Calls and Down-and-In Bonds

- Let I be the size of one tick and let N be the number of potential spot levels below the current spot, including zero. For example, if the tick size is $I = \frac{1}{8}$ and the current spot is at 2, then N is $\frac{S}{I} = 16$.
- To hedge the sale of a lookback call, an investor should go long one (zero cost) forward contract and buy I units of each of the $N - 1$ different down-and-in bonds which have positive barriers below the current spot. In the previous example, the investor buys $\frac{1}{8}$ of a down-and-in bond with barrier $1\frac{7}{8}$, $\frac{1}{8}$ of a down-and-in bond with barrier $1\frac{6}{8} = 1\frac{3}{4}$, and so on, with the last $\frac{1}{8}$ of a down-and-in bond having a barrier at $\frac{1}{8}$.
- If the spot never falls below its initial level over the life of the option (see path B of Figure 3), then the forward contract pays $S_T - S_0$, while the down-and-in bonds expire worthless. Since S_0 is the minimum stock price in this case, this payoff matches that of the lookback.

Figure 3

Figure 3

- If instead the minimum spot ends up being one tick below the initial spot, then only the first down-and-in bond comes in, and the payoff is $S_T - S_0 + I = S_T - (S_0 - I)$, which again matches that of the lookback.
- As the spot reaches each new low, another down-and-in bond knocks in, enhancing the payoff to match that of the lookback.
- Equating the lookback value to the cost of the replicating portfolio gives:

$$LC = I \sum_{i=1}^{N-1} DIB(iI). \quad (13)$$

Lookback Calls and Standard Puts

- Recall that the lookback call could be replicated using down-and-in bonds:

$$LC = I \sum_{i=1}^{N-1} DIB(iI). \quad (14)$$

- Also recall that the down-and-in bond could be replicated using standard puts alone:

$$DIB(H) = \lim_{n \uparrow \infty} \left(n - \frac{1}{2H} \right) P \left(H + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2H} \right) P \left(H - \frac{1}{n} \right). \quad (15)$$

- Substituting (15) in (14) implies that the lookback call can be valued in terms of standard puts alone:

$$LC = I \sum_{i=1}^{N-1} \left[\lim_{n \uparrow \infty} \left(2n - \frac{1}{iI} \right) P \left(iI + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(2n + \frac{1}{iI} \right) P \left(iI - \frac{1}{n} \right) \right]. \quad (16)$$

Barrier Options Under Nonzero Carrying Costs

- We now relax the assumption that carrying costs are zero.
- When the dividend yield differs from the interest rate, the forward differs from the spot.
- Let $\hat{H} \equiv He^{(r-\delta)T}$ be the initial “forward barrier”, i.e. if $S = H$ at initiation, then the forward $F = \hat{H}$ by the cost of carry relation.
- When carrying costs are positive i.e. $r > \delta$, then the forward is above the spot, $F > S$, and similarly, the initial forward barrier is above the spot barrier, $\hat{H} > H$. In this case, when the barrier of a down-and-in call is below its strike, (2) becomes:

$$\frac{K}{H}P\left(\frac{H^2}{K}\right) \leq DIC(K, H) \leq \frac{K}{\hat{H}}P\left(\frac{\hat{H}^2}{K}\right), \quad H \leq K. \quad (17)$$

- Table 1 shows the tightness of the bounds for various parameter values.
- Figures 4 to 9 show the effect of varying each parameter on the bounds.
- When the barrier of a down-and-in call is above its strike, (11) becomes:

$$\begin{aligned} & (H - K) \left[\lim_{n \uparrow \infty} \left(n - \frac{1}{2H} \right) P \left(H + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2H} \right) P \left(H - \frac{1}{n} \right) \right] \\ & \leq DIC(K, H) - P(K) \leq \\ & (\hat{H} - K) \left[\lim_{n \uparrow \infty} \left(n - \frac{1}{2\hat{H}} \right) P \left(\hat{H} + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2\hat{H}} \right) P \left(\hat{H} - \frac{1}{n} \right) \right]. \end{aligned}$$

