Why Be Backward?

Forward Equations for American Options*

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I. Introduction

Valuing and hedging derivatives consistent with the volatility smile has been a major research focus for over a decade. A breakthrough occurred in the mid-nineties with the recognition that in certain models, European option values satisified forward evolution equations in which the independent variables are the options' strike and maturity. More specifically, Dupire (1994) showed that under deterministic carrying costs and a diffusion process for the underlying price, no arbitrage implies that European option prices satisfy a certain partial differential equation (PDE), now called the Dupire equation. Assuming that one could observe European option prices of all strikes and maturities, then this forward PDE can be used to explicitly determine the underlying's instantaneous volatility as a function of the underlying's price and time. Once this volatility function is known, the value function for European, American, and many exotic options can be determined by a wide array of standard methods. As this value function relates theoretical prices of these instruments to the underlying's price and time, it can also be used to determine many greeks of interest as well.

Aside from their use in determining the volatility function, forward equations also serve a second useful purpose. Once one knows the volatility function either by an explicit specification or by a prior calibration, the forward PDE can be numerically solved to efficiently value a collection of European options of different strikes and maturities all written on the same underlying asset. Furthermore, as pointed out in Andreasen (1998), all the greeks of interest satisfy the same forward PDE and hence can also be efficiently determined in the same way.

Since the original development of forward equations for European options in continuous models, several extensions have been proposed. For example, Esser and Schlag (2002) develop forward equations for European options written on the forward price rather than the spot price. Forward equations for European options in jump diffusion models were developed in Andersen and Andreasen (1999) and extended by Andreasen and Carr (2002). It is straightforward to develop the relevant forward equations for European binary options or for European power options by differentiating or integrating the forward equation for standard European options.

Buraschi and Dumas (2001) develop forward equations for compound options[1]. In contrast to the PDE's determined by others, their evolution equation is an ordinary differential equation whose sole independent variable is the intermediate maturity date.

Given the close relationship between compound options and American options, it seems plausible that there might be a forward equation for American options. The development of such an equation has important practical implications since all listed options on individual stocks are American-style. The Dupire equation cannot be used to infer the volatility function from market prices of American options, nor can it be used to efficiently value a collection of American options of differing strikes and maturities.

The purpose of this paper is to develop forward equations for standard American options. This problem is addressed for American calls on stocks paying discrete dividends in Buraschi and Dumas (2001) and it is also considered in a lattice setting in Chriss (1996). We direct our attention to the more difficult problem of pricing continuously exerciseable American puts in continuous time models. To do so, we depart from the diffusive models which characterize most of the previous research on forward equations in continuous time. To capture the smile, we assume that prices jump rather than assuming that the instantaneous volatility is a function of stock price and time. Dumas, Fleming, and Whaley (1998) find little empirical support for the Dupire model whereas there is a long history of empirical support for jump-diffusion models[2]. In particular, we assume that the returns on the underlying asset have stationary independent increments, or in other words that the log price is a Lévy process. Besides the Black and Scholes (1973) model, our framework includes as special cases the variance gamma (VG) model of Madan, Carr, and Chang (1998), the CGMY model of Carr, Geman, Madan, and Yor (2002), the finite moment logstable model of Carr and Wu (2002), the Merton (1976) and Kou (2002) jump diffusion models, and the hyperbolic models of Eberlein, Keller, and Prause (1998). In all of these models except Black Scholes, the existence of a jump compo-

[1]However, their definition of a compound option is non-standard in that the critical stock price is specified in the contract.

[2]For example, three recent papers documenting support for such models are Anderson, Benzoni, and Lund (2002), Carr, Geman, Madan, and Yor (2002), and Carr and Wu (2002).

nent implies that the backward and forward equations contain an integral in addition to the usual partial derivatives. Despite the computational complications introduced by this term, we use finite differences to solve both of these fundamental partial integro differential equations (PIDE's). To illustrate that our forward PIDE is a viable alternative to the traditional backward approach, we calculate American option values in the diffusion extended VG[3] option pricing model and find very close agreement.

Our approach to determining the forward equation for American options is to start with the well-known backward equation and then exploit the symmetries which essentially define Lévy processes. In the process of developing the forward equation, we also determine two hybrid equations of independent interest. The advantage of these hybrid equations over the forward equation is that they hold in greater generality. Depending on the problem at hand, these hybrid equations can also have large computational advantages over the backward or forward equations when the model has already been calibrated. In particular, the advantage of these hybrid equations over the backward equation is that they are more computationally efficient when one is interested in the variation of prices or greeks across strike or maturity at a fixed time, eg. market close.

The first of these hybrid equations has the stock price and maturity as independent variables. The numerical solution of this hybrid equation is an alternative to the backward equation in producing a spot slide, which shows how American option prices vary with the initial spot price of the underlying. If one is interested in understanding how this spot slide varies with maturity, then our hybrid equation is much more efficient than the backward equation.

Our second hybrid equation has the strike price and calendar time as independent variables. The numerical solution of this hybrid equation is an alternative to the forward equation in producing an implied volatility smile at a fixed maturity. If one is interested in understanding how the model predicts that this smile will change over time, then our hybrid equation is much more computationally efficient than the forward equation. This second hybrid equation also

[3]For details on the use of finite differences for solving the backward PIDE for American options in the VG model, see Hirsa and Madan (2002).

allows parameters to have a term structure, whereas our forward equation does not[4]. Hence, if one needs to efficiently value a collection of American options of different strikes in the time-dependent Black-Scholes model, then it is far more efficient to solve our hybrid equation than to use the standard backward equation.

The remainder of this paper is structured as follows. The next section introduces our setting and reviews the backward PIDE which governs American option values in this setting. The following section develops the first hybrid equation, while the subsequent section develops the second one. The penultimate section develops the forward equation for American options, while the final section summarizes and suggests further research.

II. Review of the Backward Free Boundary Problem

Throughout this article, we focus on (standard) American puts on stocks leaving American calls and other underlyings as an exercise for the reader. We assume perfect capital markets, continuous trading, no arbitrage opportunities, continuous dividend payments, and Markovian stock price dynamics under all martingale measures. We further assume that the spot interest rate and dividend yield are given by deterministic functions $r(t) > 0$ and $q(t) \geq 0$ respectively. Thus, we assume that under a risk-neutral measure Q, the stock price s_t satisfies the following stochastic differential equation:

$$ds_t = [r(t) - q(t)]s_{t-}dt + \sigma(s_{t-}, t)s_{t-}dW_t + \int_{-\infty}^{\infty} s_{t-}(e^x - 1)[\mu(dx, dt) - \nu(s_{t-}, x, t)dxdt], \quad (1)$$

for all $t \in [0, \bar{T}]$. Thus, the change in the stock price decomposes into three parts. The first part is the risk-neutral drift, comprised entirely of the dollar carrying cost of the stock. The second part is the diffusion part, expressed in terms of the instantaneous volatility function $\sigma(S, t)$. As usual, the term dW_t denotes increments of a standard Wiener process defined on the time set $[0, \bar{T}]$ and on a complete probability space $(\Omega, \mathcal{F}, Q)$. The third part is the jump

[4]Note however that implied volatility can have a term or strike structure in our Lévy setting.

part. The random measure $\mu(dx, dt)$ counts the number of jumps of size x in the log price at time t. The Hunt density $\{\nu(S, x, t), S > 0, x \in \Re, t \in [0, \bar{T}]\}$ is used to compensate the jump process $J_t \equiv \int_0^t \int_{-\infty}^{\infty} s_{t-}(e^x - 1)\mu(dx, ds)$, so that the last term in (1) is the increment of a Q jump martingale.[5] The jump martingale is specified in such a way that jumps to negative prices are impossible. Since the last two parts are both martingales, we have:

$$E^Q[s_t|s_0] = s_0 e^{\int_0^t [r(u)-q(u)]du},$$

where the initial stock price s_0 is positive.

