

Static Hedging of Standard Options

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Overview

- Motivation and Setting
- Hedging Options with Options
- Monte Carlo Simulation Results
- Empirical Results
- Conclusions and Extensions

Motivation

- Recent studies have provided persuasive empirical support for continuous time option pricing models in which the underlying asset price can always jump to any nonnegative level.
- Examples of such models enjoying this empirical support include:
 - Merton (1976) lognormal jump/lognormal diffusion model
 - Bates (1996) lognormal jump/stochastic vol model
 - numerous pure jump Lévy process models (e.g VG, NIG, CGMY)
 - recent time-changed Lévy process models (e.g. Carr&Wu, CGMY).
- Recent studies of the underlying time series or of option prices finds that jumps are pervasive:
 - Ait-Sahalia (2002): Total positivity test
 - Carr and Wu (2002): Near term asymptotics
 - Andersen, Benzoni and Lund (2002); Eraker (2004); Huang, Wu (2004) ...

Pricing

- When the underlying asset price can always jump to any nonnegative level, dynamic hedging with just the underlying does not remove all risk.
- Some risk is removed by diversifying across underlyings, but systematic jump risk still resides at the aggregate portfolio level.
- The absence of arbitrage implies the existence of multiple martingale measures all consistent with the prices of traded assets.
- Of all these measures, the market is using some unknown criterion to select one.
- There is much ongoing work in making stronger assumptions than no arbitrage to help find the measure used by the market.
- For example, the notion of an acceptable opportunity (Carr Geman Madan 2001) can sometimes lead to unique prices in incomplete markets.

Hedging

- When the underlying asset price can always jump to any nonnegative level, dynamic hedging with just the underlying does not remove all risk.
- If a derivative security can be spanned at all, perfect replication of its payoff requires continuous rebalancing in a continuum of options written on the same underlying asset.
- However, assuming that the risk-neutral price process for this underlying asset is Markov in itself, we show that the value of a target call at any fixed future date can be spanned by the payoff from a static portfolio of calls maturing then.
- Just as perfect replication via dynamic trading relies on a continuum of trading opportunities across time, perfect replication via static positioning relies on being able to trade in a continuum of options of all strikes at a fixed maturity.

Comparison of Three Hedging Strategies

- **Dynamic hedging:**
 - *Not robust:* Delta is model-dependent; fails in presence of jumps.
 - Works for a *wide range* of contingent claims
- **Breeden and Litzenberger (1978) static hedging (BL):**
 - *Robust:* completely model-independent.
 - Can only be used on a *limited range* of claims:
No standard options; no path-dependence.
- **Our approach:** We take the middle ground.
 - *More robust* than dynamic hedging: can deal with jumps...
 - Covers a *wider range* than BL:
Standard options; mildly path-dependent options.

Assumptions

- Two key assumptions underlying our (semi-)static hedging strategy:
 1. **Discrete Path Monitoring:** The payoff of the target contingent claim depends on only a finite number of prices of the underlying asset.
 2. **Markovian Stock Price Dynamics:** The risk-neutral evolution of the stock price, S , is Markovian in itself and time.
- Two widely used classes of processes which have the Markov property are:
 - Local volatility models: Dupire (1994)
 - Lévy processes: Merton (1976), Madan, Carr, Chang (1998), Carr, Wu (2002).
- More generally, can have jump diffusion process where the jump arrival rate and the diffusion coefficient both depend on stock price and time.
- Both key assumptions can be relaxed somewhat.

Notation

- No arbitrage implies the existence of a martingale measure \mathbb{Q} .
- Given some \mathbb{Q} , the Markov condition and a constant riskfree rate r implies the existence of the following European call pricing function:

$$C(S, t; K, T; \Theta) \equiv e^{-r(T-t)} E^{\mathbb{Q}}[(S_T - K)^+ | S_t = S], \text{ where:}$$

- $S \geq 0$ is the spot price of the underlying conditioned on at time t .
- $t \geq 0$ is the valuation time
- $K \geq 0$ is the strike price
- $T \geq t$ is the maturity
- Θ is a vector of relevant model parameters.

