

Risk Neutral Valuation

Presentation for Course
at New York University

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Part I

Review of Dynamic Hedging

I Review of Dynamic Hedging of Path-Independent Derivatives

- Let S_t denote the dollar price at time t of an underlying stock.
- We focus attention on derivative securities which have a specified final dollar payout $f(S_T)$ paid at a fixed time T , and which also have a specified intermediate dollar payout $i(S_t, t)$ paid at every $t \in [0, T]$.
- Consider the problem of dynamically hedging the sale of such a claim under the following assumptions:
 1. Frictionless markets
 2. No arbitrage
 3. Constant interest rate r
 4. Underlying pays a constant proportional dividend continuously over time:

$$\frac{\text{\$ amount of dividend over } [t, t + dt]}{dt} = \delta S_t,$$

where δ is a non-negative constant.

5. Continuous spot price process:

$$\frac{dS_t}{S_t} = m_t dt + \sigma_t dW_t, \quad t \in [0, T],$$

where the mean growth rate process m_t is adapted and the volatility process σ_t is a function of S_t and t only, i.e., there is a function σ such that:

$$\sigma_t = \sigma(S_t, t).$$

- We also assume that m_t and $\sigma(S, t)$ are chosen so as to prevent negative prices and explosions.

I-A Representing the Payoffs

- Itô's lemma applied to the function $V(S_t, t)e^{r(T-t)}$ gives:

$$V(S_T, T) = V(S_0, 0)e^{rT} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_t, t) dS_t + \int_0^T e^{r(T-t)} \left[\frac{\sigma^2(S_t, t) S_t^2}{2} \frac{\partial^2 V}{\partial S^2}(S_t, t) - rV(S_t, t) + \frac{\partial V}{\partial t}(S_t, t) \right] dt.$$

- The 1st term is a constant, while the 2nd is a stochastic integral. Thus, the 1st term can be created by depositing $V(S_0, 0)$ in the bank and the 2nd term accumulates gains on $\frac{\partial V}{\partial S}(S_t, t)$ shares held at each $t \in [0, T]$.
- However, long positions in stock are costly. If we borrow to finance the stock position, then gains from the stock are reduced by the carrying cost as follows:

$$V(S_T, T) = V(S_0, 0)e^{r(T-t)} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_t, t) [dS_t - (r - \delta)S_t dt] + \int_0^T e^{r(T-t)} \left[\frac{\sigma^2(S_t, t) S_t^2}{2} \frac{\partial^2 V}{\partial S^2}(S_t, t) + (r - \delta)S_t \frac{\partial V}{\partial S}(S_t, t) - rV(S_t, t) + \frac{\partial V}{\partial t}(S_t, t) \right] dt.$$

- Now, by choosing $V(S, t)$ to solve the following fundamental PDE:

$$\frac{\sigma^2(S, t) S^2}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + (r - \delta)S \frac{\partial V}{\partial S}(S, t) - rV(S, t) + \frac{\partial V}{\partial t}(S, t) = -i(S, t),$$

with:

$$V(S, T) = f(S),$$

we get $f(S_T) + \int_0^T e^{r(T-t)} i(S_t, t) dt$

$$= V(S_0, 0)e^{rT} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_t, t) [dS_t - (r - \delta)S_t dt].$$

Representing the Payoffs (con'd)

- Recall the representation of the final and intermediate payoffs:

$$\begin{aligned} f(S_T) + \int_0^T e^{r(T-t)} i(S_t, t) dt \\ = V(S_0, 0)e^{rT} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_t, t) [dS_t - (r - \delta)S_t dt]. \end{aligned}$$

- Thus the final and intermediate payoffs are the sum of:
 1. the future value of the initial investment $V(S_0, 0)$ and
 2. the accumulated gains from holding $\frac{\partial V}{\partial S}(S_t, t)$ shares, where all purchases are financed by borrowing and all sales are invested in the bank.
- It follows that the fair value of the payoff is $V(S_0, 0)$.
- The initial lending is $V(S_0, 0) - \frac{\partial V}{\partial S}(S_0, 0)S_0$ (which may be negative). At any time t , the lending must be $V(S_t, t) - \frac{\partial V}{\partial S}(S_t, t)S_t$. This is proven for the special case of the Black Scholes model with zero intermediate payouts in Appendix 1.

I-B Examples

- All of our examples will assume constant volatility:

$$\sigma^2(S, t) = \sigma^2.$$

Thus, we are assuming the validity of the Black-Scholes model. If the drift were also constant, then the stock price would follow geometric Brownian motion.

Example 1: Butterfly Spread

- Assume no intermediate payouts and that the final payoff is:

$$f(S) = \delta(S - K),$$

where $\delta(\cdot)$ is Dirac's delta function.