Consider an American put option on the stock with a fixed strike price $K_0 > 0$ and a fixed maturity date $T_0 \in [0, \bar{T}]$. Let p_t denote the value of the American put at time $t \in [0, T_0]$. In this general setup, it is not yet known whether the American put value is monotone in S. Hence, we further assume whatever sufficient conditions on the coefficients that are needed so that the put value is monotone in S. Then for each time $t \in [0, T_0]$, there exists a unique *critical stock price*, $\underline{s}(t)$, below which the American put should be exercised early, i.e.:

$$\text{if } s_t \leq \underline{s}_t, \text{ then } p_t = \max[0, K_0 - s_t] \tag{2}$$

$$\text{and if } s_t > \underline{s}_t, \text{ then } p_t > \max[0, K_0 - s_t]. \tag{3}$$

The exercise boundary is the time path of critical stock prices, $\underline{s}_t, t \in [0, T_0]$. This boundary is independent of the current stock price s_0 and is bounded above by K_0. It is a smooth, nondecreasing function of time t whose terminal limit is:

$$\lim_{t \uparrow T_0} \underline{s}_t = K_0 \min\left[1, \frac{r(T_0)}{q(T_0)}\right].$$

[5]The function $\nu(S, x, t)$ must have the following properties:

$$\nu(S, 0, t) = 0, \quad \int_{-\infty}^{\infty} (x^2 \wedge 1)\nu(S, x, t)dx < \infty.$$

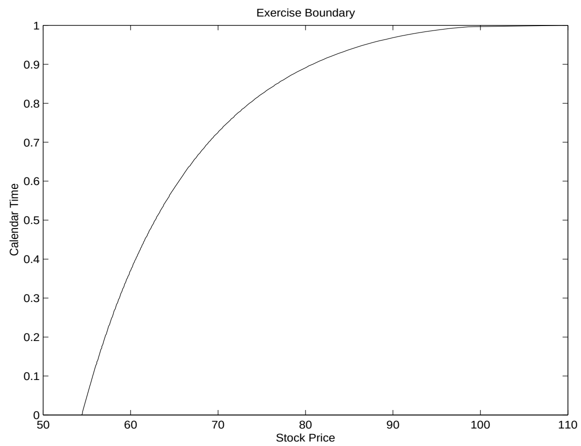


Figure 1. Exercise Boundary in the Diffusion-Extended VG Model
Critical stock prices are computed from the DEVG model for the following inputs:

$$r = .06, q = .02, \sigma = .4, s = .3, \nu = .25, \theta = -.3, K_0 = 110, T_0 = 1.$$

The finite difference scheme uses $M = 400$ space steps and $N = 200$ time steps on a domain running from 10 to 400 with initial price $S_0 = 100$.

Right at expiration, the critical stock price is the strike price, i.e. $\underline{s}_{T_0} = K_0$. Hence, when $q(T_0) > r(T_0)$, there is a discontinuity in the exercise boundary. Figure 1 plots the exercise boundary in the Diffusion Extended Variance Gamma (DEVG) model. This model extends the pure jump Variance Gamma model of Madan, Carr, and Chang (1998), by adding a diffusion component with constant volatility.

The American put value is also a function, denoted $p(s, t)$, mapping its domain $\mathcal{D} \equiv (s, t) \in [0, \infty) \times [0, T_0]$ into the nonnegative real line. The exercise boundary, $\underline{s}_t, t \in [0, T_0]$, divides this domain $\mathcal{D}$ into a *stopping region* $\mathcal{S} \equiv [0, \underline{s}_t] \times [0, T_0]$ and a *continuation region* $\mathcal{C} \equiv (\underline{s}_t, \infty) \times [0, T_0]$. Equation (2) indicates that in the stopping region, the put value function $p(s, t)$ equals its exercise value, $\max[0, K_0 - S]$. In contrast, the inequality expressed in (3) shows that in the continuation region, the put is worth more "alive" than "dead". The transition between boundaries is smooth in the following sense:

$$\lim_{s \downarrow \underline{s}_t} p(s, t) = K_0 - \underline{s}_t, \quad t \in [0, T_0] \tag{4}$$

$$\lim_{s \downarrow \underline{s}_t} \frac{\partial p(s,t)}{\partial s} = -1, \qquad t \in [0, T_0]. \tag{5}$$

The *value matching condition* (4) and (2) imply that the put value is continuous across the exercise boundary. Furthermore, the *high contact condition* (5) and (2) further imply that the put's *delta* is continuous. Equations (4) and (5) are jointly referred to as the *"smooth fit" conditions.*

The partial derivatives, $\frac{\partial p}{\partial t}$, $\frac{\partial p}{\partial s}$, and $\frac{\partial^2 p}{\partial s^2}$ exist and satisfy the following partial integro differential equation (PIDE):

$$\begin{aligned}
&\frac{\partial p(s,t)}{\partial t} + \frac{\sigma^2(s,t)s^2}{2}\frac{\partial^2 p(s,t)}{\partial s^2} + [r(t) - q(t)]s\frac{\partial p(s,t)}{\partial s} - r(t)p(s,t) \\
&+ \int_{-\infty}^{\infty} \left[p(se^x, t) - p(s,t) - \frac{\partial}{\partial s}p(s,t)s(e^x - 1)\right]\nu(s,x,t)dx \\
&+ 1(s < \underline{s}_t)\left\{r(t)K_0 - q(t)s - \int_{\ln(\underline{s}_t/s)}^{\infty} [p(se^x, t) - (K_0 - se^x)]\nu(s,x,t)dx\right\} = 0.
\end{aligned} \tag{6}$$

The last term on the left hand side (LHS) of (6) is the result of applying the integro-differential operator defined by the first two lines to the value $p(s,t) = K_0 - s$ holding in the stopping region.