Notation

- Given some \mathbb{Q} , the Markov condition also implies the existence of a risk-neutral probability density function:

$$q(S, t; K, T; \Theta) \equiv \mathbb{Q}\{S_T \in dK | S_t = S\}.$$

- In our setting, the results of Ross (1976) and Breeden & Litenberger (1979) imply:

$$q(S, t; K, T; \Theta) = e^{r(T-t)} \frac{\partial^2 C}{\partial K^2}(S, t; K, T; \Theta).$$

Spanning Options with Options

- **Theorem 1:** Under frictionless markets, no arbitrage, and Markovian stock price dynamics, the value C given $S_t = S$ at time $t \geq 0$ of a European call of strike $K > 0$ and maturity $T \geq t$ is related to the prices of a spectrum of European calls at a shorter maturity $u \in [t, T]$ by the following integral relation:

$$C(S, t; K, T; \Theta) = \int_0^{\infty} w(\mathcal{K}) C(S, t; \mathcal{K}, u; \Theta) d\mathcal{K}, \quad (1)$$

where the strike weighting density function $w(\mathcal{K})$ is given by

$$w(\mathcal{K}) = \frac{\partial^2}{\partial \mathcal{K}^2} C(\mathcal{K}, u; K, T; \Theta). \quad (2)$$

- The weight on the basis call of maturity u and strike \mathcal{K} is proportional to the gamma that the target call will have at time u , should the underlying be at price \mathcal{K} then.
- Theorem 1 actually holds without any assumption on interest rates and dividends.

Spanning Options with Options: Examples

- The **Black-Scholes** model (BS):

$$w(\mathcal{K}) = \frac{\partial^2 C(\mathcal{K}, u; K, T; \Theta)}{\partial \mathcal{K}^2} = e^{-\delta(T-u)} \frac{n(d_1(\mathcal{K}, u; K, T; r, q, \sigma))}{\mathcal{K} \sigma \sqrt{T-u}},$$

where $n(\cdot)$ is the standard normal density function.

- The **Merton** (1976) jump-diffusion model (MJD):

$$w(\mathcal{K}) = e^{-r(T-u)} \sum_{n=0}^{\infty} \Pr(n) e^{(r_n - \delta)(T-u)} \frac{n(d_{1n}(\mathcal{K}, u; K, T))}{\mathcal{K} \sigma_n \sqrt{T-u}},$$

where

$$\Pr(n) = e^{-\lambda^*(T-t)} \frac{(\lambda^*(T-t))^n}{n!},$$

$$d_{1n}(S, t; K, T) = \frac{\ln(S/K) + (r_n - \delta - \sigma_n^2/2)(T-t)}{\sigma_n \sqrt{T-t}},$$

$$r_n = r - \lambda^* g^* + n(\mu_j^* + \sigma_j^2/2)/(T-t),$$

$$\sigma_n^2 = \sigma^2 + n\sigma_j^2/(T-t).$$

Spanning Options with Options: Interpretations

- Recall:

$$C(S, t; K, T; \Theta) = \int_0^{\infty} w(\mathcal{K}) C(S, t; \mathcal{K}, u; \Theta) d\mathcal{K}, \quad (3)$$

where the strike weighting density function $w(\mathcal{K})$ is given by

$$w(\mathcal{K}) = \frac{\partial^2}{\partial \mathcal{K}^2} C(\mathcal{K}, u; K, T; \Theta). \quad (4)$$

- As the function relating a call's gamma to the underlying spot is a bell shaped curve in most models, the strike weight is centered around the strike K of the target call of maturity T .
- As $u \rightarrow T$, the gamma becomes more concentrated about K and hence the relative weight on these calls increases.
- In the limit when $u = T$, all of the weight is on the call of strike K , and (3) reduces to a tautology.

