Optimal Positioning in Derivative Securities

Overheads for Presentation

by Peter Carr, Morgan Stanley

and Dilip Madan, University of Maryland

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Introduction

Positions not Prices

- The traditional focus of derivatives research has been on pricing.
- The purpose of this research is to develop models explaining the *positions* taken in derivative securities and their underlying assets.
- In standard pricing models (eg. Black Scholes or HJM), parameters such as volatility are assumed to be known with certainty. If we take this assumption literally, then an investor either knows that the parameter is the same as implied from the option's price or knows that it differs.
  - If the known parameter is the same as the implied, then there is no unique optimal position in the option from which the parameter was implied, but rather an infinite number.
  - If the known parameter differs from the implied, then the optimal position in the option is either infinitely long or infinitely short.
- Neither possibility squares with reality. Thus, the standard models are ill-suited for determining optimal positioning in derivative securities.

Market Completion Through Options

- The problem with standard pricing models for determining optimal positioning is that markets are dynamically complete in the underlying securities; either derivatives are redundant or their underlying securities are.
- In this paper, we rule out dynamic completeness by assuming that one or more investors are unable to trade continuously.
- We consider a simple single period setting in which each investor is exposed to a single source of uncertainty called the stock.
- We rule out completeness in the riskless asset and the stock by assuming that there are more than two possible terminal prices. In fact, we assume that the stock has a continuum of possible terminal prices.
- We assume the availability of a continuum of option strikes, which means that Arrow Debreu securities can be constructed.
- Thus, there is a unique risk-neutral density and the market is (statically) complete.

The 5 P's

- We view the investment decision as the problem of determining the optimal wealth profile considered as a a function of a single source of uncertainty.
- Thus the choice variable is the wealth profile, rather than the amount to invest in each asset.
- The optimal wealth profile is determined by maximizing expected utility subject to a budget constraint. A personal probability density function is used to calculate the expectation of utility. Option prices are used to determine the risk-neutral density, which is used to calculate the cost of a possible profile.
- Once the optimal wealth profile is determined, the completeness of markets ensures a unique solution to the subproblem of determining positions in assets needed to attain this profile.
- In short, **p**rices, **p**references, and **p**riors determine **p**ayoffs. In turn, payoffs determine **p**ositions.

Overview of the Presentation

- The presentation has 3 parts:
  1. Investor Optimization Problem
  2. Optimal Positioning in Partial Equilibrium
  3. Optimal Postioning in General Equlibrium
- We first focus on how an investor would position himself given his preferences, probability beliefs, and the risk-neutral density. We derive an expression relating the optimal payoff of an investor to his preferences, his personal probability density, and the risk-neutral density.
- In the partial equilibrium, the risk-neutral density is determined from the given prices of the bond, the stock, and of all the options. We will first derive general results and then specialize to the case when personal beliefs and the risk-neutral density are both lognormal. This is consistent with the Black Scholes model except that the investor under consideration cannot trade continuously.
- In contrast, in the general equilibrium, the risk-neutral density is determined from the preferences and beliefs of all other investors. Instead of treating asset prices as observable, all asset prices are determined so that the aggregate demand meets the fixed supply. The fixed supply of the bond and options is zero. The fixed supply of the stock is one share.

Investor Optimization Problem

Market Structure

- Consider a single investor who faces a single uncertainty in a single period.
- The uncertainty is the payoff of a risky asset. In the GE section, we will treat the uncertainty as the value of the market portfolio.
- At the end of the single period, the uncertainty is resolved and all securities pay off. Markets are only open at the beginning of the single period.
- We assume that there are three types of securities available for trading at the beginning of the period:
  1. A bond costing B_0 and paying unity at the end of the period
  2. A stock costing S_0 and paying S at the end of the period
  3. European puts costing $P_0(K)$ and paying $(K - S)^+, K < S_0$ and European calls costing $C_0(K)$ and paying $(S - K)^+, K > S_0$ at the end of the period. Note that only out-of-the-money options are available.

Spanning

- Consider a terminal cash flow $f(S)$, which is a twice differentiable function of the final stock price S.
- By using the fundamental theorem of calculus twice, the paper proves that at the end of the period:

$$\begin{aligned} f(S) &= [f(S_0) - f'(S_0)S_0] + f'(S_0)S \\ &\quad + \int_0^{S_0} f''(K)(K-S)^+ dK + \int_{S_0}^{\infty} f''(K)(S-K)^+ dK. \end{aligned}$$

- This may be interpreted as a Taylor series expansion with remainder of the final payoff $f(\cdot)$ about the spot price S_0.
- The first two terms give the tangent to the payoff at S_0; the last two terms bend the tangent so as to conform to the payoff.
- The payoff of an arbitrary claim has been decomposed into the payoff from $f(S_0) - f'(S_0)S_0$ bonds, $f'(S_0)$ shares, and the spectrum of out-of-the-money options.

