

Put Call Reversal

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Assuming that the stock price process is a jump diffusion, we derive a new relation between puts and calls termed Put Call Reversal (PCR). We show how PCR gives simple new probabilistic interpretations of deltas and gammas. We also show how PCR simplifies semi-static hedging of long dated options.

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Put Call Reversal

I Introduction

There are many known results which relate put and call values to each other. These results differ in terms of the generality of the model for the underlying dynamics and in terms of whether they apply to both European and American options. For example, put call parity (PCP) implies that a put and a call have the same time value. As is well known, PCP is a completely model-free result governing European option values, but not American ones (see Merton[18]). Put Call Equivalence (PCE) implies that a put on one currency is a call on the other (Grabbe[14]). By considering a different currency more generally as a change of numeraire, PCE can be extended to options on many different underlyings (Schroder[22]). In contrast to PCP, PCE governs both European and American options (DeTemple[11]). PCE is almost model-free, but it does assume that the numeraire is never worth zero. Put Call Symmetry (PCS) is a result that relates an out-of-the-money put on one asset to an out-of-the-money call on the same asset (Bates[3]). In contrast to PCP, PCS applies to both European and American options and in contrast to PCE, PCS allows zero or negative prices for the underlying assets. However, it is much more model-dependent than PCP and PCE in that it relies on a distributional symmetry for some function of the underlying price under some risk-neutral measure. Nonetheless, PCS has unique uses as a gauge of the asymmetry of the risk-neutral distribution (Bates[4]) and as a technique for semi-static hedging of certain path-dependent options (Carr, Ellis, and Gupta[8]). Another result that apparently relies on distributional restrictions is Put Call Duality (PCD) (see Andreasen and Gurenwald[1], Peskir and Shiryaev[20], and Haug[15]). In the constant parameter Black Scholes model, PCD states that the value of a put can be obtained from a formula or algorithm for valuing a call by negating the stock price, strike price, and volatility. Peskir and Shiryaev prove that PCD holds for both European and American options in the Black Scholes model. Andreasen and Gruenewald show that PCD also holds for both types of options under the more general Merton[19] jump-diffusion model.

To this panoply of results relating put and call values to each other, this paper adds one more. We derive a new result termed Put Call Reversal¹ (PCR) which relates the value of a European call to the value of a European put written on a price process which runs backward in time. The result is model-dependent and assumes that the usual forward-running stock price process S is a jump diffusion, where lognormal volatility is bounded and where the jump part of the returns process has independent increments. We also assume that the riskfree rate and dividend yield are deterministic. Let $C(t_0, S_0; T, K)$ be the value at time t_0 given that $S_{t_0} = S_0 \geq 0$ of a European call written on process S with maturity $T \geq t_0$ and strike $K \geq 0$. Consider some economy in which time runs backwards. Since the forward running process S is Markov in itself, the backward running process \hat{S} is Markov in itself. In the backward economy, let $\hat{P}(\hat{t}_0, \hat{S}_0; \hat{T}, \hat{K})$ be the value at the reverse time \hat{t}_0 given $\hat{S}_{\hat{t}_0} = \hat{S}_0 \geq 0$ of a European put written on the process \hat{S} with maturity $\hat{T} \leq \hat{t}$ and strike $\hat{K} \geq 0$. Then PCR is:

$$C(t_0, S_0; T, K) = \hat{P}(T, K; t_0, S_0), \quad S_0, K \in \mathfrak{R}^+, T \geq t_0. \quad (1)$$

In words, at $t = t_0$, a call on the forward running process S started at $S_{t_0} = S_0$ with maturity T and strike K has the same value at the reverse time $\hat{t} = T$ as a put on a backward running process \hat{S} started at $\hat{S}_{\hat{t}_0} = K$, where the put matures at $\hat{t} = t_0$ and has strike $\hat{K} = S_0$. The valuation date for the put is the maturity date of the call and vice versa. Hence, the two options have the same time to maturity on their respective valuation dates. Note that the call's strike price is the price for the put's underlying on the put's valuation date. Analogously, the price of the call's underlying on the call's valuation date is the put's strike price. There is of course a parallel result showing that the put in the forward economy has the same value as a call in the backward economy. Thus, we may paraphrase PCR as the financial equivalent of Newton's third law: for every option, there is an equal and opposite option.

The intuition behind PCR comes from thinking of the backward variables t and S as forward variables in the reverse time economy. So long as an option's gamma is non-negative and integrable in S , the function relating gamma to t and S can be regarded as the function relating a state pricing density to its

¹When a put call parity violation is arbitrated away, the trade is said to be a conversion (from call to put) or a reversal (from put to call). The Put Call Reversal discussed in this paper is based on time reversal and should not be confused with parity reversal.

forward variables in the reverse time economy. Now, by the sifting property of Dirac's delta function², we have:

$$C(t_0, S_0; T, K) = \int_0^\infty C(t_0, S; T, K) \delta(S - S_0) dS. \quad (2)$$

Appendix 1 shows that integrating (2) by parts twice gives:

$$C(t_0, S_0; T, K) = \int_0^\infty \frac{\partial^2}{\partial S^2} C(t_0, S; T, K) (S_0 - S)^+ dS, \quad S_0, K \geq 0, T \geq t, \quad (3)$$

since $C(t_0, 0; T, K) = C_s(t_0, 0; T, K) = 0$. Suppose that we can interpret $C_{ss}(t_0, S; T, K)$ as the price of an Arrow Debreu security in a reverse time economy. This butterfly spread has maturity t_0 and strike S , and is written on an asset which at the initial time T has initial price K . We can then interpret the right hand side of (3) as the pricing equation for a *put* maturing at t_0 and struck at S_0 . The put is written on some backward running process which is conditioned to start at time T at the level K .