Spanning Options with Options: Applications

Substituting the weight function into the integral relation implies:

$$C(S, t; K, T; \Theta) = \int_0^{\infty} \frac{\partial^2}{\partial \mathcal{K}^2} C(\mathcal{K}, u; K, T; \Theta) C(S, t; \mathcal{K}, u; \Theta) d\mathcal{K}.$$

Under the twin assumptions of no arbitrage and Markovian dynamics at future dates, this result can be used:

- to detect and exploit current arbitrage opportunities.
- to hedge the call (semi-)statically.

Quadrature Approximation

$$C(S, t; K, T; \Theta) = \int_0^{\infty} \frac{\partial^2}{\partial \mathcal{K}^2} C(\mathcal{K}, u; K, T; \Theta) C(S, t; \mathcal{K}, u; \Theta) d\mathcal{K}.$$

- In practice, one can neither rebalance a portfolio continuously, nor can one form a static portfolio involving a continuum of securities.
- We approximate the above integral with a weighted sum over a finite number (N) of call values,

$$\int_0^{\infty} w(\mathcal{K}) C(S, t; \mathcal{K}, u; \Theta) d\mathcal{K} \approx \sum_{j=1}^N \mathcal{W}_j C(S, t; \mathcal{K}_j, u; \Theta), \quad (5)$$

- To determine the strikes and weights, we use a Gauss-Hermite quadrature rule.
- This rule generates a set of weights w_i and the nodes x_i , $i = 1, 2, \dots, N$ so that

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} dx = \sum_{i=1}^N w_i f(x_i) + \frac{N! \sqrt{\pi}}{2^N} \frac{f^{(2N)}(\xi)}{(2N)!} \quad (6)$$

for some $\xi \in (-\infty, \infty)$.

Quadrature Approximation

- We need to link our strike choice \mathcal{K}_j and the portfolio weight \mathcal{W}_j to the quadrature rule $[x_j, w_j]_{j=1}^N$.

- A reasonable choice of strike is given by

$$\mathcal{K}_j = Ke^{x_j\sigma\sqrt{2(T-u)}+(\delta-r-\sigma^2/2)(T-u)}, \quad (7)$$

This choice is motivated by the form of gamma in the Black-Scholes model, which involves the normal density evaluated at d_1 . Hence x_i can be regarded as $\sqrt{2}d_1$.

- The portfolio weights are given by

$$\mathcal{W}_j = \frac{w(\mathcal{K}_j)\mathcal{K}'_j(x_j)}{e^{-x_j^2}}w_j = \frac{w(\mathcal{K}_j)\mathcal{K}_j\sqrt{2}\sigma}{e^{-x_j^2}}w_j. \quad (8)$$

It reduces to

$$\mathcal{W}_j = \frac{e^{-\delta(T-u)}}{\sqrt{\pi}}w_j,$$

under the Black-Scholes model.

Simulation Exercises

- Objectives:
 - Determine the effectiveness of the quadrature approximation.
 - Compare the effectiveness of the static hedge with that of daily delta hedging under different controlled scenarios.
- Scenario Types:
 - Black-Scholes Model: known model/parameters.
 - Merton jump-diffusion model: known model/parameters.
 - Merton jump-diffusion model: we assume the model is unknown to the hedger, so hedges are formed based on the Black-Scholes model with implied volatility as inputs (*ad hoc*).

Simulation Exercises: Procedures

- Monte Carlo Simulation Procedure:
 - Simulate 1,000 daily paths of stock prices, option prices at all relevant maturities.
 - Length of each sample path is one month.
 - Static hedging of one-year call with one-month calls:
Form a static hedging portfolio at the start of the month, track the hedging error for the whole month.
 - Dynamic delta hedging of the one-year call with the underlying:
Form a delta hedge at the start of the month, rebalance daily, track the hedging error for the whole month.
 - Hedging error is defined as the difference between the value of the hedge portfolio and the value of the target call at the end of the month.