From Payoffs to Prices

- Recall the decomposition of the payoff function $f(S)$ into payoffs from bonds, shares, and options:

$$\begin{aligned} f(S) &= [f(S_0) - f'(S_0)S_0] + f'(S_0)S \\ &\quad + \int_0^{S_0} f''(K)(K-S)^+ dK + \int_{S_0}^{\infty} f''(K)(S-K)^+ dK. \end{aligned}$$

- The initial value $V_0[f]$ of the continuous payoff $f(\cdot)$ can be expressed in terms of the initial prices of the bond B_0, the stock S_0, calls $C_0(K)$, and puts $P_0(K)$ respectively:

$$\begin{aligned} V_0[f] &= [f(S_0) - f'(S_0)S_0]B_0 + f'(S_0)S_0 \\ &\quad + \int_0^{S_0} f''(K)P_0(K)dK + \int_{S_0}^{\infty} f''(K)C_0(K)dK. \end{aligned}$$

- For example, suppose we are interested in decomposing an in-the-money European call, i.e. $f(S) = (S - K_c)^+$, $K_c < S_0$. Formally using the above decomposition gives:

$$C_0(K_c) = -K_c B_0 + S_0 + P_0(K_c),$$

which is Put Call Parity.

Risk Neutral Density

- Recall that the value of an arbitrary claim decomposed into $[f(S_0) - f'(S_0)S_0]$ bonds, $f'(S_0)$ shares, and the spectrum of out-of-the-money options:

$$\begin{aligned} V_0[f] &= [f(S_0) - f'(S_0)S_0]B_0 + f'(S_0)]S_0 \\ &\quad + \int_0^{S_0} f''(K)P_0(K)dK + \int_{S_0}^{\infty} f''(K)C_0(K)dK. \end{aligned}$$

- It is well-known that market completeness implies the existence of a unique risk-neutral density $q(S)$, which is also used to value payoffs as:

$$V_0[f] = \int_0^{\infty} f(K)B_0 q(K)dK.$$

- The second equation can be obtained from the first by integrating by parts twice. The state pricing density $B_0 q(K)$ is related to option prices by:

$$B_0 q(K) = \begin{cases} \frac{\partial^2 P_0(K)}{\partial K^2} & \text{for } K \leq S_0; \\ \frac{\partial^2 C_0(K)}{\partial K^2} & \text{for } K > S_0. \end{cases}$$

- Thus, the existence of the bond, the stock, and out-of-the-money options of all strikes implies the existence of the risk-neutral density.

Preferences

- Now that we know how to price an arbitrary continuous payoff, we need to know which payoff is preferred.
- Under standard axioms on investor behavior, an investor's preferences are fully characterized by their utility function $U_i[W_i]$, where W_i denotes terminal wealth of investor i.
- As usual, we assume that the utility of terminal wealth $U_i[\cdot]$ is increasing and concave.

Personal Beliefs

- Our final assumption concerns the investor's beliefs, characterized by a personal probability density $p_i(S)$.
- We assume that $p_i(S) > 0$ for all $S > 0$. We allow discrete probability mass at $S = 0$. The probability of a negative price is zero.
- For example, the investor may believe that the underlying risky asset price has a lognormal distribution $\ell(S; \mu_i, v_i)$, with a mean μ_i and a volatility v_i which can differ from those in the risk-neutral distribution.
- These probability beliefs are used to calculate the expected utility arising from any possible payoff.

Expected Utility Maximation

- The investor selects the payoff function f which maximizes his or her expected utility:
$$\max_{f(S)} \int_0^\infty U_i[f(S)]p_i(S)dS,$$
where $U(\cdot)$ is the investor's utility of terminal wealth and recall $p_i(S)$ is the personal density.

- This maximization is subject to the budget constraint that the investor can afford the selected payoff:
$$B_0 \int_0^\infty f(S)\ q(S)dS \leq W_0^i,$$
where W_0^i is the investor's initial wealth and recall B_0 is the unit discount bond price and $q(S)$ is the risk-neutral density.

First Order Condition

- Define the Lagrangian as:

$$\mathcal{L}_i \equiv \int_0^\infty U_i[f(S)]p_i(S)dS - \lambda_i \left[B_0 \int_0^\infty f(S)\ q(S)dS - W_0^i\right].$$

- Differentiating with respect to the payoff function f gives the first order condition governing the *optimal* payoff, $\phi_i(S)$:

$$U_i'[\phi_i(S)]\frac{p_i(S)}{B_0 q(S)} = \lambda_i.$$

- The ratio $\frac{p_i(S)}{B_0 q(S)}$ is the expected return from an investment in an Arrow-Debreu security struck at S. If an investor were risk-neutral, then the investor would plunge into the state with the highest expected return.

- Since the investor is risk-averse, the attractiveness of a state is measured by the product of the expected return and the marginal utility. This product gives the rate at which the expected utility from a state increases with the investment in that state.

- In words, the optimal payoff is chosen such that the extra expected utility gained from the last initial dollar spent on each state is the same across all states. If this condition did not hold, then expected utility could be increased by re-allocating initial wealth from states where the product is low to states where it is high.

Properties of the Optimal Payoff

- Recall the first order condition:

$$U_i'[\phi_i(S)]\frac{p_i(S)}{B_0 q(S)} = \lambda_i.$$

- Note that states with relatively high expected returns $\frac{p_i(S)}{B_0 q(S)}$ must be multiplied by relatively low marginal utilities. Since marginal utility decreases with wealth, this is accomplished by having a high payoff in such states.

- Further note that the rate at which marginal utility falls with wealth is governed by the degree of risk aversion. The less risk-averse the investor, the less marginal utility falls with wealth, and the greater is the response of the payoff to higher expected return. In the limit, a risk-neutral investor plunges into the state with the highest expected return.