In order to interpret $C_{ss}(t, S; T, K)$ as the price of an Arrow Debreu security in a reverse time economy, it must be non-negative and integrable in S . Recall that we assume that the usual forward-running stock price process is a jump diffusion, where the lognormal volatility is bounded and the jump part of the returns process has independent increments. For this process, the origin is a natural boundary, so that a process started at zero remains at zero and a process started at a nonnegative level remains nonnegative. By twice differentiating the backward boundary value problem governing call values, it is not hard to show that gamma is nonnegative for such a process. By the fundamental theorem of calculus, the integral of $C_{ss}(t, S; T, K)$ in S is given by:

$$\int_0^\infty C_{ss}(t, S; T, K) dS = C_s(t, \infty; T, K) - C_s(t, 0; T, K) = e^{-\int_t^T q(u) du}, \quad (4)$$

since $C(t, \infty; T, K) \sim S e^{-\int_t^T q(u) du}$ and $C(t, 0; T, K) = 0$. Hence, we can define a reverse time transition density as:

$$\hat{q}(T, K; t, S) \equiv e^{\int_t^T q(u) du} C_{ss}(t, S; T, K). \quad (5)$$

²The Dirac delta function is a generalized function characterized by two properties:

1. $\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$
2. $\int_{-\infty}^\infty \delta(x) dx = 1$.

From (4), $\int_0^\infty \hat{q}(T, K; t, S) dS = 1$, so t and S are the forward variables in this reverse time economy. The relevant discount rate in this economy is q , the dividend yield of the forward running stock price process.

To obtain the initial forward price in the reverse time economy, we may future value the price of a zero strike call on the backward running process at the appropriate riskfree rate:

$$\hat{E}[\hat{S}_{t_0} | \hat{S}_T = K] = e^{\int_{t_0}^T q(u) du} \hat{C}(T, K; 0, t_0).$$

By call put reversal, a zero strike call in the reverse time economy has the same value as a put on a bankrupt stock in the forward time economy. Since the put is certain to finish in-the-money, we have:

$$\hat{E}[\hat{S}_{t_0} | \hat{S}_T = K] = K e^{\int_{t_0}^T [q(u) - r(u)] du}.$$

Thus, the dividend yield on the backward running process is just the riskfree rate in the forward economy.

Summarizing to this point, the call value can be obtained from the put value by interchanging spot and strike, and by further interchanging dividend yield and the riskfree rate. While these interchanges also describe PCE, these put call relations diverge when we examine the volatility structure. To obtain the volatility structure of the backward running process, we may re-interpret the usual backward boundary value problem for the call on the usual forward running process. By interpreting this problem as a forward boundary value problem for the put on the backward running process, one can relate the diffusion coefficient and jump arrival rates of the backward running process to those of the forward running process. Alternatively, we can derive the lesser known forward boundary value problem for the call on the usual forward running process and interpret it as a backward boundary value problem for the put on the backward running process. The body of this paper takes the latter route and finds that PCR and PCE are distinct.

The distinction is most transparent when there are no jumps and when the lognormal volatility is a function of the price level and time. To convert a put's payoff to that of a call, PCE changes the numeraire for all assets which necessitates a change in risk-neutral probability measure. It is well known from Girsanov's theorem that the volatility process is invariant to this change of measure. Since the random variable underlying the put's payoff is reciprocated to determine the random variable underlying

the call's payoff, the functional relationship between the lognormal volatility of the random underlying and the price level of the random underlying must change in order to keep the lognormal volatility *process* invariant. Thus for PCE, one uses a different volatility function for the call than one uses for the put. While the volatility function changes as a function of the underlying, it does not change as a function of time, since PCE does not alter the time variable. In contrast to PCE, the functional relationship between the lognormal volatility and the time does change when we do a time reversal in PCR. Since we negate the time variable, we must change the functional relationship between the absolute value of volatility and time in order to preserve the absolute level of volatility. In further contrast to PCE, when regarded as a function of price, the lognormal volatility function does not change when we go from valuing a call to valuing a put. There are other differences between PCR and PCE as well. For example, in this paper we will only prove PCR for European options written on a restricted Markov process, while PCE is known to hold for both European and American options written on any non-negative process. In fact, we have tested whether our PCR also holds for American options and have found that it does not hold in general. When parameters are constant and lognormal volatility is independent of level, then PCR reduces to PCE and hence it holds for American options in this special case. Notwithstanding the restrictions on PCR, we will show that PCR has many applications for pricing and hedging in the context in which it holds.

