STATIC SIMPLICITY

Hedging barrier and lookback options need not be complicated. Jonathan Bowie **and** Peter Carr **provide static hedging techniques using standard options**

The ability to value and hedge exotic options normally requires an extensive knowledge of advanced mathematics. The solutions generated are seldom intuitive and rarely provide a simple hedge.

This article shows how certain types of exotic option can be valued without advanced mathematics. The hedging strategies are intuitive and simple. This simplicity arises from using standard options in the replicating portfolio[1] along with the underlying asset. As a result, analytic results and intuition developed for standard options can be easily transferred to exotics. Furthermore, in contrast to standard approaches, our hedges are static; there is no need to rebalance the portfolio dynamically, which saves the hedger both transaction costs and headaches.

We consider the valuation and hedging of barrier options and lookback options. Barrier options have the same hockey-stick payoffs as standard options, provided they are alive at expiry. A knock-in option pays nothing at expiry unless it is first brought to life by touching a barrier during the life of the option. In contrast, a knock-out option starts life as a regular option, but is killed off if the underlying touches the barrier. A knock-out sometimes has a rebate paid when the option knocks out, while a knock-in sometimes has a rebate paid at expiry if the option fails to knock in. When there are no rebates, a portfolio consisting of a knock-out and a knock-in (with the same underlying, strike, barrier and maturity) has the same payoffs and value as a standard option.[2]

A lookback call (put) is a standard option whose strike price is the minimum (maximum) price achieved by the underlying over the option's life. In contrast to standard options, lookback options never finish out of the money.[3] We assume frictionless markets and no arbitrage throughout.

We also assume zero carrying cost, and for concreteness will discuss foreign exchange options on the spot. Readers should have no difficulty in translating our results to options on spot, forward and futures prices of foreign exchange, stocks, stock indexes and commodities. The zero carrying cost implies domestic and foreign interest rates are equal over the option's life. We relax this assumption in the third section.

We initially focus on barrier options with no rebates. Consequently, it is sufficient to discuss knock-ins, since knock-outs are valued by subtracting our knock-in value from that of a standard option. We examine the valuation and hedging of down-and-in calls; up options and puts are treated analogously.

The easiest case is when the barrier of the down-and-in call is equal to the strike. In this case, the sale of a down-and-in call is hedged by going long a standard put with the same underlying, maturity and strike as the knock-in. If the underlying stays above the barrier (see path B in figure 1), then both the down-and-in call and the put will die worthless.

On the other hand, when the underlying is at the barrier (see path C in figure 1), put-call parity implies that the standard put has the same value as the standard call. The first time the underlying touches the barrier, the hedger can sell off his standard put and buy a standard call, without incurring any out-of-pocket expense. Consequently, buying a standard put at initiation is an exact hedge for writing a down-and-in call. Thus, prior to hitting the barrier, the value of a down-and-in call with strike K and barrier H, denoted DIC(K,H), is the same as that of a standard put struck at K, denoted P(K), when H = K:

$$DIC(K,H) = P(K), \quad H = K \qquad (1)$$

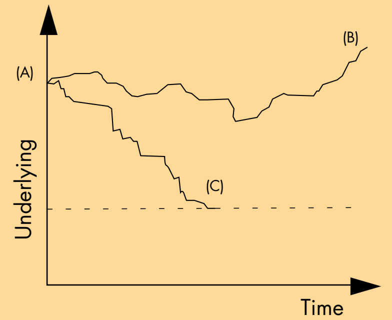
To deal with the more difficult cases where the barrier differs from the strike, we now assume the standard Garman-Kohlhagen (1983) model – frictionless markets, no arbitrage, constant interest rates r and r_f, and that the spot, S, obeys geometric Brownian motion with a constant volatility rate, σ. When dealing with options on the spot we continue to assume equal domestic and foreign interest rates, $r = r_f$.

The treatment of a down-and-in call when the strike lies below the barrier is analogous to that when the strike is equal to the barrier. We still rely on the hedge having equal value to the call at the barrier. We still execute the same hedging strategy, selling the hedge if the spot touches the barrier and doing nothing if the spot stays above it.