- Conversely, the more risk-averse the investor, the more marginal utility falls with wealth, and the lower is the response of the payoff to expected return. In the limit, the infinitely risk averse investor has an optimal payoff which is the same across all states.

Intepreting the Optimal Payoff

- Recall the first order condition:

$$U_i'[\phi_i(S)]\frac{p_i(S)}{B_0 q(S)} = \lambda_i.$$

- Another interpretation of the optimal payoff arises from multiplying by $\frac{B_0 q(S)}{\lambda_i}$ and integrating over S:

$$\frac{1}{\lambda_i}\int_0^\infty p_i(S)U_i'[\phi_i(S)]dS = B_0\int_0^\infty q(S)dS = B_0.$$

- Substituting into the top equation gives:

$$\pi_i(S) \equiv \frac{p_i(S)U_i'[\phi_i(S)]}{\int_0^\infty p_i(S)U_i'[\phi_i(S)]dS} = q(S).$$

- The LHS can be interpreted as a probability density since it is positive and integrates to one. Thus, the optimal payoff is chosen so that each investor equates his personalized risk-adjusted probability density to the risk-neutral density.

- More generally, multiplying the top equation by $\frac{B_0 q(S) f(S)}{\lambda_i}$ and integrating over S gives:

$$B_0\int_0^\infty \pi_i(S)f(S)dS = B_0\int_0^\infty f(S)q(S)dS \equiv V_0[f].$$

- For payoffs such as $f(S) = S$ or $f(S) = (S-K)^+$, the RHS is the observable market price of the stock or option. Thus, each individual chooses his optimal payoff so that the personalized value of each asset equates with the market value.

Explicit Expression for the Optimal Payoff

- Recall the first order condition:

$$U_i'[\phi_i(S)]\frac{p_i(S)}{B_0 q(S)} = \lambda_i.$$

- The optimal payoff is easily found to be:

$$\phi(S) = (U_i')^{-1}\left(\lambda_i \frac{B_0 q(S)}{p_i(S)}\right).$$

- Since $(U_i')^{-1}$ is decreasing in its argument, it is increasing in $\frac{p_i(S)}{q(S)}$. Thus, the optimal payoff in state S is increasing in the expected excess return from that state, $\frac{p_i(S)}{q(S)}$.
- If $p_i(S) = q(S)$ for all S, then the optimal payoff is that of a bond.
- If the expected return on the stock is greater than the riskfree rate, then $\int Sp_i(s)dS > \int Sq(S)dS$ and so we would anticipate that $p_i(S) > q(S)$ for most of the higher states. We cannot have $p_i(S) > q(S)$ for all states since both densities must integrate to 1. Hence $p_i(S) < q(S)$ for most of the low states. Thus, the payoff is higher in higher states and lower in lower states and so the payoff would be increasing most of the time.
- Similarly, if the expected return on the stock were lower than the riskfree rate, then the payoff would be decreasing most of the time. Note however that risk aversion would cause the payoff to be convex at higher stock prices, so as to prevent extremely negative outcomes.

Optimal Payoff(con'd)

- Recall that the optimal payoff is:

$$\phi_i(S) = (U_i')^{-1}\left(\lambda_i \frac{B_0 q(S)}{p_i(S)}\right),$$

  and that $\phi_i(S)$ is increasing in $\frac{p_i(S)}{q(S)}$.

- If the investor thinks volatility is higher than is implied in the market, then the investor has $p_i(S) > q(S)$ for states with S very high and very low, and has $p_i(S) < q(S)$ for nearby states. Thus, the optimal payoff is U-shaped.

- Conversely, if the investor thinks volatility is lower than implied, the optimal payoff will be mainly concave. However, risk aversion causes the optimal payoff to become convex in the tails so that extremely negative payoffs are avoided.

Linear Risk Tolerance

- One objective of the paper is to delineate conditions under which derivatives are not held. In this case, only the bond and stock are held.
- Cass and Stiglitz(1970) showed that a necessary condition for two fund monetary separation to hold is that investors have linear risk tolerance (LRT):

$$T_i(W) \equiv -\frac{U_i'(W)}{U_i''(W)} = \tau_i + \gamma_i W.$$

- The parameter γ_i is frequently called cautiousness in the literature.
- To avoid negative risk tolerance, the utility function is defined only for wealth levels $W_i \geq -\frac{\tau_i}{\gamma_i}$.
- For positive cautiousness, this lower bound is finite and as terminal wealth approaches it, the tolerance for risk approaches zero. Thus, LRT investors with positive cautiousness invest so as to create a floor of $-\frac{\tau_i}{\gamma_i}$ on final wealth.
- In order that this floor be attainable, we require $W_0^i \geq -B_0\tau_i/\gamma_i$.

Utility Functions Implied by LRT

- Suppose that an investor displays linear risk tolerance (LRT):

$$T_i(W) \equiv -\frac{U_i'(W)}{U_i''(W)} = \tau_i + \gamma_i W.$$

- Solving the differential equation for marginal utility $U_i'(W)$ gives:

$$U_i'(W) = \begin{cases} \rho_i \left(\tau_i + \gamma_i W\right)^{-1/\gamma_i} & \text{if } \gamma_i > 0; \\ \rho_i \exp\left(-\frac{W}{\tau_i}\right) & \text{if } \gamma_i = 0, \end{cases}$$

  where the arbitrary positive constant ρ_i can be interpreted as the individual's rate of time preference.