Simulation Results: Black-Scholes

Hedge Error	Static with Options					Dynamic with
No. of Assets	3	5	10	15	21	Underlying
Mean	0.08	0.05	-0.01	0.00	0.00	-0.00
Std Err	1.10	0.70	0.36	0.23	0.16	0.13
RMSE	1.10	0.70	0.36	0.23	0.16	0.13
MSF	-0.43	-0.28	-0.16	-0.10	-0.07	-0.05
Min	-1.95	-1.39	-0.77	-0.53	-0.37	-0.65
Max	1.87	0.94	0.42	0.26	0.19	0.31
Skewness	-0.11	-0.42	-0.50	-0.68	-0.64	-0.60
Kurtosis	1.86	1.92	1.97	2.19	2.16	3.43
Initial Cost	11.72	12.20	12.34	12.37	12.36	12.35

⇒ The effectiveness of static hedging with 21 options is similar to daily delta hedging.

C'est la vie...

Simulation Results: Merton JD Model

Hedge Error	Static with Options					Dynamic with
No. of Assets	3	5	10	15	21	Underlying
Mean	0.07	0.05	-0.03	0.01	0.00	-0.01
Std Err	0.81	0.52	0.34	0.18	0.14	1.20
RMSE	0.81	0.52	0.34	0.18	0.14	1.20
MSF	-0.27	-0.18	-0.16	-0.07	-0.05	-0.27
Min	-1.94	-1.44	-0.78	-0.68	-0.48	-15.56
Max	2.89	1.48	0.52	0.21	0.15	0.39
Skewness	0.32	-0.36	-0.04	-1.66	-1.38	-5.57
Kurtosis	4.33	3.71	1.83	5.24	4.11	40.81
Initial Cost	9.52	11.14	11.93	12.08	12.06	11.99

The effectiveness of daily delta updating deteriorates by a factor of 10. The effectiveness of static hedging remains unchanged.

Static hedging with merely 3 calls beats daily delta-hedging!