The American put value function $p(s,t)$ and the exercise boundary $\underline{s}_t$ jointly solve a backward free boundary problem (FBP), consisting of the backward PIDE (6), the smooth fit conditions (4) and (5), and the following boundary conditions:

$$p(s, T_0) = \max[0, K_0 - s], \quad s > 0 \tag{7}$$

$$\lim_{s \uparrow \infty} p(s,t) = 0, \qquad t \in [0, T_0] \tag{8}$$

$$\lim_{s \downarrow 0} p(s,t) = K_0, \qquad t \in [0, T_0]. \tag{9}$$

These Dirichlet conditions force the American put value to its exercise value along the boundaries. As the efficient implementation of a finite difference scheme usually requires the use of

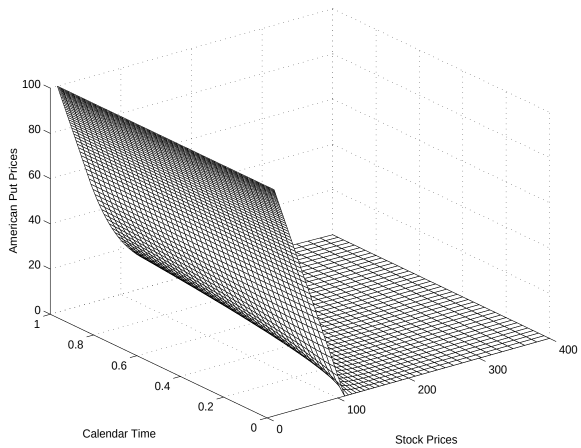


Figure 2. American Put Values in the DEVG Model
American put values are computed from the Variance Gamma model for the following inputs:

$$r = .06, q = .02, \sigma = .4, s = .3, \nu = .25, \theta = -0.3, K_0 = 110, T_0 = 1.$$

The finite difference scheme uses $M = 400$ space steps and $N = 200$ time steps on a domain running from 10 to 400. The value of the American put at the initial stock price of $S_0 = 100$ is \$23.9875.

positive finite spatial boundaries, our implementation replaces the last two conditions in the target problem by:

$$\lim_{s \uparrow \infty} p_{ss}(s,t) = 0, \quad t \in [0, T_0] \tag{10}$$

$$\lim_{s \downarrow 0} p_{ss}(s,t) = 0, \quad t \in [0, T_0]. \tag{11}$$

Hence, the put gamma is forced to zero along the spatial boundaries. Numerical experimentation suggests that imposition of the zero gamma condition on positive finite spatial boundaries tends to work better than imposing the Dirichlet conditions. The solution to this alternative specification is unique under the further condition that it be continuous along the entire boundary. Figure 2 plots American put values in the DEVG model against stock price and time.

III. Stationarity and Domain Extension in the Maturity Direction

The last section assumed that the strike K and maturity T were fixed at K_0 and T_0 respectively. To derive a hybrid FBP for American put values, we first extend the domain of the problem to all $T \in [0, \bar{T}]$, keeping the strike price K fixed at K_0.

Note that the exercise boundary depends on t, $r(t)$, $q(t)$, $\sigma(S,t)$, $\nu(S,x,t)$, T, and K_0, but not on s. Suppressing the dependence on $r(t)$, $q(t)$, $\sigma(S,t)$, $\nu(S,x,t)$, and K_0, let $\underline{s}(t;T)$ be the function relating the exercise surface to t and T:

$$\underline{s}_t = \underline{s}(t;T), \quad t \in [0,T], T \in [0,\bar{T}].$$

The extended continuation region is a 3 dimensional region denoted by Γ. This can be pictured as stacking the two-dimensional continuation regions up the Z axis as T increases from 0. For each $T \in [0, \bar{T}]$, the union of the two dimensional continuation region and the two dimensional stopping region is the plane $S > 0, t \in [0,T]$. As T increases from zero, the area covered by this plane increases. Thus, the extended domain for the backward PIDE is the wedge $S > 0, t \in [0,T], T \in [0,\bar{T}]$. We note that the backward PIDE of the last section holds on this wedge with T_0 replaced by T. Let $\Pi(s,t;T)$ be the function solving this backward PIDE:

$$\begin{aligned}
&\frac{\partial \Pi(s,t;T)}{\partial t} + \frac{\sigma^2(s,t)s^2}{2}\frac{\partial^2 \Pi(s,t;T)}{\partial s^2} + [r(t) - q(t)]s\frac{\partial \Pi(s,t;T)}{\partial s} - r(t)\Pi(s,t;T) \\
&+ \int_{-\infty}^{\infty} \left[\Pi(se^x,t;T) - \Pi(s,t;T) - \frac{\partial}{\partial s}\Pi(s,t;T)s(e^x - 1)\right]\nu(s,x,t)dx \qquad (12) \\
&+ 1(s < \underline{s}(t;T))\left\{r(t)K_0 - q(t)s - \int_{\ln(\underline{s}(t;T)/s)}^{\infty} [\Pi(se^x,t;T) - (K_0 - se^x)]\nu(s,x,t)dx\right\} = 0.
\end{aligned}$$

Now suppose stationarity, i.e. that $r(t)$, $q(t)$, $\sigma(S,t)$, $\nu(S,x,t)$ are all independent of time t. It follows that the time derivative is just the negative of the maturity derivative:

$$\frac{\partial}{\partial t}\Pi(s,t;T) = -\frac{\partial}{\partial T}\Pi(s,t;T). \tag{13}$$

Dropping the dependence of $r(t)$, $q(t)$,$\sigma(S,t)$ and $\nu(S,x,t)$ on t and substituting (13) in (12) implies that the following relation holds in the extended domain:

$$\begin{aligned}
&-\frac{\partial \Pi(s,t;T)}{\partial T} + \frac{\sigma^2(s)s^2}{2}\frac{\partial^2 \Pi(s,t;T)}{\partial s^2} + (r-q)s\frac{\partial \Pi(s,t;T)}{\partial s} - r\Pi(s,t;T) \\
&+ \int_{-\infty}^{\infty} \left[\Pi(se^x,t;T) - \Pi(s,t;T) - \frac{\partial}{\partial s}\Pi(s,t;T)s(e^x-1)\right]\nu(s,x)dx \\
&+1(s<\underline{s}(t;T))\left\{rK_0 - qs - \int_{\ln(\underline{s}(t;T)/s)}^{\infty}[\Pi(se^x,t;T)-(K_0-se^x)]\nu(s,x)dx\right\} = 0.
\end{aligned} \tag{14}$$

We note that one can fix t at t_0 and just solve the above problem in the s,T plane if desired. In this case, the initial condition is:

$$\Pi(s,t_0;t_0) = \max[0, K_0 - s], \quad s > 0. \tag{15}$$

Dirichlet boundary conditions are:

$$\lim_{s\uparrow\infty} \Pi(s,t_0;T) = 0, \quad T \in [t_0, \bar{T}] \tag{16}$$

$$\lim_{s\downarrow 0} \Pi(s,t_0;T) \sim K_0 - s, \quad T \in [t_0, \bar{T}]. \tag{17}$$