- Integrating once implies that LRT utility functions are positive linear transformations of:

$$U_i(W) = \begin{cases} \frac{1}{\gamma_i - 1} \left(\tau_i + \gamma_i W\right)^{1-1/\gamma_i}, & \text{if } \gamma_i \neq 1, 0; \\ \ln(\tau_i + W), & \text{if } \gamma_i = 1; \\ -\tau_i \exp\left(-\frac{W}{\tau_i}\right), & \text{if } \gamma_i = 0, \end{cases}$$

  for $W \geq -\frac{\tau_i}{\gamma_i}$.

Optimal Payoffs for LRT Investors

- Recall that the optimal payoff for arbitrary utility is:

$$\phi_i(S) = (U_i')^{-1}\left(\lambda_i \frac{B_0 q(S)}{p_i(S)}\right),$$

and that for LRT investors, marginal utility is:

$$U_i'(W) = \begin{cases} \rho_i\left(\tau_i + \gamma_i W\right)^{-1/\gamma_i} & \text{if } \gamma_i > 0; \\ \rho_i \exp\left(-\frac{W}{\tau_i}\right) & \text{if } \gamma_i = 0, \end{cases}$$

- Using the budget constraint $B_0 \int_0^\infty \phi(S)\, q(S) dS = W_0^i$ to substitute out λ_i gives an optimal payoff of:

$$\phi_i(S) = \begin{cases} -\frac{\tau_i}{\gamma_i} + \frac{W_0^i + B_0\tau_i/\gamma_i}{V_0\left[e^{\gamma_i D_i}\right]} e^{\gamma_i D_i(S)}, & \text{if } \gamma_i > 0; \\ \frac{W_0^i - \tau_i V_0[D_i]}{B_0} + \tau_i D_i(S), & \text{if } \gamma_i = 0, \end{cases}$$

where $D_i(S) \equiv \ln[p_i(S)/q(S)]$ is a measure of the deviation of personal beliefs from the risk-neutral density.

Optimal Payoff for Positive Cautiousness

- When $\gamma_i > 0$, the optimal payoff in each state is linear in a power of the expected return from that state $p_i(S)/q(S)$:

$$\phi_i(S) = -\frac{\tau_i}{\gamma_i} + \frac{W_0^i + B_0\tau_i/\gamma_i}{V_0\left[(p_i/q)^{\gamma_i}\right]}\left(\frac{p_i(S)}{q(S)}\right)^{\gamma_i}.$$

- The position in the riskless fund is taken to ensure that the floor of $-\frac{\tau_i}{\gamma_i}$ is preserved. Since $\gamma_i > 0$, whether the investor is long or short the fund depends on the sign of τ_i.
- All of the excess of initial wealth over the present value of this floor is invested in a long position in the risky fund with customized payoff $\left(\frac{p_i(S)}{q(S)}\right)^{\gamma_i}$. Since the risky fund has a nonnegative payoff, options must be used if the implicit position in either the bond or the stock is negative.

Optimal Payoff for Zero Cautiousness

- Recall the zero cautiousness payoff:

$$\phi_i(S) = \frac{W_0^i - \tau_i V_0[D_i]}{B_0} + \tau_i D_i(S),$$

where recall $V_0[D_i]$ is the initial cost of the customized payoff $D_i(S) \equiv \ln\left(\frac{p_i(S)}{q(S)}\right)$.

- Given the availability of this payoff, the zero cautiousness investor buys τ_i units of this risky fund, where τ_i is now the investor's constant risk tolerance.

- Thus, in contrast to the case with positive cautiousness, the investor first fixes the number of units of the risky fund and then invests all remaining wealth in the riskless fund.

- In further contrast to the case with positive cautiousness, the risky fund can have an arbitrarily large negative payoff. Since risk tolerance is independent of final wealth, low wealth realizations do not induce the zero cautiousness investor to place a floor on final wealth.

Partial Equilibrium

Lognormal Beliefs with Equal Volatilities

- We return to arbitrary preferences, but now restrict personal probabilities.
- Suppose that the Black Scholes model holds so that both the personal and risk-neutral densities are lognormal. Furthermore, the volatilities are equal.
- Consider an investor who cannot trade continuously and who maximizes expected utility of wealth at the next trading opportunity.
- In this setting, one can show that this investor's optimal payoff is increasing if the personal mean μ_i exceeds the riskfree rate r, is constant if $\mu_i = r$, and is decreasing otherwise.
- Thus, an investor who believes that the stock's expected return is above the riskfree rate, but below that required for the risk borne, should nonetheless have a payoff which is increasing with the stock.
- Furthermore, the optimal payoff will in general be non-linear, even though the investor agrees with the volatility in the market.