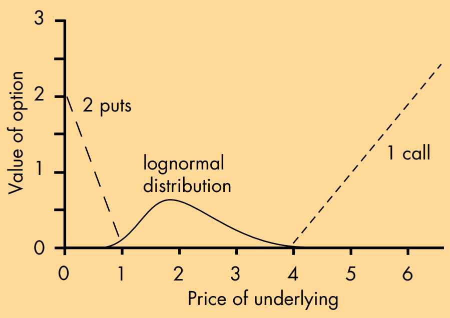
Unlike the previous situation, however, the emergent call is out of the money when the underlying touches the barrier. For this reason, a put and a call with equal strikes no longer have equal value at this point. Although it is less obvious what the hedge should be, we can use a result known as "put call symmetry'' (see Carr, 1994) to tell us how to hedge a down-and-in call.

The symmetry is depicted in figure 2, which graphs the lognormal distribution describing the spot at expiry, as well as the payoff from a call struck at 4 and from *two* puts struck at 1. When the spot is at 2, the out-of-the money call has the same value as the two out-of-the-money puts. We can imagine

1. Barrier option with no rebates



2. Put-call symmetry



[1] In complementary work Derman, Ergener and Kani (1994) consider static replication of exotic options with standard options
[2] For an exhaustive list of barrier option valuation formulas in the Black Scholes model, see Reiner and Rubinstein (1991a)
[3] For lookback option valuation formulas in the Black Scholes model, see Goldman, Sosin and Gatto (1979) and Garman (1990)

that a mirror has been placed at the spot level of 2. The reflection in this mirror of the payoff from the call struck at 4 is the payoff from the two puts struck at 1.

More generally, the reflection of the payoff from a call struck at K when the spot is at H is the payoff from K/H puts struck at H^2/K. The put strikes are chosen so that the geometric mean of the call and put strike is the barrier, for example, the geometric mean of 4 and 1 is $\sqrt{(4\times 1)}=2$. The number of puts chosen is the ratio of distances to the respective strikes when the spot is at the barrier, for example, when the spot is at 2, the distance to the call strike is $4-2=2$, while the distance to the put strike is $2-1=1$, so the number of puts purchased is $2/1=2$. This ratio ensures that the puts have the same value as a standard call *whenever* the spot is at the barrier.

Armed with this put-call symmetry result, we can see that the sale of a down-and-in call with strike K and barrier $H \le K$ is hedged by going long K/H puts, each with strike H^2/K. For example, the sale of a down-and-in call with strike 4 and barrier 2 is hedged by going long two puts, each with strike 1. Note that when $H = K$, we obtain the same hedge as in the last subsection. Moving forward through time, the hedge works in the same manner as above. When the call strike is above the barrier as assumed, the put strike must be below the barrier, so that their geometric mean equals the barrier. Consequently, if the spot never touches the barrier of 2, the hedger is assured that both the hedged item (call struck at 4) and the hedging instruments (two puts struck at 1) expire worthless. If the spot touches the barrier, then at that moment, put-call symmetry implies that the two puts struck at 1 have the same value as one call struck at 4. Consequently, the hedger can sell off both puts and buy a call without incurring any out-of-pocket expense. Since buying two standard puts is an exact hedge for writing this down-and-in call, it follows that prior to hitting the barrier, a down-and-in call has the same value and behaviour as two standard puts, or more generally:

$$DIC(K,H)=\frac{K}{H}P\left(\frac{H^2}{K}\right),\quad H\le K. \qquad (2)$$

In both of the previous examples, we dealt with a down-and-in call, where the call, once knocked in, was either at or out of the money, and therefore had zero intrinsic value at the barrier. We will now deal with a down-and-in call when the strike lies below the barrier. In this case, the knocked in call has positive intrinsic value at the barrier.

To develop a static hedge for the intrinsic value, define a down-and-in *bond* as a security that pays \$1 at expiration if and only if the underlying touches the barrier prior to expiry. Then a portfolio of $H-K$ down-and-in bonds has the same value at the barrier as the intrinsic value of the emergent call.

It turns out that a down-and-in bond can be synthesised using standard and *binary* puts. Reiner and Rubinstein (1991b) value a wide assortment of binary options in the lognormal model. A binary put struck at K, denoted BP(K), can be valued and hedged using standard puts as follows:

$$BP(K)=\lim_{n\uparrow\infty}\frac{n}{2}\left[P\left(K+\frac{1}{n}\right)-P\left(K-\frac{1}{n}\right)\right], \qquad (3)$$

In words, the investor buys a large number of standard puts struck just above the binary put strike and writes an equal number of standard puts struck just below the binary put strike.