An overview of this paper is as follows. The next section relates the assumptions underlying PCR. The following section derives the backward and forward boundary value problems governing option values. The fourth section uses the forward boundary value problem to prove PCR formally. The fifth section indicates applications of PCR to interpreting delta and gamma and to semi-static hedging. The final section summarizes and suggests further research.

II Notation and Assumptions

We assume the standard model of perfect capital markets, continuous trading, and no arbitrage opportunities. We assume that the asset underlying an option is a stock which has a nonnegative deterministic

dividend yield $q(t) \geq 0$. We assume that there is a nonnegative deterministic spot rate $r(t) \geq 0$. When a pure discount bond is used as numeraire, then it is well known that no arbitrage implies that there exists a probability measure Q under which all non-dividend paying asset prices are martingales. On the time set $[0, \bar{T}]$ and on a complete probability space (Ω, \mathcal{F}, Q) , we assume that a stock price S_t obeys the following stochastic differential equation (SDE):

$$\frac{dS_t}{S_{t-}} = [r(t) - q(t)]dt + \sigma(S_{t-}, t)dW_t + \int_{-\infty}^{\infty} (e^x - 1)[\mu(dx, dt) - \nu(x, t)dxdt], \quad \text{for all } t \in [0, \bar{T}], \quad (6)$$

where the initial stock price is known. The process is fully determined by the relative drift function $b(S, t) \equiv r(t) - q(t)$, the volatility function $\sigma(S, t)$, and the jump compensation function $\nu(x, t)$. Since these coefficients depend on at most time and the stock price, the process is Markov in itself. Although the SDE is expressed in proportional form, the dependence of the lognormal volatility σ on the stock price implies that the price process is not in general proportional.

The term dW_t in (6) denotes increments of a standard Brownian motion (SBM) W . The random measure $\mu(dx, dt)$ counts the number of jumps of size x in the log price at time t . The function $\{\nu(x, t), x \in \mathfrak{R}, t \in [0, \bar{T}]\}$ is used to compensate the jump process $J_t \equiv \int_0^t \int_{-\infty}^{\infty} S_{t-}(e^x - 1)\mu(dx, ds)$, so that the last term in (6) is the increment of a Q jump martingale.³ Since the Lévy density $\nu(x, t)$ is independent of S , the jump part of the returns process is an additive process. We bound the lognormal volatility function $\sigma(S, t)$ so that the origin serves as a natural boundary for the spot price process. Hence, (6) describes a continuous-time Markov model for the spot price dynamics, which is both arbitrage-free and consistent with limited liability. Aside from the Markov property, the main restrictions inherent in (6) are the standard assumptions that interest rates, dividend yields, and the compensator do not depend on the spot price.

³The function $\nu(x, t)$ must have the following properties:

$$\nu(0, t) = 0, \quad \int_{-\infty}^{\infty} (x^2 \wedge 1)\nu(x, t)dx < \infty, \quad t \in [0, \bar{T}].$$

III Backward and Forward Boundary Value Problems

In this section, we review the backward and forward boundary value problems governing call prices.

III-A Backward BVP

Assuming no arbitrage, the r -discounted call price must be a martingale. Using Itô's lemma for semi-martingales, it is straightforward to show that for each fixed (K, T) , the call value function C considered as a function of (S, t) satisfies the following deterministic partial integro differential equation (PIDE) in the region $S \in \mathfrak{R}^+, t \in (0, T)$:

$$\int_{-\infty}^{\infty} \left[C(t, Se^x; T, K) - C(t, S; T, K) - \frac{\partial}{\partial S} C(t, S; T, K) S(e^x - 1) \right] \nu(x, t) dx \quad (7)$$

$$+ \frac{\sigma^2(S, t) S^2}{2} \frac{\partial^2 C(t, S; T, K)}{\partial S^2} + [r(t) - q(t)] S \frac{\partial C(t, S; T, K)}{\partial S} - r(t) C(t, S; T, K) + \frac{\partial C(t, S; T, K)}{\partial t} = 0.$$

The call value function $C(t, S; T, K)$ also satisfies the following boundary conditions:

$$C(T, S; T, K) = (S - K)^+, \quad S \in \mathfrak{R}^+ \quad (8)$$

$$\lim_{S \downarrow 0} C(t, S; T, K) = 0, \quad t \in [0, T] \quad (9)$$

$$\lim_{S \uparrow \infty} C(t, S; T, K) \sim S e^{-\int_t^T q(u) du} - K e^{-\int_t^T r(u) du}, \quad t \in [0, T]. \quad (10)$$

These conditions send call value to intrinsic value along the boundaries of the domain.

By the Feynman Kac Theorem, a probabilistic representation of the solution to the backward boundary value problem (BVP) (7) to (10) is given by:

$$C(t, S; T, K) = e^{-\int_t^T r(u) du} E^Q[(S_T - K)^+ | S_t = S], \quad (11)$$

for $t \in [0, T]$ and $S \in \mathfrak{R}^+$.