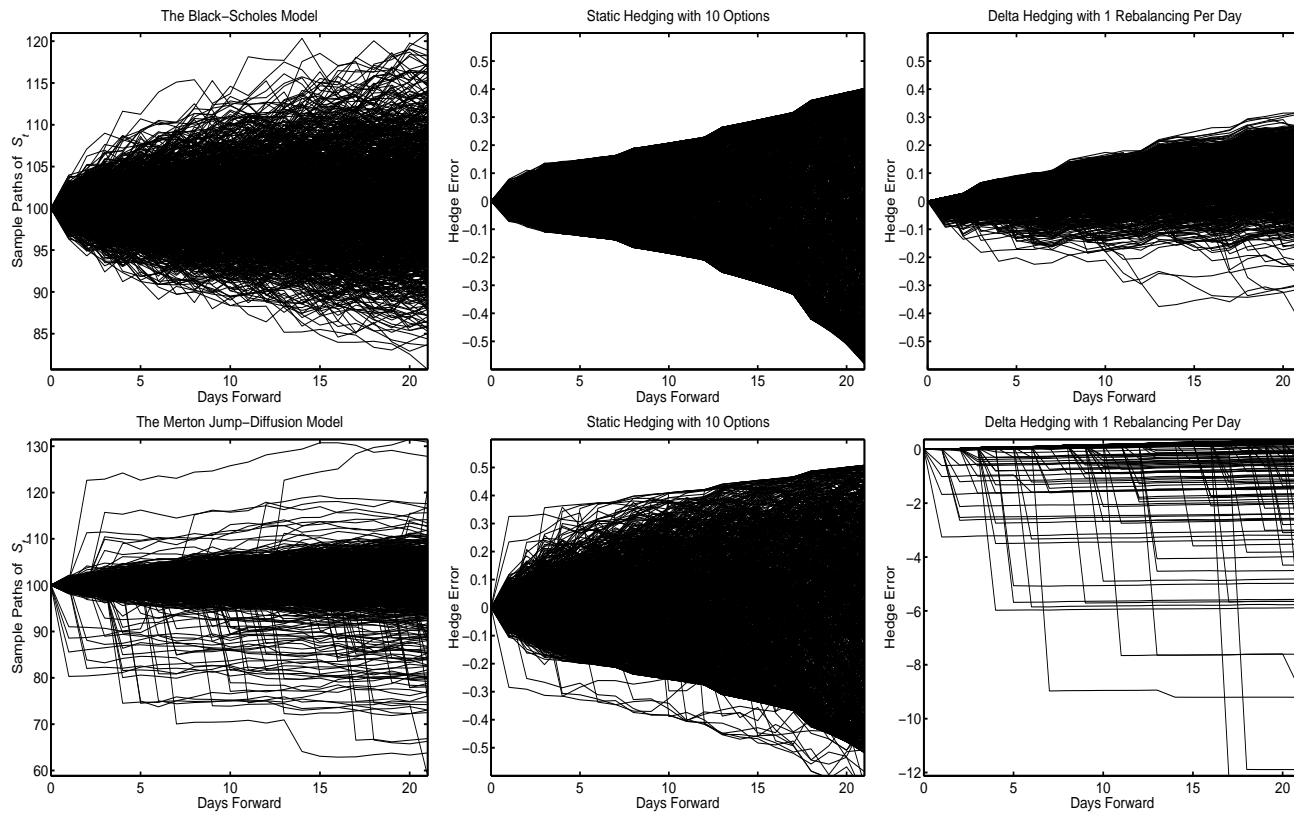
Simulation Result: Merton

Ad Hoc Black-Scholes Hedge under the Merton World

Hedge Error	Static with Options					Dynamic with
No. of Assets	3	5	10	15	21	Underlying
Mean	0.07	0.04	-0.05	-0.01	-0.02	-0.04
Std Err	0.68	0.41	0.50	0.23	0.26	1.36
RMSE	0.68	0.41	0.50	0.23	0.26	1.36
MAE	0.55	0.32	0.44	0.15	0.18	0.58
MSF	-0.24	-0.14	-0.24	-0.08	-0.10	-0.31
Min	-1.76	-1.37	-0.82	-0.99	-0.85	-13.53
Max	1.64	1.11	0.86	0.69	0.65	14.04
Initial Cost	11.53	11.96	11.84	12.07	12.03	11.99

Ad hoc delta hedge performs as well as delta-hedging when the model is known.

Sample Paths of S_t and Hedge Errors



Simulation Results: Summary

- Under the Black Scholes model, daily delta hedging 21 times over a month works as well as (if not better than) a static position in 21 calls.
- When the sample paths of the underlying price exhibit jumps of random size, the hedging effectiveness of daily stock rebalancing deteriorates miserably. The delta hedging error is not from model mis-specification, but rather from its inability to deal with jumps of random size.
- A much more effective way to hedge when random size jumps can occur is to initiate a static portfolio of options with different strikes. A buy-and-hold of just three calls performs as well as daily delta hedging in the underlying.
- Additional potential benefits of the static hedge include savings on transaction and monitoring costs.
- Static hedging is also unaffected by borrowing constraints and short selling constraints (eg. borrow costs).

Hedging S&P 500 Index Options

- Objectives:
 - Examine the historical effectiveness of the two types of hedging strategies.
 - Infer the type of process that the index follows:
 - * Pure diffusion or with random jumps
 - * Existence of additional sources of risk besides future index levels.
- Data:
 - OptionMetrics: from January 96 to August 2002, Daily closing option prices and other relevant information.
 - S&P 500 index cash-settled spot European options.
 - Options mature on the Saturday after the third Friday of each month.