LRT and Lognormal Beliefs with Equal Volatilities

- We now assume both linear risk tolerance, and that the personal and risk-neutral density are both lognormal with the same volatility σ.
- Under positive cautiousness, the optimal payoff is linear in a power function:

$$\phi_i(S) = -\frac{\tau_i}{\gamma_i} + \frac{W_0^i + B_0\tau_i/\gamma_i}{V_0[S^{\gamma_i \mathcal{S}_i}]} S^{\gamma_i \mathcal{S}_i},$$

  where recall γ_i is the cautiousness parameter and $\mathcal{S}_i$ is the Sharpe ratio $\mathcal{S}_i \equiv \frac{\mu_i - r}{\sigma^2}$.
- In a continuous time setting, the power function payoff can be achieved by keeping a constant fraction $\gamma_i \mathcal{S}_i$ of the investor's "risk-capital" $R_0^i \equiv W_0^i + B_0\tau_i/\gamma_i$ in stock and investing the rest in bonds. In practice, the power payoff is more easily achieved by investing the risk capital in a static position in a derivative with payoff $S^{\gamma_i \mathcal{S}_i}$.
  - If $\mu_i > r + \gamma_i\sigma^2$, then the optimal payoff is increasing and convex.
  - $\mu_i = r + \gamma_i\sigma^2$, then the optimal payoff is the terminal stock price.
  - If $\mu_i \in (r, r + \gamma_i\sigma^2)$, then the optimal payoff is increasing and concave.
  - If $\mu_i = r$, then the optimal payoff is flat.
  - If $\mu_i < r$, then the optimal payoff is decreasing and convex.
- Thus, options are used as part of the portfolio except in the unlikely event that $\mu_i = r + \gamma_i\sigma^2$ or $\mu_i = r$.
- The results for zero cautiousness are similar, except that the absence of a floor allows for losses of unlimited size.

LRT and Lognormal Beliefs with Unequal Volatilities

- Under linear risk tolerance and lognormal personal beliefs and risk-neutral density, the optimal payoff is:

$$\phi_i(S) = \begin{cases} -\frac{\tau_i}{\gamma_i} + \frac{W_0^i + B_0\tau_i/\gamma_i}{V_0[e^{\gamma_i(B_i x - C_i x^2/2)}]} e^{\gamma_i\left(B_i x - C_i\frac{x^2}{2}\right)}, & \text{if } \gamma_i > 0; \\ \frac{W_0^i - \tau_i(B_i V_0[x] - C_i V_0[x^2/2])}{B_0} + \tau_i\left(B_i x - C_i\frac{x^2}{2}\right), & \text{if } \gamma_i = 0, \end{cases}$$

where:

$$\begin{aligned} B_i &= \frac{\mu_i}{v_i^2} - \frac{r}{\sigma^2} \\ C_i &= \frac{1}{v_i^2} - \frac{1}{\sigma^2}. \end{aligned}$$

- If the investor's personal volatility is greater than the implied volatility, then the optimal payoff is U-shaped. This can be achieved by buying a bond and buying straddles and strangles.

- Conversely, if the investor's personal volatility is less than the implied volatility, then the shape of the optimal payoff depends on the cautiousness.

- For positive cautiousness, the shape is similar to a lognormal density. Beliefs cause the payoff to be concave for final stock prices near the initial one. Risk aversion forces the payoff to be convex in the tails.

- For zero cautiousness, the absence of a floor allows global concavity.

- Note that whenever the risk-neutral and personal volatilities differ, the investor optimally holds a non-linear payoff, mandating the use of options.

General Equilibrium

Overview

- Our analysis indicates that it is very unusual for an investor to regard a linear payoff as optimal.
  - The only situations in which bonds are the only securities held are either if the investor happens to have $p_i(S) = q(S) \forall S$, or else is infinitely risk-averse.
  - Under lognormal beliefs and risk-neutral density, a simple stock position is optimal only if the investor's personal volatility agrees with the risk-neutral volatility and if the investor's expected return on the stock is the sum of the riskfree rate and the *investor's* risk premium.
- Thus, in most cases derivatives are always held. However, we recognize that *in aggregate*, no derivatives are held and the market is fully invested in the stock. Thus, the aggregate results are directly orthogonal to those obtained at the individual level.
- A priori, the requirement that derivatives be in zero net supply is likely to dampen optimal derivative positions. We therefore turn to an equilibrium analysis.

Market Clearing Conditions

- Consider an economy in which multiple investors simultaneously optimize their holdings.
- We no longer take option prices as given, and so the the form of the risk-neutral density must be solved for endogeneously.
- We require that the risk-neutral density must price the bond:

$$B_0 \int_0^\infty 1 q(S) dS = B_0,$$

  or equivalently, that the risk-neutral density $q(\cdot)$ integrates to one.
- We further require that the risk-neutral density must price the stock:

$$B_0 \int_0^\infty S q(S) dS = S_0,$$

  or equivalently, that the risk-neutral mean stock return is the riskless rate.
- Finally, we assume that bonds and options are in zero net supply and thus in the aggregate, it is just the stock that is held:

$$\sum_{i=1}^{n} \phi_i(S) = S,$$

  which implies that the sum of the exposures is unity:

$$\sum_{i=1}^{n} \phi_i'(S) = 1.$$

- The above conditions ensure that the bond, stock, and derivative markets are cleared in equilibrium.

The Risk Neutral Density in General Equilibrium

- The paper shows that the risk-neutral density is given by:

$$q(S) = q(0)\exp\left\{-\int_0^S \frac{1}{T(Z)}dZ\right\}\exp\left\{\int_0^S \sum_{i=1}^n \frac{T_i[\phi_i(Z)]}{T(Z)}\frac{p_i'(Z)}{p_i(Z)}dZ\right\},$$

  where $T(S) \equiv \sum_{i=1}^n T_i[\phi_i(S)]$ is the total risk tolerance in state S.