III-B Forward BVP

Appendix 2 shows that for (S, t) fixed at (S_0, t_0) , the call value function C considered as a function of (K, T) satisfies the following deterministic *forward* PIDE in the region $K \in \mathfrak{R}^+, T \in (t_0, \bar{T}]$:

$$\int_{-\infty}^{\infty} \left[C(t_0, S_0; T, Ke^y) - C(t_0, S_0; T, K) - \frac{\partial}{\partial K} C(t_0, S_0; T, K) K(e^y - 1) \right] e^{-y\nu(-y, T)} dy + \frac{\sigma^2(K, T)K^2}{2} \frac{\partial^2}{\partial K^2} C(t_0, S_0; T, K) - [r(T) - q(T)]K \frac{\partial}{\partial K} C(t_0, S_0; T, K) - q(T)C(t_0, S_0; T, K) - \frac{\partial}{\partial T} C(t_0, S_0; T, K) = 0. \quad (12)$$

For future use, note that this forward PIDE has the same form as the backward PIDE (7), but the coefficients are different. Also note that put values satisfy the same PIDE's as call values. The initial condition on the call value is:

$$C(t_0, S_0; t_0, K) = (S_0 - K)^+, \quad K > 0. \quad (13)$$

Two boundary conditions are:

$$\lim_{K \downarrow 0} C(t_0, S_0; T, K) = S_0 e^{-\int_{t_0}^T q(u) du}, \quad T \in [t_0, \bar{T}] \quad (14)$$

and:

$$\lim_{K \uparrow \infty} C(t_0, S_0; T, K) = 0, \quad T \in [t_0, \bar{T}]. \quad (15)$$

This forward BVP first appeared in Andersen and Andreasen[2], who had generalized to jumps the original result of Dupire[12].

IV Derivation of Put Call Reversal

Differentiating the forward or backward BVP twice with respect to S yields a BVP for gamma. It is straightforward to prove that the solution to either BVP is non-negative. As a result, the call's gamma can be normalized into a transition density for a backward running process, as shown in the introduction.

Under the SDE (6), we have:

$$\lim_{S \uparrow \infty} C_{ss}(t_0, S; T, K) = \lim_{S \downarrow 0} C_{ss}(t_0, S; T, K) = 0. \quad (16)$$

Thus, the origin is a natural boundary for this backward running process. Since $\lim_{T \rightarrow t_0} C(S, t_0; K, T) = (S - K)^+$, we have $\lim_{t_0 \uparrow T} C_{ss}(t_0, S; T, K) = \delta(S - K)$. Thus, the backward running process is conditioned to start at K .

The Markov property of the forward running process says roughly that given the present, the future and the past are independent. Thus, the backward running process is also a Markov process. We let \hat{Q} denote the probability measure induced by the call's gamma. Since the $r - q$ discounted forward running spot price process is a Q -martingale, the results in the introduction imply that the $q - r$ discounted backward running process is a \hat{Q} martingale. To determine the coefficients of the SDE governing this martingale, we will negate the time variable in the forward call BVP (12) to (15) to put it in a form suitable for application of the Feynman Kac Theorem.

Let $\hat{t} \equiv \bar{T} - T$ be a change of the time variable in the forward call BVP. Let $\hat{T} \equiv \bar{T} - t_0$ denote the time on the new clock when the original maturity T is t_0 . Let:

$$\hat{P}(\hat{t}, \hat{S}; \hat{T}, \hat{K}) \equiv e^{-\int_{\hat{t}}^{\hat{T}} q(\bar{T}-u)du} E^{\hat{Q}}[(\hat{K} - \hat{S}_{\hat{T}})^+ | \hat{S}_{\hat{t}} = \hat{S}] \quad (17)$$

be the value of the put on the backward running process \hat{S} . To determine the coefficients of the SDE governing this process, suppose:

$$\hat{P}(\hat{t}, K; \hat{T}, S_0) \equiv C(\bar{T} - T, S_0; \bar{T} - t_0, K). \quad (18)$$

Under the change of variables, the forward call PIDE (12) becomes:

$$\begin{aligned} \int_{-\infty}^{\infty} \left[\hat{P}(Ke^y, \hat{t}; \hat{T}, S_0) - \hat{P}(\hat{t}, K; \hat{T}, S_0) - \hat{P}_{\hat{S}}(\hat{t}, K; \hat{T}, S_0)K(e^y - 1) \right] e^{-y\nu(-y, \bar{T} - \hat{t})} dy \\ + \frac{\sigma^2(K, \bar{T} - \hat{t})K^2}{2} \hat{P}_{\hat{S}\hat{S}}(\hat{t}, K; \hat{T}, S_0) \\ - [r(\bar{T} - \hat{t}) - q(\bar{T} - \hat{t})]K \hat{P}_{\hat{S}}(\hat{t}, K; \hat{T}, S_0) - q(\bar{T} - \hat{t})\hat{P}(\hat{t}, K; \hat{T}, S_0) + \hat{P}_{\hat{t}}(\hat{t}, K; \hat{T}, S_0) = 0, \end{aligned} \quad (19)$$

for $K \in \mathfrak{R}^+$, $\hat{t} \in [0, \bar{T}]$.