- In this case, the fair value $V(S, t)$ must solve:

$$\frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + (r - \delta)S \frac{\partial V}{\partial S}(S, t) - rV(S, t) + \frac{\partial V}{\partial t}(S, t) = 0,$$

with the terminal condition:

$$V(S, T) = \delta(S - K).$$

- Let \hat{V} be the forward price of the derivative:

$$\hat{V}(S, T) = e^{r(T-t)}V(S, t).$$

- Then \hat{V} must satisfy:

$$\frac{\sigma^2 S^2}{2} \frac{\partial^2 \hat{V}}{\partial S^2}(S, t) + (r - \delta)S \frac{\partial \hat{V}}{\partial S}(S, t) + \frac{\partial \hat{V}}{\partial t}(S, t) = 0,$$

with the same terminal condition

$$\hat{V}(S, T) = \delta(S - K).$$

Notice that we have eliminated the potential term $-rV(S, t)$ from the PDE.

Butterfly Spread Valuation (con'd)

- Recall the PDE for the forward price of the butterfly spread:

$$\frac{\sigma^2 S^2}{2} \frac{\partial^2 \hat{V}}{\partial S^2}(S, t) + (r - \delta) S \frac{\partial \hat{V}}{\partial S}(S, t) + \frac{\partial \hat{V}}{\partial t}(S, t) = 0,$$

and the terminal condition:

$$\hat{V}(S, T) = \delta(S - K).$$

- Now let us change the spatial independent variable as:

$$\begin{aligned} x &= \ln S \\ u(x, t) &= \hat{V}(S, t). \end{aligned}$$

- Then the PDE for $u(x, t)$ is:

$$\frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}(x, t) + \mu \frac{\partial u}{\partial x}(x, t) + \frac{\partial u}{\partial t}(x, t) = 0,$$

where $\mu \equiv r - \delta - \frac{\sigma^2}{2}$, and its terminal condition is:

$$u(x, T) = \frac{1}{K} \delta(x - \ln K),$$

where the division by K is a consequence of the requirement that the delta function in x integrates to 1.

Butterfly Spread Valuation (con'd)

- Recall that the PDE for the forward price of the butterfly spread written as a function of $x = \ln S$ is:

$$\frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}(x, t) + \mu \frac{\partial u}{\partial x}(x, t) + \frac{\partial u}{\partial t}(x, t) = 0,$$

where $\mu \equiv r - \delta - \frac{\sigma^2}{2}$, and its terminal condition is:

$$u(x, T) = \frac{1}{K} \delta(x - \ln K).$$

- Recognizing this PDE as the Kolmogorov backward equation for arithmetic Brownian motion with constant drift rate μ and constant diffusion rate σ , we can write the solution as:

$$u(x, t) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}K} \exp \left\{ -\frac{1}{2} \left[\frac{\ln K - (x + \mu(T-t))}{\sigma\sqrt{T-t}} \right]^2 \right\}.$$

- Then:

$$\hat{V}(S, t) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}K} \exp \left\{ -\frac{1}{2} \left[\frac{\ln K - (\ln S + \mu(T-t))}{\sigma\sqrt{T-t}} \right]^2 \right\}$$

$$V(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}K} \exp \left\{ -\frac{1}{2} \left[\frac{\ln K - (\ln S + \mu(T-t))}{\sigma\sqrt{T-t}} \right]^2 \right\}.$$

- Notice that $\hat{V}(S, t)$ is a lognormal density function and that $V(S, t)$ is the Green's function for the fundamental PDE, which we started with.

Example 2: Binary Call

- A binary call has no intermediate payouts and has a final payoff which can be written as:

$$f(S) = 1(S > K) = \int_K^\infty \delta(S - L)dL.$$

- From the integral representation of the indicator function and the previous example, we easily get the price of the binary call:

$$V(S, t) = \int_K^\infty \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}L} \exp\left(-\frac{1}{2}\left[\frac{\ln L - (\ln S + \mu(T-t))}{\sigma\sqrt{T-t}}\right]^2\right)dL,$$

where $\mu \equiv r - \delta - \frac{\sigma^2}{2}$.

- Let:

$$d_2(L) \equiv \frac{\ln\left(\frac{S}{L}\right) + \mu(T-t)}{\sigma\sqrt{T-t}}$$

be a standardizing transformation. Then the fair value of the binary call becomes:

$$V(S, t) = e^{-r(T-t)}N(d_2(K)),$$

where $N(x)$ is the distribution function of a standard normal random variable.