Alternatively, these Dirichlet conditions can be replaced by the following zero gamma conditions:

$$\lim_{s\uparrow\infty} \Pi_{ss}(s,t_0;T) = 0, \quad T \in [t_0, \bar{T}] \tag{18}$$

$$\lim_{s\downarrow 0} \Pi_{ss}(s,t_0;T) = 0, \quad T \in [t_0, \bar{T}]. \tag{19}$$

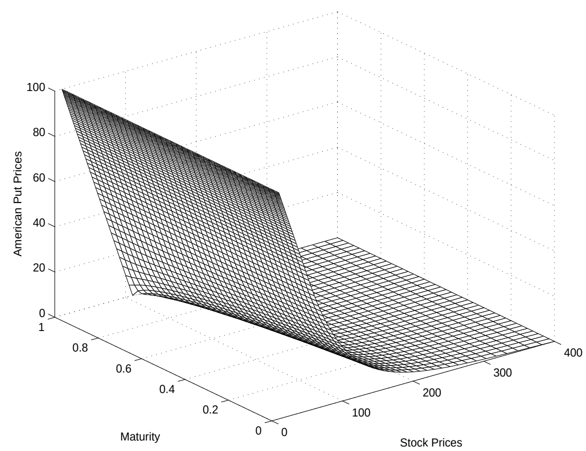


Figure 3. American Put Values in the DEVG Model
American put values are computed from the DEVG model for the following inputs:

$$r = .06, q = .02, \sigma = .4, s = .3, \nu = .25, \theta = -0.3, K_0 = 110, T_0 = 1.$$

The finite difference scheme uses $M = 400$ space steps and $N = 200$ time steps on a domain running from 10 to 400.

The smooth fit conditions are:

$$\lim_{s \downarrow \underline{s}(t_0;T)} \Pi(s, t_0, T) = K_0 - \underline{s}(t_0;T), \quad T \in [t_0, \bar{T}] \tag{20}$$

$$\lim_{s \downarrow \underline{s}(t_0;T)} \frac{\partial \Pi(s, t_0; T)}{\partial s} = -1, \qquad T \in [t_0, \bar{T}]. \tag{21}$$

Figure 3 plots American put values in the DEVG model against stock price and maturity.

IV. Additivity and Domain Extension in the Strike Direction

The last section assumed that the strike K was fixed at K_0 and that $r(t)$, $q(t)$, $\sigma(S,t)$, $\nu(S,x,t)$ are all independent of time t. To derive a new hybrid PIDE for American put values, we further extend the domain of the problem to all $K > 0$. We also restore the dependence on t of $r(t)$,

$q(t)$, $\sigma(S,t)$, and $\nu(S,x,t)$. On this larger domain, let $\underline{s}(t;T,K)$ be the function relating the exercise surface to t, T, and K:

$$\underline{s}_t = \underline{s}(t;T,K), \quad t \in [0,T], T \in [0,\bar{T}], K > 0.$$

We note that the backward PIDE (12) holding on the three dimensional domain of the last section holds on the larger four dimensional domain with K_0 replaced by all $K > 0$. Let $\Pi(s,t;K,T)$ be the function solving this backward PIDE on the extended four dimensional domain:

$$\begin{aligned}
&\frac{\partial \Pi(s,t;K,T)}{\partial t} + \frac{\sigma^2(s,t)s^2}{2}\frac{\partial^2 \Pi(s,t;K,T)}{\partial s^2} + [r(t)-q(t)]s\frac{\partial \Pi(s,t;K,T,)}{\partial s} - r(t)\Pi(s,t;K,T) \\
&+ \int_{-\infty}^{\infty} \left[\Pi(se^x,t;K,T) - \Pi(s,t;K,T) - \frac{\partial}{\partial s}\Pi(s,t;K,T)s(e^x-1)\right]\nu(s,x,t)dx \qquad (22) \\
&+1(s < \underline{s}(t;T,K))\left\{r(t)K - q(t)s - \int_{\ln(\underline{s}(t;T,K)/s)}^{\infty} [\Pi(se^x,t;K,T) - (K-se^x)]\nu(s,x,t)dx\right\} = 0.
\end{aligned}$$

We now assume that the log price process has independent increments i.e. is additive or equivalently that $\sigma(S,t)$ and $\nu(S,x,t)$ are both independent of the stock price S. Then for each fixed t and T, the exercise boundary is a linearly homogeneous function of the strike price:

$$\underline{s}(t;T,\lambda K) = \lambda \underline{s}(t;T,K), \text{ for all } \lambda \geq 0.$$

Setting $\lambda = \frac{1}{K}$ implies that:

$$\underline{s}(t;T,K) = K\underline{s}(t;T,1). \qquad (23)$$

For each fixed s, t, and T, the condition $s > \underline{s}(t;T,K)$ is thus equivalent to the condition $K < \frac{s}{\underline{s}(t;T,1)} = \frac{sK}{\underline{s}(t;T,K)} \equiv \bar{K}(s,t;T)$. We refer to the output of this function as the *critical strike price*. For each fixed s, t, and T, the critical strike price is the lowest strike price K at which the put is exercised early. Note that the critical strike price depends on s but is independent of K. For an American put, the critical strike price is bounded above by s. Also note that the geometric mean of the two critical prices is just the geometric mean of the stock price and strike price:

$$\sqrt{\underline{s}(t;T,K)\bar{K}(s,t;T)} = \sqrt{sK}. \tag{24}$$

The additivity of the log price process implies that the function $\Pi(s,t;K,T)$ is linearly homogeneous in s and K. It follows from Euler's theorem that:

$$\Pi(s,t,K,T) = s\frac{\partial}{\partial s}\Pi(s,t;K,T) + K\frac{\partial}{\partial K}\Pi(s,t;K,T). \tag{25}$$

Differentiation w.r.t. s and K and some obvious algebra establishes that:

$$s^2\frac{\partial^2}{\partial s^2}\Pi(s,t;K,T) = K^2\frac{\partial^2}{\partial K^2}\Pi(s,t;K,T). \tag{26}$$

Dropping the dependence of $\sigma(S,t)$ and $\nu(S,x,t)$ on S and substituting (25) and (26) in (22) implies:

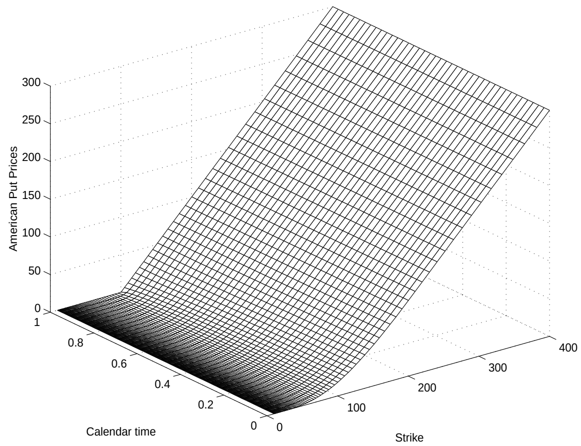
$$\begin{aligned}
&\frac{\partial\Pi(s,t;K,T)}{\partial t} + \frac{\sigma^2(t)K^2}{2}\frac{\partial^2\Pi(s,t;K,T)}{\partial K^2} - [r(t)-q(t)]K\frac{\partial\Pi(s,t;K,T,)}{\partial K} - q(t)\Pi(s,t;K,T)\\
&+\int_{-\infty}^{\infty}\left[\Pi(s,t;Ke^{-x},T) - \Pi(s,t;K,T) - \frac{\partial}{\partial K}\Pi(s,t;K,T)K(e^{-x}-1)\right]e^x\nu(x,t)dx \\
&+1(k>\bar{k}(s,t;T))\left\{r(t)K - q(t)s - \int_{\ln(\bar{k}(s,t;T)/K)}^{\infty}[\Pi(s,t;Ke^{-x},T) - (Ke^{-x}-s)]e^x\nu(x,t)dx\right\} = 0.
\end{aligned} \tag{27}$$

We note that one can fix s and T at say s_0 and T_0 and just solve the above problem in the K,t plane if desired. In this case, the terminal condition is:

$$\Pi(s_0,T_0;K,T_0) = \max[0,K-s_0], \quad K>0. \tag{28}$$

Dirichlet boundary conditions are:

$$\lim_{K\uparrow\infty}\Pi(s_0,t;K,T_0) =\sim K - s_0, \quad t\in[0,T_0] \tag{29}$$



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$$\lim_{K \downarrow 0} \Pi(s_0, t; K, T_0) = 0, \quad t \in [0, T_0]. \tag{30}$$

Alternatively, these Dirichlet conditions can be replaced by:

$$\lim_{K \uparrow \infty} \Pi_{kk}(s_0, t; KT_0) = 0, \quad t \in [0, T_0] \tag{31}$$

$$\lim_{K \downarrow 0} \Pi_{kk}(s_0, t; K, T_0) = 0, \quad t \in [0, T_0]. \tag{32}$$

The smooth fit conditions are:

$$\lim_{K \uparrow \bar{K}(s,t;T_0)} \Pi(s_0, t; K, T_0) = \bar{K}(s_0, t; T_0) - s_0, \quad t \in [0, T_0] \tag{33}$$

$$\lim_{K \uparrow \bar{K}(s,t;T_0)} \frac{\partial \Pi(s_0, t; K, T_0)}{\partial K} = 1, \qquad t \in [0, T_0]. \tag{34}$$

Figure 4 plots American put values in the DEVG model against strike price and calendar time.

We note that setting jumps to zero reduces the PIDE to a PDE arising in the special case of the time-dependent Black Scholes model. If one wishes to value American options in this model for multiple strikes and maturities and with fixed time and spot, it is much more efficient to solve the hybrid problem of this section once for each T than it is to solve the usual backward problem once for each K and once for each T as is usually done.

V. The Forward Free Boundary Problem

We now assume that we have both stationarity and additivity. In other words, the log price is a Lévy process and $r(t)$, $q(t)$, $\sigma(S,t)$, $\nu(S,x,t)$ are all independent of both time t and the stock price S. Stationarity implies that the function $\Pi(s,t;K,T)$ depends on t and T only through $T-t$. It thus follows that:

$$\frac{\partial}{\partial t}\Pi(s,t;K,T) = -\frac{\partial}{\partial T}\Pi(s,t;K,T). \tag{35}$$

Substituting (35) in (27) implies:

$$\begin{aligned}
&-\frac{\partial \Pi(s,t;K,T)}{\partial T} + \frac{\sigma^2 K^2}{2}\frac{\partial^2 \Pi(s,t;K,T)}{\partial K^2} - (r-q)K\frac{\partial \Pi(s,t;K,T,)}{\partial K} - q\Pi(s,t;K,T) \\
&+\int_{-\infty}^{\infty}\left[\Pi(s,t;Ke^{-x},T) - \Pi(s,t;K,T) - \frac{\partial}{\partial K}\Pi(s,t;K,T)K(e^{-x}-1)\right]e^{x}\nu(x)dx \\
&+1(k>\bar{k}(s,t;T))\left\{rK - qs - \int_{\ln(\bar{k}(s,t;T)/K)}^{\infty}[\Pi(s,t;Ke^{-x},T) - (Ke^{-x}-s)]e^{x}\nu(x)dx\right\} = 0.
\end{aligned} \tag{36}$$

We note that one can fix s and t at say s_0 and t_0 and just solve the above problem in the K,T plane if desired. In this case, the initial condition is:

$$\Pi(s_0,t_0;K,t_0) = \max[0, K-s_0], \quad K>0. \tag{37}$$

Dirichlet boundary conditions are:

$$\lim_{K\uparrow\infty} \Pi(s_0, t_0; K, T) \sim K - S_0, \quad T \in [t_0, \bar{T}] \tag{38}$$

$$\lim_{K\downarrow 0} \Pi(s_0, t_0; K, T) = 0, \quad T \in [t_0, \bar{T}]. \tag{39}$$

Alternatively, these Dirichlet conditions can be replaced by:

$$\lim_{K\uparrow\infty} \Pi_{kk}(s_0, t_0; K, T) = 0, \quad T \in [t_0, \bar{T}] \tag{40}$$

$$\lim_{K\downarrow 0} \Pi_{kk}(s_0, t_0; K, T) = 0, \quad T \in [t_0, \bar{T}]. \tag{41}$$

The smooth fit conditions are:

$$\lim_{K\uparrow\bar{K}(s,t_0;T)} \Pi(s_0, t_0; K, T) = \bar{K}(s_0, t_0; T) - s_0, \quad T \in [t_0, \bar{T}] \tag{42}$$

$$\lim_{K\uparrow\bar{K}(s,t_0;T)} \frac{\partial\Pi(s_0,t_0;K,T)}{\partial K} = 1, \qquad T \in [t_0, \bar{T}]. \tag{43}$$

Figure 5 plots American put values in the DEVG model against strike price and maturity. The value of the American put at the initial stock price of $S_0 = 100$ is \$23.9875 from the backward problem and \$23.9785 from the forward problem. The small difference is due to numerical error since the difference gets even smaller as we increase the number of time and spatial steps. Figure 6 plots critical strike prices against maturity using the same inputs.