- Since the optimal payoff ϕ_i depends on q, this is not an explicit expression. However, under homogeneous beliefs, we do get an explicit expression:

$$q(S) = q(0)\exp\left\{-\int_0^S \frac{1}{T(Z)}dZ\right\}p(S).$$

- Furthermore, under heterogeneous beliefs, the top equation indicates that the equilibrium risk-neutral density is the product of a factor reflecting total risk tolerance i.e. $\exp\left\{-\int_0^S \frac{1}{T(Z)}dZ\right\}$ and a factor reflecting the personal beliefs, which we term the market view.

- The greater the risk tolerance of a given investor, the more his probability density gets reflected in the market view.

- The first factor is a positive declining function of S which changes the mean in the market view to the riskless rate, and may add negative skewness.

The Optimal Exposure in General Equilibrium

- Recall the first order condition $U_i'[\phi_i(S)]\frac{p_i(S)}{B_0 q(S)} = \lambda_i$.
- Taking the logarithmic derivative with respect to the stock price implies that the investor's optimal *exposure* decomposes into the product of his preferences and beliefs:

$$\phi_i'(S) = T_i[\phi_i(S)]\frac{d}{dS}\ln[p_i(S)/q(S)].$$

- Substituting in the expression for the risk-neutral density on the last page gives the optimal exposure as:

$$\phi_i'(S) = \frac{T_i[\phi_i(S)]}{T(S)} + T_i[\phi_i(S)]\left[\frac{d\ln p_i(S)}{dS} - \sum_{i=1}^{n}\frac{T_i[\phi_i(S)]}{T(S)}\frac{d\ln p_i(S)}{dS}\right].$$

- The first term reflects the investor's risk tolerance relative to the population total.
- The second term is a composite of the investor's risk tolerance and the extent to which the investor's beliefs differ from a risk tolerance weighted average of the beliefs of other investors in the economy.

Optimal Exposure Under Homogenous Beliefs

- Recall the expression for the optimal exposure:

$$\phi_i'(S) = \frac{T_i[\phi_i(S)]}{T(S)} + T_i[\phi_i(S)] \left[\frac{d \ln p_i(S)}{dS} - \sum_{i=1}^{n} \frac{T_i[\phi_i(S)]}{T(S)} \frac{d \ln p_i(S)}{dS} \right].$$

- If investors have homogeneous beliefs (i.e. $p_i(S) = p(S)\ \forall i$), then the second term vanishes:

$$\phi_i'(S) = \frac{T_i[\phi_i(S)]}{T(S)}.$$

- Since the right side is positive, homogeneous beliefs imply that all investors must have an increasing payoff. The greater the investor's risk tolerance relative to the total, the greater the exposure of the investor's position.

- To solve this nonlinear o.d.e., recall that under homogeneous beliefs:

$$q(S) = q(0) \exp \left\{ - \int_0^S \frac{1}{T(Z)} dZ \right\} p(S).$$

- Substituting into the optimal payoff $\phi(S) = (U_i')^{-1} \left(\lambda_i \frac{B_0 q(S)}{p_i(S)} \right)$ gives:

$$\phi_i(S) = (U_i')^{-1} \left(\lambda_i B_0 q(0) \exp \left\{ - \int_0^S \frac{1}{T(Z)} dZ \right\} \right).$$

- Thus under homogeneous beliefs, the actual form of the common proprietary density function does not affect the optimal payoff.

Optimal Payoffs Under Homogeneous Beliefs and Opposite Cautiousness

- We will soon show that LRT investors with homogeneous beliefs and identical cautiousness will not hold derivatives, even though they differ in risk aversion.
- Thus, an interesting open question is whether homogeneous beliefs induce linear payoffs, even though investors differ in risk aversion.
- Here, we show that derivatives are held in an economy with two LRT investors with homogeneous beliefs but *opposite* cautiousness. In particular, if $T_1[W_1] = \tau_1 + \gamma W_1$ and $T_2[W_2] = \tau_2 - \gamma W_2$, then:

$$\begin{aligned}
\phi_1(S) &= \frac{S}{2} - \frac{\tau}{2\gamma} + \sqrt{\left(\frac{S}{2} + \frac{\tau_2 - \tau_1}{2\gamma}\right)^2 + k^2} \\
\phi_2(S) &= \frac{S}{2} + \frac{\tau}{2\gamma} - \sqrt{\left(\frac{S}{2} + \frac{\tau_2 - \tau_1}{2\gamma}\right)^2 + k^2},
\end{aligned}$$

  where $\tau \equiv \tau_1 + \tau_2$ and k is an arbitrary constant.

- Thus, in this simple economy, a three fund separation occurs in which each investor holds equal positions in the stock and offsetting positions in the bond and the derivative.
- This simple example shows that derivatives may be optimally held when investors differ only in preferences.

Linear Risk Tolerance with Identical Cautiousness

- As mentioned previously, a necessary condition for two fund monetary separation to hold under homogeneous beliefs is that all investors display linear risk tolerance with identical cautiousness:

$$T_i[\phi_i(S)] = \tau_i + \gamma\phi_i(S), \qquad \gamma \geq 0.$$

- In order to solve for the equilibrium holdings, we assume linear risk tolerance with identical cautiousness, but allow for heterogeneous beliefs.
- The heterogeneity of beliefs implies that two fund monetary separation will not hold. Instead, we derive sufficient conditions for $m > 2$ fund monetary separation under heterogeneous beliefs.