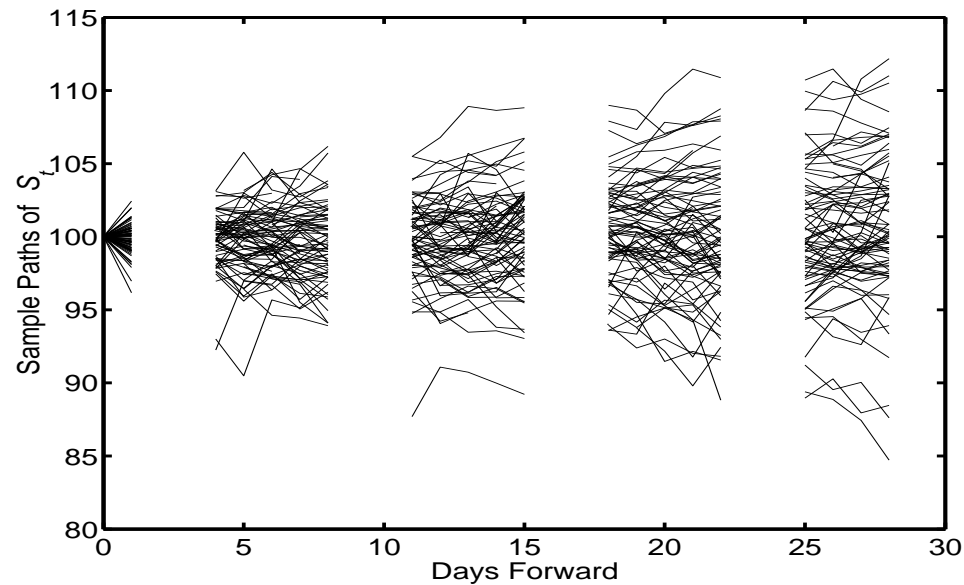
Hedging S&P 500 Index Options

- Procedure:
 - Start a hedging experiment for each month 30 or 29 days before the next expiry date (Thursday or Friday).
 - 79 starting dates (hence 79 experiments) over the 6 plus years of data.
 - Form various static option portfolios at each starting date, monitor the hedging error daily over one month.
 - Form delta hedging portfolios at each starting date, rebalance daily, and monitor hedging errors daily over one month.

Hedging S&P 500 Index Options

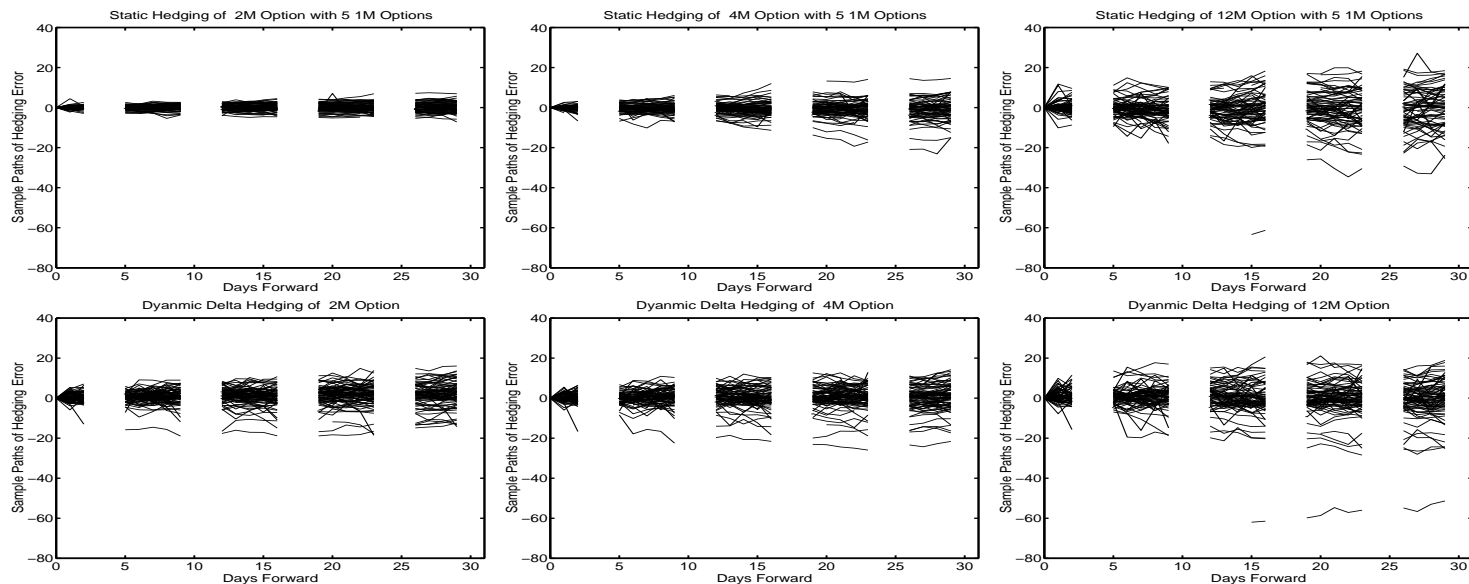
- Maturities of call being hedged:
 - (1) Two months (2) Four-six months (3) 12 months or longer
- Instruments used for hedging:
 - Zeros (for financing)
 - Futures (for delta hedging)
 - Basis Calls (for static hedge):
 - (1) One month, (2) Two months, (3) Four-six months.