In table 1, we determine how many options are required to obtain reasonable convergence. For now, it suffices to note that the above centered difference approximation converges faster than either a forward or backward difference.

To synthesise a down-and-in bond with barrier H, an investor should buy *two* binary puts struck at H and write 1/H standard puts struck at H. If the spot stays above the barrier, then the down-and-in bond and the hedging instruments all expire worthless. If the spot touches the barrier, then at that moment, the (risk-neutral) probabilities of finishing above and below the barrier are *roughly* equal. If the distribution were symmetric, then when the spot is at the barrier, the two at-the-money binary puts could be liquidated for proceeds sufficient to buy a bond paying \$1 at expiry. However, the assumed lognormal distribution is left-skewed. As a result, when the spot is at the barrier, the (risk-neutral) probability of finishing above the barrier is less than the probability of finishing below. Consequently, to purchase a bond paying \$1 at expiry the hedger needs less money than that generated by the sale of the two at-the-money binary puts. By writing 1/H of a standard put initially and covering when the spot touches the barrier, the hedger ensures that the transition at the barrier is self-financing. As a result, the down-and-in bond value is given by:

$$DIB(H)=2BP(H)-\frac{1}{H}P(H).$$

Since the binary put is more efficiently replicated using standard puts struck just above and below H, less strikes are traded if we replace P(H) with an average of puts struck just above and below H:

$$DIB(H)=2BP(H)-\frac{1}{H}\lim_{n\uparrow\infty}\frac{P\left(H+\frac{1}{n}\right)+P\left(H-\frac{1}{n}\right)}{2}. \qquad (4)$$

Substituting (3) in (4) implies that the down-and-in bond can be valued in terms of standard puts alone:

$$DIB(H)=\lim_{n\uparrow\infty}\left(n-\frac{1}{2H}\right)P\left(H+\frac{1}{n}\right)-\lim_{n\uparrow\infty}\left(n+\frac{1}{2H}\right)P\left(H-\frac{1}{n}\right) \qquad (5)$$

An immediate bonus of this analysis is that *rebates* associated with knock-in options can be hedged and therefore valued. The rebate attached to a down-and-in call is paid at expiry if the underlying stays above the barrier. Clearly, a rebate of R attached to a down-and-in call is synthesised by going long R standard bonds paying \$1 at expiry and shorting R down-and-in bonds.

$$DIR(H)=Re^{-rT}-R\cdot DIB(H) \qquad (6)$$

Substituting (5) in (6) implies that the down-and-in rebate can be valued in terms of standard puts alone.

We are now able to value a down-an-in call when the strike lies below the barrier. The initial sale of a down-and-in call with strike K and barrier $H \ge K$ is hedged by going long $H-K$ down-and-in bonds and long one standard

Table 1. Speed of convergence of hedge as a function of n

$S = 2$, $K = 1.8$, $H = 1.9$, $r = r_f = 4\%$, $\sigma = 15\%$, $T = 1$ year

n	DIC(K,H)	= P(K)	+ (H – K)(n – 1/2H)	× P(H + 1/n)	– (n + 1/2H)	× P(H – 1/n)
10	0.110836796	0.038849078	0.973684211	0.11186	1.026315789	0.039871
100	0.110994179	0.038849078	9.973684211	0.74105	10.02631579	0.668902
1,000	0.110984592	0.038849078	99.97368421	7.08037	100.0263158	7.008238
10,000	0.111051003	0.038849078	999.9736842	70.4786	1000.026316	70.40637

put struck at K. Simply speaking, the down-and-in bonds are purchased to provide the intrinsic value of the call at the barrier, while the put is purchased to provide the time value of the call at the barrier.