VI. Summary and Future Research

We first reviewed the backward PIDE governing the arbitrage-free price of an American put option when the underlying spot price process is Markov in itself. By imposing various restrictions on the process, we then derived three new PIDE's for American put values. In particular, by assuming stationarity, we derived a forward PIDE in maturities with spot price still an independent variable. By alternatively assuming that the evolution coefficients for the

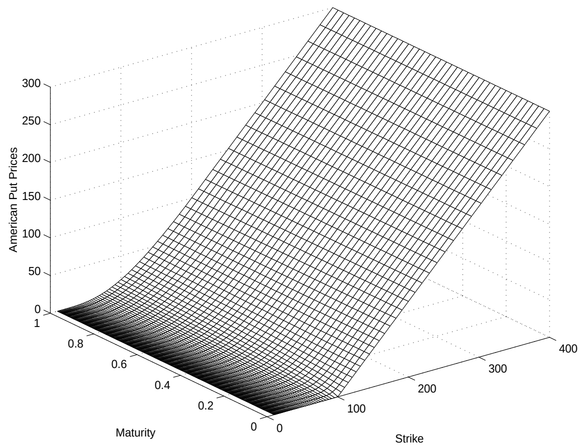

Figure 5. American Put Values in the DEVG Model

American put values are computed from the DEVG model for the following inputs:

$$r = .06, q = .02, \sigma = .4, s = .3, \nu = .25, \theta = -0.3, K_0 = 110, T_0 = 1.$$

The finite difference scheme uses $M = 400$ space steps and $N = 200$ time steps on a domain running from 10 to 400. The value of the American put at the initial stock price of $S_0 = 100$ is \$23.9785.

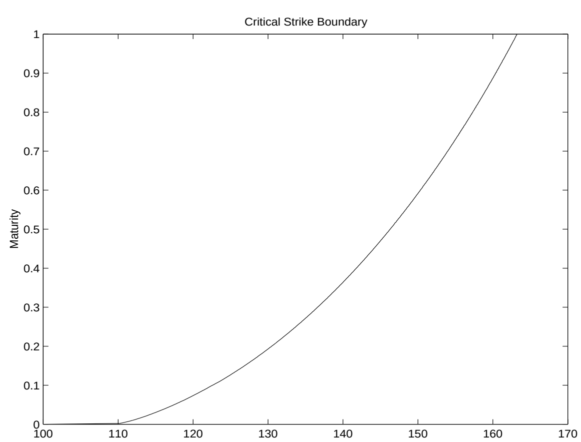

Figure 6. Critical Strike Prices in the DEVG Model

proportional process are independent of spot, we derived a backward PIDE with the strike price as an independent variable. Finally, by assuming that the log price of the underlying is a Lévy process, we derived the forward PIDE for arbitrage-free American put values. We numerically solved this forward PIDE for the case of the diffusion extended VG model and found very close agreement to the numerical soltution of the backward PIDE. A longer version of this paper downloadable from www.math.nyu.edu/research/carrp/papers/pdf contains an appendix detailing the finite difference scheme used to numerically solve the forward PIDE for American put options.

It is clear how to apply our analysis to American calls or more generally to payoffs which are both monotone and linearly homogeneous in spot and strike. It should be possible to extend our analysis to barrier options in which the payoff is linearly homogeneous in some subset of spot, strike, barrier, or rebate. An open problem is the forward equation for American options when the evolution parameters depend on stock price and/or time. It would also be interesting to extend our univariate approach to additional state variables besides the stock price. If the extra state variable is another asset price, then bivariate American options could be handled. If the extra state variable is a path statistic, then many path-dependent options could be handled. If the extra state variable is the current level of a randomly evolving volatility process, then our approach would encompass stochastic volatility and GARCH models for which there is considerable empirical support. In the interests of brevity, we defer this research to future work.

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Appendix: Discretization of Forward Equation for American Options

This appendix shows how finite differences can be used to numerically solve the following forward PIDE governing American put values:

$$\frac{\partial P(s,t;K,T)}{\partial T} - \frac{\sigma^2}{2}K^2\frac{\partial^2 P(s,t;K,T)}{\partial K^2} + (r-q)K\frac{\partial P(s,t;K,T)}{\partial K} + qP(s,t;K,T) \tag{44}$$

$$-\int_{-\infty}^{+\infty}\left[P(s,t;Ke^{-y},T) - P(s,t;K,T) - \frac{\partial P(s,t;K,T)}{\partial K}K(e^{-y}-1)\right]e^y\nu(y)dy \tag{45}$$

$$-\mathbf{1}_{K>\bar{K}(s,t;T)}\left\{rK - qs - \int_{\ln(K/\bar{K}(s,t;T))}^{\infty}\left[P(s,t;Ke^{-y},T) - (Ke^{-y}-s)\right]e^y\nu(y)dy\right\} = 0 \tag{46}$$

We illustrate the solution in the diffusion extended VG model for which the Lévy density has the form:

$$\nu(y)dy = \frac{\exp(\theta y/\sigma^2)}{\nu\,|y|}\exp\left(-\frac{\sqrt{\frac{2}{\nu}+\frac{\theta^2}{\sigma^2}}}{\sigma}|y|\right). \tag{47}$$

Notice that this Lévy density explodes as y approaches zero from either direction. As a result, special measures will have to be taken when approximating the integral containing this Lévy density. One can show that:

$$\int_{-\infty}^{+\infty}(e^{-y}-1)e^y\nu(y)dy = \omega$$

where:

$$\omega \equiv \frac{1}{\nu}\ln(1-\theta\nu-\sigma^2\nu/2). \tag{48}$$

Dropping the arguments s and t to simplify notation, we can re-write (46) as:

$$\frac{\partial P(K,T)}{\partial T} - \frac{\sigma^2}{2}K^2\frac{\partial^2 P(K,T)}{\partial K^2} + (r-q+\omega)K\frac{\partial P(K,T)}{\partial K} + qP(K,T)$$
$$-\int_{-\infty}^{+\infty}\big((P(Ke^{-y},T) - P(K,T)\big)e^y\nu(y)dy$$
$$-\mathbf{1}_{K>\bar{K}(T)}\left\{rK - qs - \int_{\ln(K/\bar{K}(T))}^{\infty}\left[P(Ke^{-y},T) - (Ke^{-y}-s)\right]e^y\nu(y)dy\right\} = 0$$