Under the time reversal, the initial condition (13) becomes the following terminal condition on the new clock:

$$\hat{P}(\hat{T}, K; \hat{T}, S_0) = (S_0 - K)^+, \quad K \in \mathfrak{R}^+. \quad (20)$$

The boundary conditions (14) and (15) become:

$$\lim_{K \downarrow 0} \hat{P}(\hat{t}, K; \hat{T}, S_0) = S_0 e^{-\int_{\hat{t}}^{\hat{T}} q(\bar{T}-u) du}, \quad \hat{t} \in [0, \hat{T}] \quad (21)$$

and:

$$\lim_{K \uparrow \infty} \hat{P}(\hat{t}, K; \hat{T}, S_0) = 0, \quad \hat{t} \in [0, \hat{T}]. \quad (22)$$

By the Feynman Kac Theorem, a probabilistic representation of the solution of (19) to (22) is given by:

$$\hat{P}(\hat{t}, K; \hat{T}, S_0) = e^{-\int_{\hat{t}}^{\hat{T}} q(\bar{T}-u) du} E^{\hat{Q}}[(S_0 - \hat{S}_{\hat{T}})^+ | \hat{S}_{\hat{t}} = K], \quad (23)$$

for $\hat{t} \in [0, \hat{T}]$, $K \in \mathfrak{R}^+$, where the superscript \hat{Q} on the expectations operator E is the probability measure under which the process \hat{S} solves the following SDE:

$$\frac{d\hat{S}_{\hat{u}}}{\hat{S}_{\hat{u}-}} = [q(\bar{T}-\hat{u}) - r(\bar{T}-\hat{u})] d\hat{u} + \sigma(\hat{S}_{\hat{u}-}, \bar{T}-\hat{u}) d\hat{B}_{\hat{u}} + \int_{-\infty}^{\infty} (e^y - 1) [\hat{\mu}(dy, d\hat{u}) - e^{-y} \nu(-y, \hat{u}) dy d\hat{u}], \quad \hat{u} \in [\hat{t}, \hat{T}], \quad (24)$$

where $\hat{S}_{\hat{t}} = K$. Here, \hat{B} is a \hat{Q} standard Brownian motion and $\hat{\mu}(dy, d\hat{u})$ is the counting measure for jumps in \hat{S} . The SDE implies that the backward running stock price process \hat{S} differs from the time reversed process, although the two processes are indexed by the same clock. The distinction between the two processes is clearest when the forward running process S starts at $t = 0$ from some fixed level. Then as shown in Haussmann and Pardoux[16] in the diffusion case, the time reversed process is a bridge process which terminates at this fixed level with probability one. In contrast, the coefficients of our backward running process are completely independent of this level.

From (23), \hat{P} is naturally interpreted as the value of a *put* on the \hat{S} process with fixed strike S_0 and fixed maturity \hat{T} . In words, (11), (18), and (23) imply that at time $t = t_0$, the call on the forward running

process S with maturity T has the same value as a put at time $\hat{t}_0 \equiv \bar{T} - T$ written on the backward running process \hat{S} :

$$C(t_0, S_0; T, K) = \hat{P}(\bar{T} - T, K; \bar{T} - t_0, S_0), \quad S_0, K \in \mathfrak{R}^+, 0 \leq t_0 \leq T \leq \bar{T}. \quad (25)$$

This equation formally establishes Put Call Reversal which is the fundamental result of this paper. The right hand side of (25) is best understood as evaluating the function $\hat{P}(\hat{t}, \hat{S}; \hat{T}, \hat{K})$ defined in (17) at the point $\hat{S} = K, \hat{t} = \bar{T} - T, \hat{K} = S_0, \hat{T} = \bar{T} - t_0$. As indicated by (25), the forward running process S underlying the call is conditioned to start at S_0 , while the backward running process \hat{S} underlying the put is conditioned to start at K . The call on the forward running process pays the excess, if any, of S_T over K . In contrast, the put on the backward running process pays the shortfall, if any, of $\hat{S}_{\bar{T}}$ under S_0 . In the forward running economy, for each $t \in [t_0, \bar{T}]$, the stock price S is expected to rise under Q at the relative rate $r(t) - q(t)$. In the backward running economy, the reversed process evaluated at the corresponding time is expected to *drop* under \hat{Q} at the same relative rate $r(t) - q(t)$, or equivalently, to rise at the relative rate $q(t) - r(t)$. Clearly, the change in sign in expected relative velocity is purely due to the use of different clocks. The diffusion coefficient is the same function of the relevant spot price for the forward and backward running process. So if the volatility declines as the forward running stock price rises, then the volatility of the backward running process also declines as the backward running process rises. If the volatility of the forward running process declines over time (i.e. gets smaller as the call's maturity nears), then the volatility of the backward running process rises over time (i.e. gets larger as the put's maturity nears). Like the volatility, a downward sloping term structure of jump arrival rates in the forward running economy becomes an upward sloping term structure of arrival rates in the backward economy. From (24), the arrival rate of a jump of size x in the forward economy is multiplied by e^x to obtain the arrival rate of a jump of size $-x$ in the backward economy.