To illustrate, the initial sale of a down-and-in call with strike 4 and barrier 6 is hedged by going long $6 - 4 = 2$ down-and-in bonds with barrier 6 and long 1 standard put struck at 4. Moving forward through time, the hedge proceeds as follows. If the spot stays above the barrier of 6, then the down-and-in call and the hedging intruments all expire worthless. If the spot touches the barrier of 6, then the 2 down-and-in bonds knock in. By put-call parity, these bonds and the standard put struck at 4 can be sold to finance the purchase of a standard call struck at 4. Since buying 2 down-and-in bonds with barrier 6 and buying a standard put struck at 4 is an exact hedge for writing the down-and-in call, it follows that prior to hitting the barrier, a down-and-in call has the same value and behaviour as this portfolio, or more generally:

$$DIC(K,H) = (H - K)DIB(H) + P(K), H \geq K. \quad (7)$$

Substituting (5) in (7) implies that the down-and-in call can be valued in terms of standard puts alone.

Table 1 shows the speed of convergence of the hedge as a function of n. With $n = 10$, the relative error is $-5/8\%$, which is certainly within the bid-offer spread for these types of options.

A lookback call pays off the difference between the ending and minimum spot over the option's life. Goldman, Sosin and Gatto (1979) indicate how to value and hedge a lookback call using a dynamic strategy in straddles when

$$r = r_f + 0.5\,\sigma^2.$$

We now show how to value and hedge a lookback call using a static strategy in standard puts when $r = r_f$. Furthermore, we explicitly account for the discreteness of the tick size. Lookback puts can be valued and hedged analogously.

The intuition for this example lies in the ability to rephrase the question, "what was the minimum stock price?'' as several questions. For example, when the underlying starts at 2, we could find the minimum underlying price by asking the following questions:

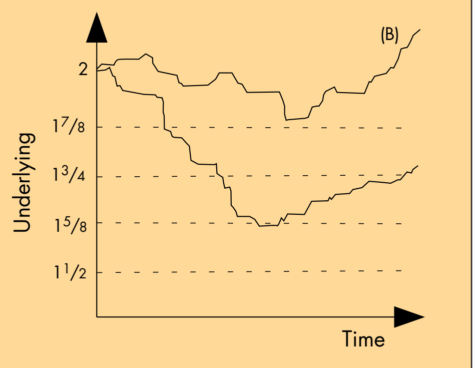
1 Did the underlying get as low as 1/8?
2 Did the underlying get as low as 1/4?
3 Did the underlying get as low as 3/8? etc.

The payoff from a down-and-in bond with barrier H can be interpreted as the answer to the logical question "Did the underlying get as low as H?'' The payoff is \$1 if the answer is yes and zero if the answer is no. This intuition suggests a relationship between lookback options and down-and-in bonds which we now explore.

Let I be the size of one tick and let N be the number of potential spot levels below the current spot, including zero. For example, if the tick size is $I = 1/8$ and the current spot is at 2, then N is $S/I = 16$. To hedge the sale of a lookback call, an investor should go long one (zero cost) forward contract and buy I units of each of the $N - 1$ different down-and-in bonds which have positive[4] barriers below the current spot. In the previous example, the investor buys 1/8 of a down-and-in bond with barrier 1 7/8, 1/8 of a down-and-in bond with barrier 1 6/8 = 1¾, and so on, with the last 1/8 of a down-and-in bond having a barrier at 1/8.

[4] In the Garman-Kohlhagen model which we are using, it is theoretically impossible for the spot to reach zero. Consequently, there is no need to take a position in the down-and-in bond with a barrier at zero

3. Lookback option



If the spot never falls below its initial level over the life of the option (see path B of figure 3), then the forward contract pays $S_T - S_0$, while the down-and-in bonds expire worthless. Since S_0 is the minimum stock price in this case, this payoff matches that of the lookback.

If instead the minimum spot ends up being one tick below the initial spot, then only the first down-and-in bond comes in, and the payoff is $S_T - S_0 + I = S_T - (S_0 - I)$, which again matches that of the lookback. As the spot reaches each new low, another down-and-in bond knocks in, enhancing the payoff to match that of the lookback. Equating the lookback value to the cost of the replicating portfolio gives:

$$LC = I \sum_{i=1}^{N-1} DIB(iI) \quad (8)$$

Substituting (5) in (8) implies that the lookback call can be valued in terms of standard puts alone.