By making the change of variable $x = \ln K$ we have

$$\begin{aligned}
p(x,T) &= P(K,T), \\
\frac{\partial p}{\partial x}(x,T) &= K\frac{\partial P}{\partial K}(K,T), \\
\frac{\partial^2 p}{\partial x^2}(x,T) - \frac{\partial p}{\partial x}(x,T) &= K^2\frac{\partial^2 P}{\partial K^2}(K,T), \\
p(x-y,T) &= P(Ke^{-y},T),
\end{aligned}$$

and hence we obtain the following PIDE for $p(x,T)$,

$$\begin{aligned}
\frac{\partial p}{\partial T}(x,T) - \frac{\sigma^2}{2}\frac{\partial^2 p(x,T)}{\partial x^2} + (r - q + \frac{\sigma^2}{2} + \omega)\frac{\partial p}{\partial x}(x,T) + qp(x,T) \\
- \int_{-\infty}^{+\infty} (p(x-y,T) - p(x,T))\,\tilde{\nu}(y)dy \\
-\mathbf{1}_{x>\bar{x}(T)}\left\{re^x - qs - \int_{x-\bar{x}(T)}^{\infty} \big(p(x-y,T) - (e^{x-y} - s)\big)\,\tilde{\nu}(y)dy\right\} &= 0,
\end{aligned}$$

where:

$$\begin{aligned}
\tilde{\nu}(y) &= \frac{e^{-\tilde{\lambda}_p y}}{\nu y}\mathbf{1}_{y>0} + \frac{e^{-\tilde{\lambda}_n |y|}}{\nu|y|}\mathbf{1}_{y<0}, \\
\tilde{\lambda}_p &= \left(\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}\right)^{\frac{1}{2}} - \frac{\theta}{\sigma^2} - 1, \\
\tilde{\lambda}_n &= \left(\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}\right)^{\frac{1}{2}} + \frac{\theta}{\sigma^2} + 1.
\end{aligned}$$

This PIDE is solved subject to the initial condition

$$p(x,0) = (e^x - s)^+, \tag{49}$$

and the (zero gamma) boundary conditions

$$\frac{\partial^2 p}{\partial x^2}(-\infty,T) - \frac{\partial p}{\partial x}(-\infty,T) = 0 \;\; \forall T, \tag{50}$$

$$\frac{\partial^2 p}{\partial x^2}(+\infty,T) - \frac{\partial p}{\partial x}(+\infty,T) = 0 \;\; \forall T. \tag{51}$$

Discretization of PIDE

In our finite difference discretization, we adopt a mixed approach. For the jump terms, we use an explicit approach so that the matrix to be inverted at each time step is tri-diagonal. To evaluate the integrals, we apply an analytical approach to handle the singularity at zero. On the rest of the PIDE, a fully implicit approach is used. We consider M equally spaced sub-intervals in T-direction. For the x-direction, we assume N equally spaced sub-intervals on $[x_{\text{min}}, x_{\text{max}}]$. Thus, we have the following mesh on $[x_{\text{min}}, x_{\text{max}}] \times [0, \bar{T}]$

$$\begin{aligned} D = \big\{ &(x_i, T_j) \in \mathbb{R}^+ \times \mathbb{R}^+ \,\big| x_i = (x_{\text{min}} + i\Delta x, i = 0, 1, \ldots, N, \\ &T_j = j\Delta T, j = 0, 1, \ldots, M, \ \Delta x = (x_{\text{min}} - x_{\text{max}})/N, \Delta T = \bar{T}/M \big\}. \end{aligned}$$

Let $p_{i,j}$ be the finite difference approximation of the values of $p(x_i, T_j)$ on D. We obtain the following difference equation at point (x_i, T_{j+1})

$$\begin{aligned} &\frac{1}{\Delta T}(p_{i,j+1} - p_{i,j}) - \frac{\sigma^2}{2} \frac{p_{i+1,j+1} - 2p_{i,j+1} + p_{i-1,j+1}}{\Delta x^2} + q p_{i,j+1} \\ &+ (r - q + \frac{\sigma^2}{2} + \omega) \frac{1}{2\Delta x}(p_{i+1,j+1} - p_{i-1,j+1}) - \int_{-\infty}^{+\infty} (p(x_i - y, T_j) - p(x_i, T_j)) \tilde{\nu}(y) dy \\ &- \mathbf{1}_{x_i > x(T_j)} \left\{ rK - q e^{x_i} - \int_{x_i - x(T_j)}^{\infty} \left[p(x_i - y, T_j) - (K - e^{x_i + y}) \right] \tilde{\nu}(y) dy \right\} = 0. \end{aligned}$$

Equivalently, we have:

$$\begin{aligned} &(-B - A) p_{i-1,j+1} + (1 + 2B + q\Delta T) p_{i,j+1} + (-B + A) p_{i+1,j+1} = \\ &\quad p_{i,j} + \Delta T \int_{-\infty}^{+\infty} (p(x_i - y, T_j) - p_{i,j})) \tilde{\nu}(y) dy + \\ &\quad \Delta T \times \mathbf{1}_{x_i > x(T_j)} \left\{ r e^{x_i} - q s - \int_{x_i - x(T_j)}^{\infty} \left[p(x_i - y, T_j) - (e^{x_i - y} - s) \right] \tilde{\nu}(y) dy \right\}, \qquad (52) \end{aligned}$$

where:

$$\begin{aligned} A &= (r - q + \frac{\sigma^2}{2} + \omega)\frac{\Delta T}{2\Delta x}, \\ B &= \frac{\sigma^2}{2}\frac{\Delta T}{\Delta x^2}, \end{aligned}$$

$$\begin{aligned} p_{i,0} &= (e^{x_i} - s)^+, \\ x(T_0) &= \ln s, \end{aligned}$$

and

$$x(T_j) = \min_{x_i}\{x_i : p_{i,j} - (e^{x_i} - s)^+ < 0\} \text{ for } j = 1, \ldots, M.$$

For the first integral on the RHS of (52), we decompose the range of integration into 6 parts:

$$\begin{aligned} \int_{-\infty}^{+\infty} (p(x_i - y, T_j) - p_{i,j})\,\tilde{\nu}(y)dy &= \int_{-\infty}^{x_i - x_N} (p(x_i - y, T_j) - p_{i,j})\,\tilde{\nu}(y)dy \\ &+ \int_{x_i - x_N}^{-\Delta x} (p(x_i - y, T_j) - p_{i,j})\,\tilde{\nu}(y)dy \\ &+ \int_{-\Delta x}^{0} (p(x_i - y, T_j) - p_{i,j})\,\tilde{\nu}(y)dy \\ &+ \int_{0}^{+\Delta x} (p(x_i - y, T_j) - p_{i,j})\,\tilde{\nu}(y)dy \\ &+ \int_{+\Delta x}^{x_i - x_0} (p(x_i - y, T_j) - p_{i,j})\,\tilde{\nu}(y)dy \\ &+ \int_{x_i - x_0}^{+\infty} (p(x_i - y, T_j) - p_{i,j})\,\tilde{\nu}(y)dy \end{aligned}$$