V Applications of PCR

V-A Interpreting Call Deltas and Gammas

Put Call Reversal gives simple new probabilistic representations of a call's delta and gamma. Differentiating (25) w.r.t. S_0 implies:

$$\frac{\partial}{\partial S} C(t_0, S_0; T, K) = \frac{\partial}{\partial \hat{K}} \hat{P}(\bar{T} - T, K; \bar{T} - t_0, S_0). \quad (26)$$

Thus, the delta of the call on the forward running process is the first strike derivative of the put on the reversed process. The latter is just the q -discounted probability that the reversed process finishes below S_0 when it starts from K .

Differentiating (26) w.r.t. S_0 implies:

$$\frac{\partial^2}{\partial S^2} C(t_0, S_0; T, K) = \frac{\partial^2}{\partial \hat{K}^2} \hat{P}(\bar{T} - T, K; \bar{T} - t_0, S_0). \quad (27)$$

Thus, the gamma of the call on the forward running process is the second strike derivative of the put on the reversed process. The latter is just the q -discounted PDF under \hat{Q} of the event that the reversed process finishes at S_0 when it starts at K . As discounted probabilities are always nonnegative, so is a call's delta and gamma. The non-negativity of a call's delta and gamma in the diffusion context is already known (see Bergman, Grundy, and Weiner[5] and El Karoui, Jeanblanc-Picque, and Shreve[13]). By using PCR, we have easily shown that this nonnegativity extends to our jump setting.

V-B Semi-Static Hedging of Long-Dated Options

The last subsection showed how backward relationships such as the call's delta and gamma can be given convenient probabilistic interpretations using the reverse clock. This subsection shows how some new forward relationships among call values can be conveniently derived and interpreted using well-known backward relations for puts in the reverse economy. By forward relations, we are referring to relationships among calls of different strikes and maturities. Using PCR, we will see that these may be understood in

terms of familiar relationships among a single put's values as we vary its underlying's price and (reverse) time.

To begin, we note that when interest rates and dividends are zero, the forward PIDE (12) indicates that the decrease in call value as maturity is decreased slightly is entirely due to a reduction in the expected gamma trading profit at maturity. Using PCR, this value decrease can alternatively be interpreted in terms of the well understood tradeoff implicit in the usual backward PIDE. In particular, the call's value decrease as maturity is shortened can be interpreted as the initial decrease in put value due to time decay along $\hat{S} = K$.

More generally, we may be interested in the relationship among call values as the maturity is decreased by more than an infinitesimal amount. It is a tautology in the backwards economy that:

$$\hat{P}(\hat{t}_0, K; \hat{T}, S_0) = \int_0^\infty \frac{\partial^2}{\partial \hat{K}^2} \hat{P}(\hat{t}_0, K; \bar{T} - u, L) \hat{P}(\bar{T} - u, L; \hat{T}, S_0) dL, \quad (28)$$

where $u \in [t_0, T]$. Substituting Put Call Reversal (25) for both put values implies:

$$\begin{aligned} C(t_0, S_0; T, K) &= \int_0^\infty \frac{\partial^2}{\partial \hat{K}^2} \hat{P}(\bar{T} - T, K; \bar{T} - u, L) C(t_0, S_0; u, L) dL \\ &= \int_0^\infty e^{-\int_u^T q(v) dv} \hat{Q}\{\hat{S}_{\bar{T}-u} \in dL | \hat{S}_{\hat{t}_0} = K\} C(t_0, S_0; u, L) \quad S_0, K \geq 0, t_0 \in [0, u], T \in [u, \bar{T}]. \end{aligned}$$

Thus, a call maturing at T with strike K has the same value for all $t_0 \in [0, u]$ as a portfolio of calls of all strikes L and the single maturity u . The weight on each call of strike L is given by the q -discounted risk-neutral PDF of the event that \hat{S} is at L when the shorter dated call matures, given that \hat{S} started at K at $\hat{t} = \hat{t}_0$ (or equivalently $t = T$).

Without loss of generality, suppose that the T maturity call to be valued is initially out-of-the-money, i.e. $S_0 < K$. Then instead of forming a replicating portfolio of calls of the same maturity but different strikes, one can instead form a replicating portfolio of calls of different maturities and all strikes $L \leq H$, where $H \in (S_0, K)$. As the call on the forward running process is initially out-of-the-money, the put on the backward running process is also initially out-of-the-money. The put's initial value is the expected value under \hat{Q} of the q -discounted values the put will have at all the possible first passage times to H in

$[\hat{t}_0, \hat{T}]$:

$$\hat{P}(\hat{t}_0, K; \hat{T}, S_0) = \int_{\hat{t}_0}^{\hat{T}} e^{-\int_{\hat{u}}^{\hat{T}} \hat{r}(\hat{v})d\hat{v}} \int_0^H \hat{P}(\hat{u}, L; \hat{T}, S_0) \hat{Q}\{\hat{\tau}_H \in d\hat{u}, \hat{S}_{\hat{u}} \in dL | \hat{S}_{\hat{t}_0} = K\}. \quad (29)$$