Negative Carrying Costs

- In contrast, when carrying costs are negative, i.e. $r < \delta$, then the forward is below the spot, $F < S$, and similarly, the initial forward barrier is below the spot barrier, $\hat{H} < H$.
- In this case, when the barrier of a down-and-in call is below its strike, (2) becomes:

$$\frac{K}{\hat{H}} \left(\frac{\hat{H}^2}{K} \right) \leq DIC(K, H) \leq \frac{K}{H} P \left(\frac{H^2}{K} \right), \quad H \leq K. \quad (18)$$

- When the barrier of a down-and-in call is above its strike, (11) becomes:

$$\begin{aligned} & (\hat{H} - K) \left[\lim_{n \uparrow \infty} \left(n - \frac{1}{2\hat{H}} \right) P \left(\hat{H} + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2\hat{H}} \right) P \left(\hat{H} - \frac{1}{n} \right) \right] \\ & \leq DIC(K, H) - P(K) \leq \\ & (H - K) \left[\lim_{n \uparrow \infty} \left(n - \frac{1}{2H} \right) P \left(H + \frac{1}{n} \right) - \left(n + \frac{1}{2H} \right) \lim_{n \uparrow \infty} P \left(H - \frac{1}{n} \right) \right]. \end{aligned}$$

Arbitrage

- If any of these inequalities is violated, an arbitrage opportunity arises.
- For example, recall the inequality:

$$DIC(K, H) \leq \frac{K}{H} P\left(\frac{H^2}{K}\right), \quad H \leq K.$$

- If this upper bound is violated, then the down-and-in call should be sold and hedged by buying $\frac{K}{H}$ standard puts struck at $\frac{H^2}{K}$.
- If the spot never hits the barrier, then the down-and-in call and the standard puts expire worthless as usual.
- If the spot hits the barrier, then the puts should be sold off at that time with the proceeds used to purchase one call.
- In contrast to the zero carry case, there will be funds left over.
- As a result, the down-and-in call value is bounded above by the cost of this “super” replicating portfolio of puts.

Down-and-In Rebates

- When carrying costs are positive i.e. $r > \delta$, then the forward is above the spot, $F > S$, and similarly, the initial forward barrier is above the spot barrier, $\hat{H} > H$.
- In this case, (9) describing the rebate of a down-and-in call becomes:

$$\begin{aligned}
 & R \left[e^{-rT} - \lim_{n \uparrow \infty} \left(n - \frac{1}{2H} \right) P \left(H + \frac{1}{n} \right) + \lim_{n \uparrow \infty} \left(n + \frac{1}{2H} \right) P \left(H - \frac{1}{n} \right) \right] \\
 & \leq DIR(H) \leq \\
 & R \left[e^{-rT} - \lim_{n \uparrow \infty} \left(n - \frac{1}{2\hat{H}} \right) P \left(\hat{H} + \frac{1}{n} \right) + \lim_{n \uparrow \infty} \left(n + \frac{1}{2\hat{H}} \right) P \left(\hat{H} - \frac{1}{n} \right) \right].
 \end{aligned}$$

- In contrast, when carrying costs are negative i.e. $r < \delta$, then the forward is below the spot, $F < S$, and similarly, the initial forward barrier is below the spot barrier, $\hat{H} < H$. In this case, (9) describing the rebate of a down-and-in call becomes:

$$\begin{aligned}
 & R \left[e^{-rT} - \lim_{n \uparrow \infty} \left(n - \frac{1}{2\hat{H}} \right) P \left(\hat{H} + \frac{1}{n} \right) + \lim_{n \uparrow \infty} \left(n + \frac{1}{2\hat{H}} \right) P \left(\hat{H} - \frac{1}{n} \right) \right] \\
 & \leq DIR(H) \leq \\
 & R \left[e^{-rT} - \lim_{n \uparrow \infty} \left(n - \frac{1}{2H} \right) P \left(H + \frac{1}{n} \right) + \lim_{n \uparrow \infty} \left(n + \frac{1}{2H} \right) P \left(H - \frac{1}{n} \right) \right].
 \end{aligned}$$