Generalized Logarithmic Utility

- Under identical cautiousness of one, then utility is generalized log, $U(W_i) = \ln(\tau_i + W)$, and the optimal payoff becomes:

$$\phi_i(S) = -\tau_i + \frac{R_0^i p_i(S)\tau}{\Sigma_{i=1}^n R_0^i p_i(S)} + \frac{R_0^i p_i(S) S}{\Sigma_{i=1}^n R_0^i p_i(S)},$$

  where $R_0^i \equiv W_0^i + B_0\tau_i$ is the "risk capital" of investor i.

- Thus, each logarithmic utility investor first establishes a floor at $-\tau_i$ and then invests all remaining wealth in the two limited liability funds.

- The higher is W_0^i or τ_i, the higher is the investor's risk tolerance, and the larger is his position in each customized derivative.

Separation Under Generalized Log Utility

- To obtain a separation result under generalized log utility, assume that each investor's personal density can be represented as:

$$p_i(S) = p(S)\left(\sum_{k=1}^{m} c_{ik} f_k(S)\right), \quad i = 1, \ldots, n,$$

where $p(S)$ is the unknown true density and $\{f_k(S), k = 1, \ldots, m\}$ is a collection of basis functions.

- In words, each investor's density differs from the true density by a multiplicative error, which can be represented by a finite number of basis functions.

- When the top equation holds, then the optimal payoff is:

$$\phi_i(S) = -\frac{\tau_i}{\gamma} + \sum_{k=1}^{m} b_i c_{ik} \frac{f_k(S)\tau/\gamma}{\sum_{k=1}^{m} \theta_k f_k(S)} + \sum_{k=1}^{m} b_i c_{ik} \frac{f_k(S) S}{\sum_{k=1}^{m} \theta_k f_k(S)},$$

where $\theta_k \equiv \sum_{i=1}^{n} b_i c_{ik}$.

- Thus, each investor's holdings separate into $2m+1$ funds. The first fund is the riskless fund, which is used to establish the floor of $-\frac{\tau_i}{\gamma}$. Each investor holds $b_i c_{ik}$ units of each of the $2m$ derivative funds, where the funds have a payoff of $\left\{\frac{(\tau/\gamma)^{1-l} S^l f_k(S)}{\sum_{k=1}^{m} \theta_k f_k(S)}, k = 1, \ldots, m, l = 0, 1\right\}$.

- No one holds the stock individually, although the collective holdings sum to the stock.

Zero Cautiousness

- Suppose that each investor has constant risk tolerance, i.e. $T_i[\phi(S)] = \tau_i \,\forall i$.
- Then each investor has an optimal exposure of the form:

$$\phi_i'(S) = \frac{\tau_i}{\tau} + \tau_i \left[\frac{d \ln p_i(S)}{dS} - \sum_{i=1}^{n} \frac{\tau_i}{\tau} \frac{d \ln p_i(S)}{dS} \right],$$

  where $\tau \equiv \sum_{i=1}^{n} \tau_i$ is the total risk tolerance.

- Integration gives the optimal payoff in terms of bonds, stocks, and derivatives:

$$\phi_i(S) = \kappa_i + \frac{\tau_i}{\tau} S + \tau_i d_i(S),$$

  where $d_i(S) \equiv \ln p_i(S) - \sum_{i=1}^{n} \frac{\tau_i}{\tau} \ln p_i(S)$.

- The constant of integration κ_i is determined by substituting this payoff into the budget constraint.

$$\kappa_i = \frac{W_0^i - \frac{\tau_i}{\tau} S_0 - \tau_i V_0[d_i]}{B_0}.$$

Analysis of Zero Cautiousness Economy

- Recall that the optimal payoff for each investor is:

$$\phi_i(S) = \frac{W_0^i - \frac{\tau_i}{\tau} S_0 - \tau_i V_0[d_i]}{B_0} + \frac{\tau_i}{\tau} S + \tau_i d_i(S),$$

  where $d_i(S) \equiv \ln p_i(S) - \sum_{i=1}^{n} \frac{\tau_i}{\tau} \ln p_i(S)$.

- In this economy, each investor's stock and derivatives position does not depend on his initial wealth. Thus, the bond position is used to finance the positions in stocks and derivatives. The magnitude of this position in stock and derivatives depends on their risk tolerance. The greater the risk tolerance, the greater the exposure to stocks and derivatives.

- Each investor's stock position does not depend on his beliefs. In contrast, each investor's derivatives position depends mainly on the extent to which his beliefs differ from those in the market. Thus, the open interest in derivatives markets is primarily a reflection of the heterogeneity of beliefs.

- If investors have homogeneous beliefs but differing risk aversion, then they do not hold derivatives. Differences in risk aversion under homogeneous beliefs affect only the division between the riskless asset and the stock.

Separation Under Constant Risk Tolerance

- Recall that the optimal payoff for each investor is:

$$\phi_i(S) = \frac{W_0^i - \frac{\tau_i}{\tau} S_0 - \tau_i V_0[d_i]}{B_0} + \frac{\tau_i}{\tau} S + \tau_i d_i(S),$$

  where $d_i(S) \equiv \ln p_i(S) - \sum_{i=1}^{n} \frac{\tau_i}{\tau} \ln p_i(S)$.