The six integrals are evaluated as:

$$\int_{-\Delta x}^{0} (p(x_i - y, T_j) - p_{i,j})\,\tilde{\nu}(y)dy \cong \frac{1}{\nu \Delta x \tilde{\lambda}_n}(1 - e^{-\tilde{\lambda}_n \Delta x})(p_{i+1,j} - p_{i,j}),$$

$$\int_0^{\Delta x} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y) dy \cong \frac{1}{\nu \Delta x \tilde{\lambda}_p}(1 - e^{-\tilde{\lambda}_p \Delta x})(p_{i-1,j} - p_{i,j}).$$

$$\begin{aligned}
&\int_{x_i - x_N}^{-\Delta x} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y) dy \\
&= \frac{1}{\nu} \sum_{k=1}^{N-i-1} \left(p_{i+k,j} - p_{i,j} - k(p_{i+k+1,j} - p_{i+k,j})\right) \left\{\text{expint}(k \Delta x \tilde{\lambda}_n) - \text{expint}((k+1)\Delta x \tilde{\lambda}_n)\right\} \\
&+ \frac{1}{\tilde{\lambda}_n \nu \Delta x} \sum_{k=1}^{N-i-1} \left(p_{i+k+1,j} - p_{i+k,j}\right) \left(e^{-\tilde{\lambda}_n k \Delta x} - e^{-\tilde{\lambda}_n (k+1) \Delta x}\right)
\end{aligned}$$

where:

$$\text{expint}(x) \equiv \int_x^{\infty} \frac{e^{-t}}{t} dt \tag{53}$$

is the exponential integral.

$$\begin{aligned}
&\int_{\Delta x}^{x_i - x_0} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y) dy \\
&= \sum_{k=1}^{i-1} \frac{1}{\nu} \left(p_{i-k,j} - p_{i,j} - k(p_{i-k-1,j} - p_{i-k,j})\right) \left\{\text{expint}(k \Delta x \lambda_p) - \text{expint}((k+1)\Delta x \lambda_p)\right\} \\
&+ \sum_{k=1}^{i-1} \frac{p_{i-k-1,j} - p_{i-k,j}}{\lambda_p \nu \Delta x} \left(e^{-\lambda_p k \Delta x} - e^{-\lambda_p (k+1) \Delta x}\right).
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{x_i - x_N} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y) dy &= \frac{e^{x_i}}{\nu} \text{expint}((N-i)\Delta x(\tilde{\lambda}_n - 1)) \\
&- \frac{s + p_{i,j}}{\nu} \text{expint}((N-i)\Delta x \tilde{\lambda}_n).
\end{aligned}$$

$$\int_{x_i - x_0}^{\infty} \left(p(x_i - y, T_j) - p_{i,j}\right) \tilde{\nu}(y) dy = -\frac{1}{\nu} p_{i,j} \text{expint}(i \Delta x \lambda_p).$$

The integral inside the Heaviside term in (52) is treated in the same manner as the other integral. Therefore, we have:

$$
\begin{aligned}
&\int_{x_i - x(T_j)}^{\infty} \left[p(x_i - y, T_j) - (e^{x_i - y} - s)\right] \tilde{\nu}(y) dy \\
&= \frac{1}{\nu} \sum_{k=i-l}^{i-1} (p_{i-k,j} - k(p_{i-k-1,j} - p_{i-k,j})) \left(\text{expint}(k\Delta x \tilde{\lambda}_p) - \text{expint}((k+1)\Delta x \tilde{\lambda}_p)\right) \\
&+ \sum_{k=i-l}^{i-1} \frac{p_{i-k-1,j} - p_{i-k,j}}{\tilde{\lambda}_p \nu \Delta x} \left(e^{-\tilde{\lambda}_p k \Delta x} - e^{-\tilde{\lambda}_p (k+1)\Delta x}\right) \\
&- \frac{1}{\nu} \left\{ e^{x_i} \text{expint}((i-l)\Delta x(\tilde{\lambda}_p + 1)) - s \ \text{expint}((i-l)\Delta x \tilde{\lambda}_p) \right\}
\end{aligned}
$$

A. Difference Equation

Putting all of the pieces together, we obtain the following difference equation at the point (x_i, T_{j+1})

$$
E p_{i-1,j+1} + F p_{i,j+1} + G p_{i+1,j+1} = p_{i,j} + \frac{\Delta T}{\nu} R_{i,j} + \Delta T \mathbf{1}_{x_i > x(T_j)} H_{i,j} \tag{54}
$$

where

$$
\begin{aligned}
E &= -A - B - B_p, \\
F &= 1 + q\Delta T + 2B + B_n + B_p + \frac{\Delta T}{\nu} \left(\text{expint}(i\Delta x \tilde{\lambda}_p) + \text{expint}((N-i)\Delta x \tilde{\lambda}_n)\right), \\
G &= A - B - B_n,
\end{aligned}
$$

$$
\begin{aligned}
R_{i,j} &= \sum_{k=1}^{N-i-1} (p_{i+k,j} - p_{i,j} - k(p_{i+k+1,j} - p_{i+k,j})) \left\{\text{expint}(k\Delta x \lambda_n) - \text{expint}((k+1)\Delta x \lambda_n)\right\} \\
&+ \sum_{k=1}^{N-i-1} \frac{p_{i+k+1,j} - p_{i+k,j}}{\lambda_n \Delta x} \left(e^{-\tilde{\lambda}_n k \Delta x} - e^{-\tilde{\lambda}_n (k+1)\Delta x}\right) \\
&+ \sum_{k=1}^{i-1} (p_{i-k,j} - p_{i,j} - k(p_{i-k-1,j} - p_{i-k,j})) \left\{\text{expint}(k\Delta x \lambda_p) - \text{expint}((k+1)\Delta x \lambda_p)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{k=1}^{i-1} \frac{p_{i-k-1,j} - p_{i-k,j}}{\lambda_p \Delta x} \left(e^{-\tilde{\lambda}_p k \Delta x} - e^{-\tilde{\lambda}_p (k+1) \Delta x} \right) \\
&+ e^{x_i} \text{expint}((N-i)\Delta x(\lambda_n - 1)) - s \ \text{expint}((N-i)\Delta x \lambda_n),
\end{aligned}
$$

$$
\begin{aligned}
H_{i,j} &= re^{x_i} - qs \\
&- \sum_{k=i-l}^{i-1} \frac{1}{\nu} \left(p_{i-k,j} - k(p_{i-k-1,j} - p_{i-k,j}) \right) \left(\text{expint}(k \Delta x \lambda_p) - \text{expint}((k+1)\Delta x \lambda_p) \right) \\
&- \sum_{k=i-l}^{i-1} \frac{p_{i-k-1,j} - p_{i-k,j}}{\lambda_p \nu \Delta x} \left(e^{-\tilde{\lambda}_p k \Delta x} - e^{-\tilde{\lambda}_p (k+1) \Delta x} \right) \\
&+ \frac{1}{\nu} \left\{ e^{x_i} \text{expint}((i-l)\Delta x(\lambda_p + 1)) - s \ \text{expint}((i-l)\Delta x \lambda_p) \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
B_n &= \frac{\Delta T}{\nu \Delta x \tilde{\lambda}_n} \left(1 - e^{-\tilde{\lambda}_n \Delta x} \right), \\
B_p &= \frac{\Delta T}{\nu \Delta x \tilde{\lambda}_p} \left(1 - e^{-\tilde{\lambda}_p \Delta x} \right).
\end{aligned}
$$

The initial condition (49) and boundary conditions (50) and (51) are discretized in the usual manner. A standard finite difference solver can then be used to solve the boundary value problem.