Down-and-Out Rebates

- The rebate of a down-and-out call is paid if and when the option knocks out.
- If the rebate pays R the first time that the spot hits the barrier H , then the value of this down-and-out rebate $DOR(H)$ is bracketed as follows:

$$R \geq DOR(H)$$

$$DOR(H) \geq R \lim_{n \uparrow \infty} \left(n - \frac{1}{2H} \right) P \left(H + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2H} \right) P \left(H - \frac{1}{n} \right).$$

- When carrying costs are positive i.e. $r > \delta$, then the forward is above the spot, $F > S$, and similarly, the initial forward barrier is above the spot barrier, $\hat{H} > H$.
- In this case, the rebate of a down-and-out call becomes:

$$R \geq DOR(H)$$

$$DOR(H) \geq R \left[\lim_{n \uparrow \infty} \left(n - \frac{1}{2H} \right) P \left(H + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2H} \right) P \left(H - \frac{1}{n} \right) \right].$$

- In contrast, when carrying costs are negative i.e. $r < \delta$, then the forward is below the spot, $F < S$, and similarly, the initial forward barrier is below the spot barrier, $\hat{H} < H$.
- In this case, the rebate of a down-and-out call becomes:

$$R \geq DOR(H)$$

$$DOR(H) \geq R \left[\lim_{n \uparrow \infty} \left(n - \frac{1}{2\hat{H}} \right) P \left(\hat{H} + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2\hat{H}} \right) P \left(\hat{H} - \frac{1}{n} \right) \right].$$

Lookback Options Under Nonzero Carry

- In analogy to the above, let $\hat{I} \equiv Ie^{(r-\delta)T}$ be the “forward” tick size.
- When carrying costs are positive, i.e. $r > \delta$, then the forward is above the spot, $F > S$, and similarly, the forward tick size is above the spot tick size, $\hat{I} > I$.
- In this case, (16) becomes:

$$\begin{aligned} & I \sum_{i=1}^{N-1} \left[\lim_{n \uparrow \infty} \left(n - \frac{1}{2iI} \right) P \left(iI + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2iI} \right) P \left(iI - \frac{1}{n} \right) \right] \\ \leq LC & \leq \hat{I} \sum_{i=1}^{N-1} \left[\lim_{n \uparrow \infty} \left(n - \frac{1}{2i\hat{I}} \right) P \left(i\hat{I} + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2i\hat{I}} \right) P \left(i\hat{I} - \frac{1}{n} \right) \right]. \end{aligned}$$

- In contrast, when carrying costs are negative, i.e. $r < \delta$, then the forward is below the spot, $F < S$, and similarly, the forward tick size is below the spot tick size, $\hat{I} < I$.
- In this case, (14) becomes:

$$\begin{aligned} & \hat{I} \sum_{i=1}^{N-1} \left[\lim_{n \uparrow \infty} \left(n - \frac{1}{2i\hat{I}} \right) P \left(i\hat{I} + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2i\hat{I}} \right) P \left(i\hat{I} - \frac{1}{n} \right) \right] \\ \leq LC & \leq I \sum_{i=1}^{N-1} \left[\lim_{n \uparrow \infty} \left(n - \frac{1}{2iI} \right) P \left(iI + \frac{1}{n} \right) - \lim_{n \uparrow \infty} \left(n + \frac{1}{2iI} \right) P \left(iI - \frac{1}{n} \right) \right]. \end{aligned}$$

Summary and Conclusions

- This paper has developed static hedges for barrier and lookback options, by using standard options as hedging vehicles.
- When the underlying has no carrying cost, the portfolio of standard options exactly replicates the path-dependent option's payoff, providing exact valuation formulas.
- When the underlying has a carrying cost, we provided tight lower and upper bounds on the exotic option's value.