We now relax the assumption that carrying costs are zero. When interest rates differ, the forward differs from the spot. Let $\hat{H} \equiv He^{(r-r_f)T}$ be the initial "forward barrier'', that is, if $S = H$ at initiation, then the forward $F = \hat{H}$ by interest rate parity. When carrying costs are positive, that is, $r > r_f$, then the forward is above the spot, $F > S$, and similarly, the initial forward barrier is above the spot barrier, $\hat{H} > H$. In this case, when the barrier of

2. Valuing a down-and-in European call option on foreign exchange

Option on 100,000 units of the underlying currency
Spot rate = \$1; strike rate = \$1; barrier = \$0.80; time to maturity = 1 year;
domestic interest rate (r) = 5%; foreign interest rate (r_f) = 4%; volatility = 15%

Change in parameter	No of puts	Put value (\$)	Lower bound value (\$)	Value of down-and-in call (\$)	Upper bound value (\$)	Put) value (\$)	No of puts
Spot rate							
0.85	1.25	100.09	125.11	167.48	176.75	142.82	1.24
0.90	1.25	35.39	44.24	58.69	65.17	52.66	1.24
0.95	1.25	11.82	14.78	19.47	22.68	18.33	1.24
1.00	1.25	3.77	4.71	6.17	7.52	6.08	1.24
1.25	1.25	0.01	0.01	0.01	0.01	0.01	1.24
Strike							
0.85	1.06	117.75	125.11	144.95	176.74	168.02	1.05
0.90	1.13	39.32	44.24	53.45	65.18	58.52	1.11
1.00	1.25	3.77	4.71	6.17	7.51	6.07	1.24
1.10	1.38	0.31	0.43	0.60	0.74	0.54	1.36
1.25	1.56	0.01	0.01	0.01	0.01	0.01	1.55
Barrier							
1.00	1.00	5,250.19	5,250.19	6,206.19	6,206.20	6,268.57	0.99
0.90	1.11	402.71	447.46	546.82	600.51	545.89	1.10
0.80	1.25	3.77	4.71	6.17	7.52	6.08	1.24
0.75	1.33	0.12	0.16	0.21	0.28	0.21	1.32
0.70	1.43	0.00	0.00	0.00	0.00	0.00	1.41
Time							
0.25	1.25	0.00	0.00	0.00	0.00	0.00	1.25
0.50	1.25	0.02	0.03	0.03	0.04	0.03	1.24
1.00	1.25	3.77	4.72	6.17	7.52	6.08	1.24
3.00	1.25	236.80	296.74	427.67	521.49	429.90	1.21
10.00	1.25	1,469.89	1,841.96	3,313.41	4,040.34	3,572.21	1.13
Volatility							
10.00	1.25	0.00	0.00	0.01	0.01	0.01	1.24
15.00	1.25	3.77	4.71	6.17	7.52	6.08	1.24
20.00	1.25	58.89	73.61	87.74	98.10	79.27	1.24
30.00	1.25	635.73	794.66	879.86	924.59	747.11	1.24
40.00	1.25	1,910.18	2,387.73	2,565.29	2,637.85	2,131.49	1.24
(r-r_f)*							
–5.00	1.25	14.36	17.95	4.65	1.72	1.31	1.31
–2.00	1.25	7.52	9.40	5.48	3.69	2.89	1.28
–1.00	1.25	6.02	7.53	5.74	4.71	3.73	1.26
–0.25	1.25	5.08	6.35	5.93	5.64	4.50	1.25
0.00	1.25	4.79	5.99	5.99	5.99	4.79	1.25
0.25	1.25	4.51	5.64	6.04	6.35	5.09	1.25
1.00	1.25	3.77	4.71	6.17	7.52	6.08	1.24
2.00	1.25	2.95	3.69	6.32	9.40	7.67	1.23
5.00	1.25	1.38	1.73	6.66	17.94	15.09	1.19

* For $r < r_f$, the lower bound becomes the upper bound and vice versa

a down-and-in call is below its strike, (2) becomes:

$$\frac{K}{H}P\left(\frac{H^2}{K}\right) \le DIC(K,H) \le \frac{K}{\hat{H}}P\left(\frac{\hat{H}^2}{K}\right), H \le K. \quad (9)$$

Table 2 shows the tightness of the bounds for various parameter values.