If we use PCR to convert both put values to call values, we arrive at the following result:

$$C(t_0, S_0; T, K) = \int_{t_0}^T e^{-\int_u^T q(v)dv} \int_0^H C(t_0, S_0; u, L) \hat{Q}\{\hat{\tau}_H \in d(\bar{T} - u), \hat{S}_{\bar{T}-u} \in dL | \hat{S}_{t_0} = K\}, \quad (30)$$

where τ_H is the first passage time to H (defined to be infinite if H is never reached). Thus, the weight on each call maturity u from t_0 to T is just the q -discounted PDF of the event that the first passage time to H of \hat{S} is that maturity, and that the stock price at the call's maturity is L , where L is the call's strike. Note that if the forward-running stock price cannot jump up, then the backward running stock price can not jump down, and the double integral in (30) collapses to a single integral over call maturities. This result generalizes to our setting a result derived previously by Chou and Georgiev[10] for the constant parameter Black Scholes model. As the first passage time PDF is known explicitly in that model, they obtained a closed form solution for the call weights at every maturity. More generally, one would have to numerically solve a PDE for the first passage time PDF. This numerical solution is much more computationally effective than the numerical solution of the integral equation which arises in the direct approach.

The above results generalize to any curved barrier $H(t)$ in space-time. So long as the forward running stock price must hit the barrier to finish in-the-money, the long-dated call can be replicated by a portfolio of shorter dated calls whose strikes lie along and below the curved barrier. For each $u \in [t_0, T]$, the weight of the call struck at $H(u)$ is given by the q -discounted joint probability that the first passage time of the reversed process to the curved barrier corresponds to u and that the stock price after crossing is the call's strike. Using coordinate changes and the reflection principle, there are many models other than Black Scholes in which these probabilities can be determined in closed form.

VI Summary and Future Research

Working in a jump diffusion setting, we derived a new result termed Put Call Reversal. We showed how this result differs from previous results relating calls and puts. We discussed the implications of Put Call

Reversal for interpreting the hedge positions in both a dynamic and semi-static context.

One possible research direction is to extend or alter the class of stochastic processes for which PCR is valid. For example, one can investigate to what extent PCR holds when the arrival rate of jumps in the log price is state dependent. One can also investigate the extent to which modifications to PCR are needed to extend it to American options, multi-variate options, and to path-dependent options. In the interests of brevity, these research directions are best left for future research⁴.

⁴or historical research under a time reversal!

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Appendix 1: Verification of Integration by Parts

Consider integrating (2) by parts:

$$\begin{aligned} u &= C(t_0, S; T, K) & dv &= \delta(S_0 - S)dS \\ du &= \frac{\partial}{\partial S}C(t_0, S; T, K)dS & v &= -1(S < S_0). \end{aligned}$$

The boundary terms vanish since $C(t_0, 0; T, K) = 0$:

$$C(t_0, S_0; T, K) = \int_0^\infty \frac{\partial}{\partial S}C(t_0, S; T, K)1(S < S_0)dS, \quad K \geq 0, T \in [t_0, \bar{T}]. \quad (31)$$

Consider integrating (31) by parts:

$$\begin{aligned} u &= \frac{\partial}{\partial S}C(t_0, S; T, K) & dv &= 1(S < S_0)dS \\ du &= \frac{\partial^2}{\partial S^2}C(t_0, S; T, K)dS & v &= -(S_0 - S)^+. \end{aligned}$$

The boundary terms vanish since $C_s(t_0, 0; T, K) = 0$:

$$C(t_0, S_0; T, K) = \int_0^\infty \frac{\partial^2}{\partial S^2}C(t_0, S; T, K)(S_0 - S)^+dS, \quad K \geq 0, T \in [t_0, \bar{T}]. \quad (32)$$

Appendix 2: Derivation of Forward PIDE

Consider the product $e^{\int_t^T r(u)du}(S_t - K)^+$. By the Tanaka Meyer formula (see Protter[21]):

$$\begin{aligned} (S_T - K)^+ &= e^{\int_{t_0}^T r(u)du} (S_0 - K)^+ + \int_{t_0}^T e^{\int_t^T r(u)du} 1_{(S_{t-} > K)} dS_t \\ &+ \int_{t_0}^T e^{\int_t^T r(u)du} \left\{ \frac{\sigma^2(S_{t-}, t) S_{t-}^2}{2} \delta(S_{t-} - K) - r(t) (S_{t-} - K)^+ \right\} dt \\ &+ \int_{t_0}^T e^{\int_t^T r(u)du} \int_{-\infty}^{\infty} \left[(S_{t-} e^x - K)^+ - (S_{t-} - K)^+ - 1_{(S_{t-} > K)} S_{t-} (e^x - 1) \right] \mu(dx, dt). \end{aligned} \quad (33)$$