Sample Paths of the Index



- Gaps are weekends; also possible line breaks due to holidays, missing data.
- Monitoring stops on the close of the third Thursday of each month, the last full trading day in the life of the expiring calls.
- We observe both small and large moves in the underlying index.

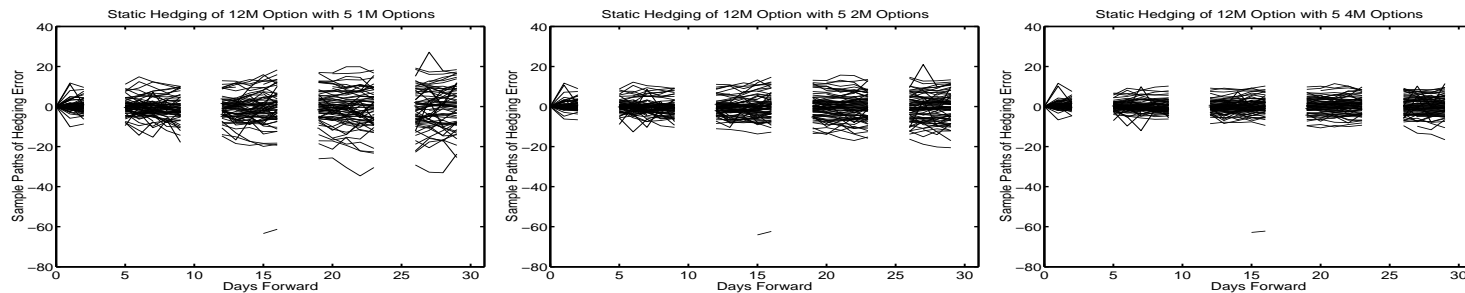
Sample Paths of the Hedging Error



Static hedging with 5 one-month calls outperforms daily delta hedging, more so when hedging short term options \Rightarrow jumps of random size.

Effectiveness of the static hedge declines as the maturity of the call being hedged increases, given a fixed maturity for calls in the hedge portfolio.

Sample Paths of the Static Hedging Error



Given a fixed maturity for the target call (one-year), the hedge performance improves as the maturity of the calls used in the hedge portfolio increases.

Hedge performance improves as we shrink the difference in maturities between the hedging calls and the target call

⇒ additional sources of risk.

Hedging of S&P 500 index options

Target Mat	2	4	12	12	12	2	4	12	12	12	2	4	12
Strategy	Static with Three Options					Static with Five Options					Daily Delta		
Instruments	1	1	1	2	4	1	1	1	2	4	Futures		
Mean	-0.03	-0.13	-0.03	-0.09	0.00	-0.00	-0.10	-0.01	-0.07	0.01	0.16	0.03	-0.01
Std Err	0.27	0.62	1.06	0.74	0.44	0.23	0.50	0.87	0.64	0.42	0.57	0.65	0.88
RMSE	0.27	0.63	1.05	0.74	0.44	0.23	0.51	0.86	0.64	0.42	0.59	0.65	0.87
MAE	0.21	0.50	0.85	0.61	0.35	0.18	0.38	0.69	0.52	0.33	0.47	0.49	0.64
MSF	-0.12	-0.31	-0.44	-0.35	-0.17	-0.09	-0.24	-0.35	-0.29	-0.16	-0.15	-0.23	-0.32
Min	-0.79	-1.78	-2.90	-1.95	-1.55	-0.62	-1.55	-2.32	-1.97	-1.58	-1.28	-2.23	-2.51
Max	0.55	1.34	1.90	1.34	0.76	0.50	1.09	1.56	1.10	0.86	1.45	1.42	1.76
Skewness	-0.19	-0.21	-0.35	-0.22	-0.87	-0.28	-0.51	-0.46	-0.45	-0.74	-0.49	-0.78	-0.76
Kurtosis	3.00	3.20	2.81	2.51	4.16	3.12	3.70	2.91	2.89	4.51	3.34	4.47	3.83
Target Call	3.63	5.92	10.49	10.41	11.47	3.68	6.16	10.94	10.76	11.52	3.65	6.14	11.58