Figure 4 shows the effect of varying each parameter on the bounds. When the barrier of a down-and-in call is above its strike, (7) becomes:

$$(H-K)DIB(H)+P(K) \le DIC(K,H) \le (\hat{H}-K)DIB(\hat{H})+P(K). \quad (10)$$

In contrast, when carrying costs are negative, that is, $r < r_f$, then the forward is below the spot, $F < S$, and similarly, the initial forward barrier is below the spot barrier, $\hat{H} < H$. In this case, when the barrier of a down-and-in call is below its strike, (2) becomes:

$$\frac{K}{\hat{H}}\left(\frac{\hat{H}^2}{K}\right) \le DIC(K,H) \le \frac{K}{H}P\left(\frac{H^2}{K}\right), H \le K. \quad (11)$$

When the barrier of a down-and-in call is above its strike, (7) becomes:

$$(\hat{H}-K)DIB(\hat{H})+P(K) \le DIC(K,H) \le (H-K)DIB(H)+P(K).$$

If any of these inequalities is violated, an arbitrage opportunity arises. For example, if the upper bound in (11) is violated, then the down-and-in call should be sold and hedged by buying K/H standard puts struck at H^2/K. If the spot never hits the barrier, then the down-and-in call and the standard puts expire worthless as usual. If the spot hits the barrier, then the puts should be sold off at that time with the proceeds used to purchase one call. In contrast to the zero carry case, there will be funds left over. As a result, the down-and-in call value is bounded above by the cost of this "super'' replicating portfolio of puts.

When carrying costs are positive, that is, $r > r_f$, then the forward is above the spot, $F > S$, and similarly, the initial forward barrier is above the spot barrier, $\hat{H} > H$. In this case, (6) describing the rebate of a down-and-in call becomes:

$$R\left[e^{-rT} - DIB(H)\right]$$
$$\le DIR(H) \le R\left[e^{-rT} - DIB(\hat{H})\right].$$

In contrast, when carrying costs are negative, that is, $r < r_f$, then the forward is below the spot, $F < S$, and similarly, the initial forward barrier is below the spot barrier, $\hat{H} < H$. In this case, (6) describing the rebate of a down-and-in call becomes:

$$R\left[e^{-rT} - DIB(\hat{H})\right]$$
$$\le DIR(H) \le R\left[e^{-rT} - DIB(H)\right].$$

The rebate of a down-and-out call is paid if and when the option knocks out. If the rebate pays R the first time that the spot hits the barrier H, then the value of this down-and-out rebate DOR(H) is bracketed as follows:

$$R \ge DOR(H) \ge R \cdot DIB(H).$$

When carrying costs are positive, that is, $r > r_f$, then the forward is above the spot, $F > S$, and similarly, the initial forward barrier is above the spot barrier, $\hat{H} > H$. In this case, the rebate of a down-and-out call becomes:

$$R \ge DOR(H) \ge R \cdot DIB(H). \tag{12}$$

In contrast, when carrying costs are negative,that is, $r < r_f$, then the forward is below the spot, $F < S$, and similarly, the initial forward barrier is below the spot barrier, $\hat{H} < H$. In this case, the rebate of a down-and-out call becomes:

$$R \ge DOR(H) \ge R \cdot DIB(\hat{H}). \tag{13}$$

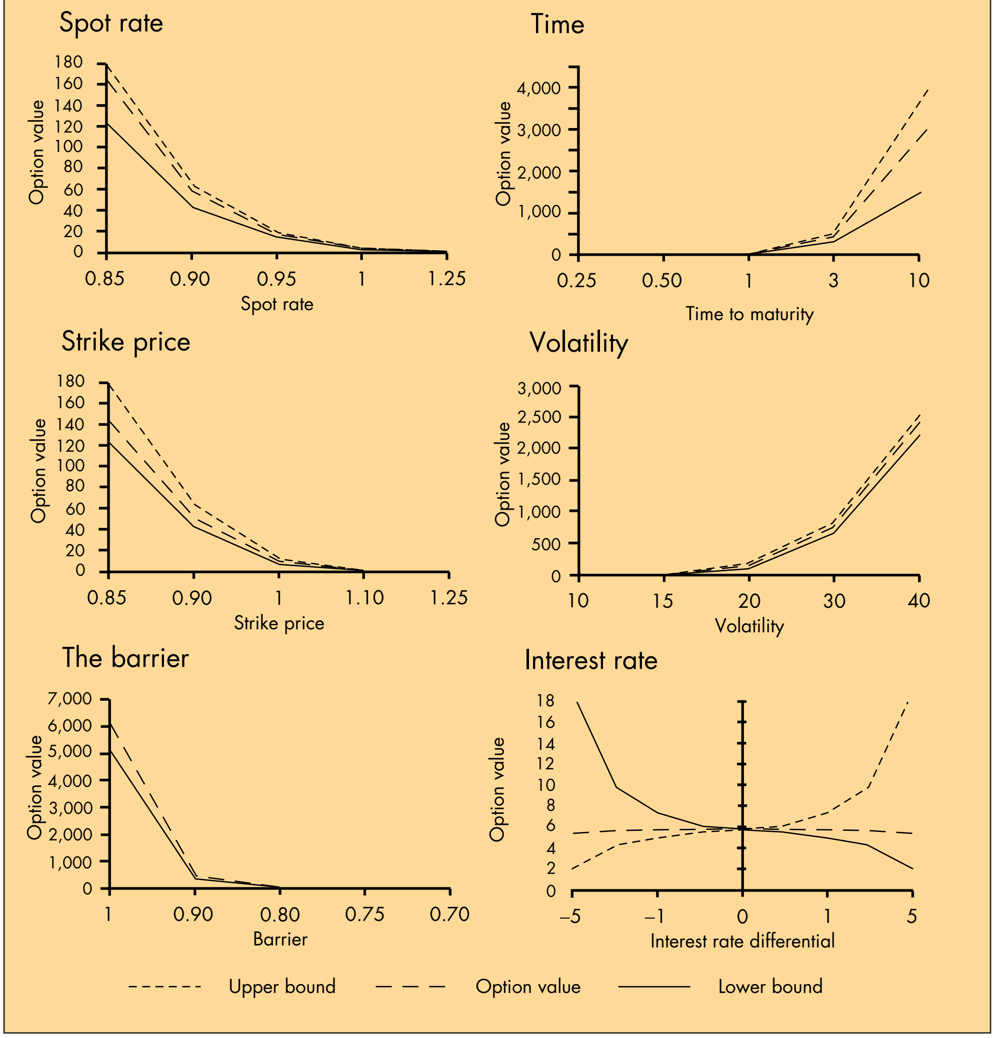
In analogy to the above, let $\hat{I} \equiv I\, e^{(r-r_f)T}$ be the "forward" tick size. When carrying costs are positive, ie $r > r_f$, then the forward is above the spot, $F > S$, and similarly, the forward tick size is above the spot tick size, $\hat{I} > I$. In this case, (8) becomes:

$$I\sum_{i=1}^{N-1}\left[DIB(iI)\right] \le LC \le \hat{I}\sum_{i=1}^{N-1}\left[DIB\left(i\hat{I}\right)\right].$$

In contrast, when carrying costs are negative, that is, $r < r_f$, then the forward is below the spot, $F < S$, and similarly, the forward tick size is below the spot tick size, $\hat{I} < I$. In this case, (11) becomes:

$$\hat{I}\sum_{i=1}^{N-1}\left[DIB\left(i\hat{I}\right)\right] \le LC \le I\sum_{i=1}^{N-1}\left[DIB(iI)\right].$$

4. Change in option value with respect to:



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REFERENCES

Carr, P, 1994, *European put call symmetry*, Cornell University working paper

Derman, E, D Ergener and I Kani, 1994, *Static options replication*, Goldman Sachs working paper

Garman, M, 1989, *Recollection in tranquility*, *Risk* March 1989, pages 16–19

Garman, M, and S Kohlhagen, 1983, *Foreign currency option values*, Journal of International Money and Finance, 2, pages 231–237

Goldman, M, H Sosin and MA Gatto, 1979, *Path dependent options: buy at the low, sell at the high*, Journal of Finance 34, pages 1,111–1,127

Reiner, E and M Rubinstein, 1991a, *Breaking down the barriers*, *Risk* September 1991, pages 28–35

Reiner, E and M Rubinstein, 1991b, *Unscrambling the binary code*, *Risk* October 1991, pages 75–83