Multiplying by $e^{-\int_{t_0}^T r(u)du}$ and taking conditional expectations on both sides under an equivalent martingale measure Q , we have:

$$\begin{aligned} C(K, T) &= (S_0 - K)^+ + \int_{t_0}^T e^{-\int_{t_0}^t r(u)du} E_0^Q \{ 1_{(S_{t-} > K)} [r(t) - q(t)] S_{t-} \} dt \\ &+ \int_{t_0}^T e^{-\int_{t_0}^t r(u)du} \left\{ \frac{\sigma^2(K, t) K^2}{2} E_0^Q \delta(S_{t-} - K) - r(t) E_0^Q (S_{t-} - K)^+ \right\} dt \\ &+ \int_{t_0}^T e^{-\int_{t_0}^t r(u)du} E_0^Q \int_{-\infty}^{\infty} \left[(S_{t-} e^x - K)^+ - (S_{t-} - K)^+ - 1_{(S_{t-} > K)} S_{t-} (e^x - 1) \right] \nu(x, t) dx dt, \end{aligned} \quad (34)$$

where $C(K, T) \equiv e^{-\int_{t_0}^T r(u)du} E^Q[(S_T - K)^+ | S_{t_0} = S_0] \equiv e^{-\int_{t_0}^T r(u)du} E_0^Q(S_T - K)^+$. Differentiating w.r.t. T implies:

$$\begin{aligned} \frac{\partial}{\partial T} C(K, T) &= e^{-\int_{t_0}^T r(u)du} E_0^Q \{ 1_{(S_{T-} > K)} [r(T) - q(T)] S_{T-} \} \\ &+ \frac{\sigma^2(K, T) K^2}{2} e^{-\int_{t_0}^T r(u)du} E_0^Q \delta(S_{T-} - K) - r(T) e^{-\int_{t_0}^T r(u)du} E_0^Q (S_{T-} - K)^+ \\ &+ e^{-\int_{t_0}^T r(u)du} E_0^Q \int_{-\infty}^{\infty} \left[(S_{T-} e^x - K)^+ - (S_{T-} - K)^+ - 1_{(S_{T-} > K)} S_{T-} (e^x - 1) \right] \nu(x, T) dx. \end{aligned} \quad (35)$$

Subtracting and adding $e^{-\int_{t_0}^T r(u)du} E_0^Q \{ [r(T) - q(T)] K 1_{(S_{T-} > K)} \}$:

$$\begin{aligned} \frac{\partial}{\partial T} C_0(K, T) &= e^{-\int_{t_0}^T r(u)du} E_0^Q \{ 1_{(S_{T-} > K)} [r(T) - q(T)] (S_{T-} - K) \} \\ &+ e^{-\int_{t_0}^T r(u)du} E_0^Q \{ [r(T) - q(T)] K 1_{(S_{T-} > K)} \} + \frac{\sigma^2(K, T) K^2}{2} \frac{\partial^2}{\partial K^2} C(K, T) - r(T) C(K, T) \\ &+ e^{-\int_{t_0}^T r(u)du} E_0^Q \int_{-\infty}^{\infty} \left[e^x (S_{T-} - K e^{-x})^+ - 1_{(S_{T-} > K)} (S_{T-} - K + S_{T-} e^x - S_{T-}) \right] \nu(x, T) dx \\ &= [r(T) - q(T)] C(K, T) - [r(T) - q(T)] K \frac{\partial}{\partial K} C(K, T) \end{aligned}$$

$$\begin{aligned}
& + \frac{\sigma^2(K, T)K^2}{2} \frac{\partial^2}{\partial K^2} C_0(K, T) - r(T)C(K, T) \\
& + e^{-\int_{t_0}^T r(u)du} E_0^Q \int_{-\infty}^{\infty} e^x \left[(S_{T-} - Ke^{-x})^+ - 1(S_{T-} > K)(S_{T-} - Ke^{-x} + K - K) \right] \nu(x, T) dx \\
= & -q(T)C(K, T) - [r(T) - q(T)]K \frac{\partial}{\partial K} C(K, T) + \frac{\sigma^2(K, T)K^2}{2} \frac{\partial^2}{\partial K^2} C(K, T) + \\
& e^{-\int_{t_0}^T r(u)du} E_0^Q \int_{-\infty}^{\infty} \left[(S_{T-} - Ke^{-x})^+ - (S_{T-} - K)^+ - \frac{\partial}{\partial K} (S_{T-} - K)^+ K(e^{-x} - 1) \right] e^x \nu(x, T) dx \\
= & -q(T)C(K, T) - [r(T) - q(T)]K \frac{\partial}{\partial K} C(K, T) + \frac{\sigma^2(K, T)K^2}{2} \frac{\partial^2}{\partial K^2} C(K, T) \\
& + \int_{-\infty}^{\infty} \left[C(Ke^{-x}, T) - C(K, T) - \frac{\partial}{\partial K} C(K, T)K(e^{-x} - 1) \right] e^x \nu(x, T) dx. \tag{36}
\end{aligned}$$

Letting $y = -x$ be a change of variable in the last integral, the forward PIDE becomes:

$$\begin{aligned}
\frac{\partial}{\partial T} C(K, T) = & -q(T)C(K, T) - [r(T) - q(T)]K \frac{\partial}{\partial K} C(K, T) + \frac{\sigma^2(K, T)K^2}{2} \frac{\partial^2}{\partial K^2} C(K, T) \\
& + \int_{-\infty}^{\infty} \left[C(Ke^y, T) - C(K, T) - \frac{\partial}{\partial K} C(K, T)K(e^y - 1) \right] e^{-y} \nu(-y, T) dy. \tag{37}
\end{aligned}$$