Hedging S&P 500 Options: Summary

- Static hedging with merely five options beat daily delta hedging with volatility updating in all of the cases investigated.
- This result, together with the simulation results, supports the existence of jumps of random size (Carr and Wu (2002)).
- The effectiveness of static hedging deteriorates moderately as we increase the maturity gap between the hedge portfolio and the target call, suggesting the likely existence of additional sources of risk, e.g. stochastic volatility.
- There is ample room left for improving the implementation of the static hedge by different choices of maturity, strikes, and strike weights.

Summary and Concluding Remarks

- Assuming no arbitrage and that the underlying price is Markov in itself, we develop a static hedge for a given call using a set of shorter-term calls.
- We used a quadrature method to approximate the static hedge.
- Monte Carlo simulation illustrates its equivalence to dynamic delta hedging in the diffusive Black-Scholes environment, but the static hedge enjoys a definite advantage when the underlying price path exhibits random jumps.
- In hedging S&P 500 index options, we find that the performance of a static strategy with just five shorter term calls can outperform the ad hoc daily delta hedge, lending support to the existence of jumps of random size in describing index moves.

Further Conclusions and Future Research

- The effectiveness of the static hedge declines as we increase the maturity gap between the static hedge portfolio and the target call, suggesting the existence of additional risk factors.
- Future research: using static or semi-static positions in multiple maturities and strikes to hedge these additional risks.

Semi-Static Hedge of Path-Dependent Options

- Consider a contingent payoff that depends on a finite number ($n < \infty$) of points in the price path:

$$V_T = f(S_{t_0}, S_{t_1}, \dots, S_{t_n}),$$

- Assume a single summary statistic captures the history recursively,

$$V_T = \phi(H_T), \tag{9}$$

where

$$H_{t_i} = g_i(H_{t_{i-1}}, S_{t_{i-1}}, S_{t_i}), \quad i = 1, \dots, n, \tag{10}$$

where $\phi(\cdot)$ and $g_i(\cdot)$ are known functions, H is the single summary statistic, and H_0 and S_0 are known constants.

Example: Cliquet

A globally-floored, locally-capped, compounding five-year cliquet with annual monitoring:

- The payoff

$$V_T = S_0 \max[L, H_T], \quad (11)$$

with

$$H_{t_i} = H_{t_{i-1}} \left[\left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) \wedge U \right], \quad i = 1, \dots, n, \quad (12)$$

where L is the global floor, $U > 1$ is the local cap, and n denotes the number of monitoring periods. Here, $H_0 = 1$, and S_0 is known. In practice, L is typically chosen to be one so that the annualized return is always positive. A typical value of the local cap U is 1.35 so that the maximum return for any year cannot exceed 35 percent.

Semi-Static Hedging: Heuristic Procedure

- At time 0, we use a portfolio of European options maturing at time t_1 to span the value function of the claim.
- At time t_1 , we collect the receipts from the expiring options in the hedge portfolio and form another hedge portfolio maturing at time t_2 .
- This procedure continues until time $T = t_n$, when the payoff from the hedge portfolio formed at time t_{n-1} matches the payoff from the path-dependent claim.
- The hedging is static and no portfolio rebalancing is needed in between monitoring times.
- But at each monitoring step, the options in the hedge portfolio expire and a new hedge portfolio needs to be formed.
- Thus, the rebalancing frequency matches the monitoring frequency, reflecting the semi-static nature of the strategy.