Example 3: Plain Vanilla Call

- A plain vanilla call has no intermediate payouts and has a final payoff which can be written as the integral of the binary call:

$$f(S) = (S - K)^+ = \int_K^\infty 1(S - L)dL.$$

- Using integration by parts,

$$\begin{aligned} V(S, t) &= \int_K^\infty e^{-r(T-t)} N(d_2(L)) dL \\ &= e^{-r(T-t)} \left[LN(d_2(L)) \Big|_K^\infty - \int_K^\infty LN'(d_2(L)) dL \right] \\ &= e^{-r(T-t)} \left[-KN(d_2(K)) - \int_K^\infty Se^{(r-\delta)(T-t)} N'(d_1(L)) dL \right], \end{aligned}$$

since by completing the square $LN'(d_2(L)) = Se^{(r-\delta)(T-t)} N'(d_1(L))$, where:

$$d_1(L) = d_2(L) + \sigma\sqrt{T-t}.$$

- Rewriting the last integral in terms of the standard normal distribution function, we get the (award-winning!) Black-Scholes formula:

$$V(S, t) = -Ke^{-r(T-t)} N(d_2(K)) + Se^{-\delta(T-t)} N(d_1(K)).$$

I-C The Martingale Measure

- Recall that the cost of creating $\int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_t, t) [dS_t - (r - \delta)S_t dt]$ paid at T was zero, given that V satisfied the fundamental PDE.
- Viewed as a process in T , the absence of arbitrage clearly requires that this stochastic integral have realizations on both sides of zero for all T (or else be zero).
- Consequently, one can define a measure $Q^\$$ such that the integral has zero mean under $Q^\$$ for all T .
- Since the integral is a $Q^\$$ -martingale by the definition of $Q^\$$, $Q^\$$ is called a martingale measure.
- Given our assumptions and given the initial prices of the stock and bond, this martingale measure is uniquely determined.
- Under $Q^\$$, the integrator $dS_t - (r - \delta)S_t dt$ has zero mean and has variance $\sigma^2 S_t^2 dt$. Consequently, there exists a unique $Q^\$$ -Brownian motion $W_t^\$$ such that:

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t^\$, \quad t \in [0, T], \text{ where } S_0 = S.$$

Risk-Neutral Stock Price Process

- Recall that the absence of arbitrage has allowed us to define a unique martingale measure $Q^\$$ and a unique standard Brownian motion $\{W_t^\$; t \in [0, T]\}$ such that:

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t^\$, \quad t \in [0, T], \text{ where } S_0 = S.$$

- The drift of this process is simply the cost of carrying the underlying and has no greater significance. By sheer coincidence, it is also the drift which would arise in equilibrium if all investors were risk-neutral, and for this reason, the process is also called the “risk-neutral” process. The martingale measure is also called the risk-neutral measure. These are terribly mis-leading terms, since we are definitely *not* assuming that investors are risk-neutral.
- The volatility of the risk-neutral process is the same as the volatility of the assumed process. This arises whenever one has continuous sample paths, but does *not* necessarily follow when prices can jump.
- The risk-neutral process simply tells us the market’s unique arbitrage-free forward price for delta function type payoffs defined over various path bundles.

Risk-Neutral Valuation

- The examples hopefully made it clear that the key to valuing derivatives with no intermediate payouts and any final (path-independent) payoff was to find the value of a butterfly spread.
- The forward price of a butterfly spread with payoff $\delta(S_T - K)$ at T is the risk-neutral density of the terminal stock price. Thus:

$$V(S, t) = e^{-r(T-t)} \int_0^\infty f(K) Q^\$ \{S_T \in dK | S_t = S\}.$$

- Note that the payoff $f(K)$ is unitless, while $Q^\$$ is measured in time T dollars.
- When the payoffs are possibly path-dependent (eg. for barrier options), we can write:

$$V_t = e^{-r(T-t)} E^{Q^\$} \{V_T | \mathcal{F}_t\}.$$

- This says that to value a path-dependent derivative, we first determine the forward price of each path from $Q^\$$ and then we determine the payout along each path from V_T . The value is given by multiplying the payout along each path by its price and then summing (integrating) over paths.
- For example, given that we are at time t with the stock price at S , the forward price of the security paying $\delta(S_T - K)$ at T is simply the sum(integral) of the forward prices of all securities paying off if a given path occurs, where each such path starts at (t, S) and ends at (T, K) . The total measure of this path bundle is well-known to be:

$$Q\{S_T \in dK | S_t = S\} = \frac{dK}{\sqrt{2\pi\sigma^2(T-t)}K} \exp \left\{ -\frac{1}{2} \left[\frac{\ln(K/S) - \mu(T-t)}{\sigma\sqrt{T-t}} \right]^2 \right\}$$