- To obtain separation results under constant risk tolerance, suppose that the log of each personal density can be written as a linear combination of basis functions:

$$\ln p_i(S) = \sum_{k=1}^{m} c_{ik} f_k(S).$$

- Then the optimal payoff separates into $m+2$ funds:

$$\phi_i(S) = \kappa_i + \frac{\tau_i}{\tau} S + \tau_i \sum_{k=1}^{m} \left[\left(c_{ik} - \sum_{i=1}^{n} \frac{\tau_i}{\tau} c_{ik} \right) f_k(S) \right].$$

- Furthermore, the m derivative funds are the m basis functions which make up the log of the density. The optimal holding in the $k-th$ fund is $\tau_i \left(c_{ik} - \sum_{i=1}^{n} \frac{\tau_i}{\tau} c_{ik} \right)$.

- Thus, if investors agree on the coefficient of $\ln p$ on the $j-th$ basis function i.e. $c_{ij} = c_j$, then that fund is not held by anyone.

Risk Neutral Density Under Constant Risk Tolerance

- Under constant risk tolerance, the risk-neutral density simplifies to:

$$q(S) = \kappa \exp(-S/\tau) \prod_{i=1}^{n} [p_i(S)]^{\frac{\tau_i}{\tau}},$$

  where κ is a normalizing constant given by the requirement that q integrates to 1.

- Thus, the market view is a risk-tolerance weighted geometric average of the individual densities.

- Given a specification of probability beliefs and an array of risk tolerances, it is straightforward to use this risk-neutral density to value an option or any other derivative.

- Note that under homogeneous beliefs, the risk-neutral density simplifies to:

$$q(S) = \kappa \exp(-S/\tau) p(S).$$

- Thus, if $p(S)$ is normal, then $q(S)$ is also normal with the same variance and with mean equal to the forward price as shown in Brennan(1979).

Zero Cautiousness and Lognormal Beliefs

- When all investors have lognormal beliefs, the log of each density is quadratic in $x \equiv \ln(S/S_0)$:

$$\ln \ell(S; \mu_i, v_i) = -\ln(\sqrt{2\pi} v_i S_0) - x - \frac{1}{2}\left[\frac{x - (\mu_i - v_i^2/2)}{v_i}\right]^2.$$

- Thus, the optimal payoff under lognormal beliefs and constant risk tolerance is:

$$\phi_i(S) = \frac{W_0^i - m_i V_0[x] + p_i V_0[x^2/2] - S_0\tau_i/\tau}{B_0} + \frac{\tau_i}{\tau} S + m_i x - \frac{p_i}{2}\ln^2 x,$$

where $m_i \equiv \tau_i \left(\frac{\mu_i}{v_i^2} - \Sigma_{i=1}^n \frac{\tau_i}{\tau}\frac{\mu_i}{v_i^2}\right)$, $p_i \equiv \tau_i \left(\frac{1}{v_i^2} - \Sigma_{i=1}^n \frac{\tau_i}{\tau}\frac{1}{v_i^2}\right)$, and for the lognormal risk-neutral density, $V_0[x] = r - \sigma^2/2$ and $V_0[x^2/2] = \frac{\sigma^2 + (r - \sigma^2/2)^2}{2}$.

- The optimal payoff for each investor involves just two derivatives, one paying the log of the stock price and the other paying its square.

- The log contract is used to speculate on expected return, while the squared log contract is used to speculate on variance.

Risk Neutral Density Under Constant Risk Tolerance and Lognormal Beliefs

- The equilibrium risk-neutral density is:

$$q(S) = \kappa \exp(-S/\tau) \frac{\exp\left[-\frac{1}{2}(\frac{\ln S - \zeta}{\theta})^2\right]}{S\sqrt{2\pi}\theta},$$

  where the aggregate precision $\frac{1}{\theta^2} = \sum_{i=1}^{n} \frac{\tau_i}{\tau} \frac{1}{v_i^2}$ is a risk tolerance weighted average of the individual precisions, and $\zeta = \frac{\sum_{i=1}^{n}(\tau_i/v_i^2)(\ln S_0 + \mu_i - v_i^2/2)}{\sum_{i=1}^{n}(\tau_i/v_i^2)}$ is a weighted average of the individual means for the log price, where the weights are given by the ratio of the risk tolerance to the risk.

- We note that $q(S)$ is not a lognormal density even though each investor believes that the stock price is lognormally distributed. However, the market view is lognormal since it is a geometric average of the lognormal individual views.

- The negative exponential adds negative skewness to this lognormal density. As a result, a graph of Black Scholes implied volatilities against strike prices will slope down, as is observed in equity index option markets.

Conclusions

- The main point is that heterogeneous beliefs and/or preferences induce investors to hold derivatives individually, even if they are not held in aggregate.
- The only exception we uncovered is that under linear risk tolerance with identical cautiousness and beliefs, investors do not hold derivatives, even though they differ in risk aversion.
- We also found that the equilibrium risk-neutral density is usually in a separate class from the market view. However, if at least one investor is risk-neutral, then the risk-neutral density would be in the same class.

Extensions

- The analysis has been further extended to a continuous time setting, in which the stock price process is a pure jump process, with a continuous jump size distribution and an infinite arrival rate
- We determine the optimal wealth exposure in this setting and show how dynamic trading strategies in derivatives are able to deliver this exposure, while dynamic trading in stocks and bonds cannot.
- It should also be possible to extend the analysis to a setting with stochastic local volatility. If the volatility at every strike and maturity follows its own process, then dynamic trading in the continuum of options would be needed to achieve optimal positions.
- Finally, it would also be interesting to extend the analysis to the situation where markets are incomplete even in the presence of derivatives.