

# The Finite Moment Logstable Process And Option Pricing

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# Background

- In many options markets, the implied volatility smile/smirk flattens out as maturity increases.
- The conventional wisdom is that the Central Limit Theorem has kicked in:

The normal law of error stands out in the experience of mankind as one of the broadest generalizations of natural philosophy. It serves as the guiding instrument in researches in the physical and social sciences and in medicine, agriculture and engineering. It is an indispensable tool for the analysis and the interpretation of the basic data obtained by observation and experiment.

W.J. Youden  
Experimentation and Measurement  
p. 55

# Why Be Normal?

Everybody believes in the normal approximation, the experimenters because they believe it is a mathematical theorem, the mathematicians because they believe it is an experimental fact!

G. Lippman  
Quoted in D'Arcy Thompson's  
*On Growth and Form*  
Volume I, p. 121

to quote a statement of Poincaré, who said (partly in jest no doubt) that there must be something mysterious about the normal law since mathematicians think it is a law of nature whereas physicists are convinced that it is a mathematical theorem.

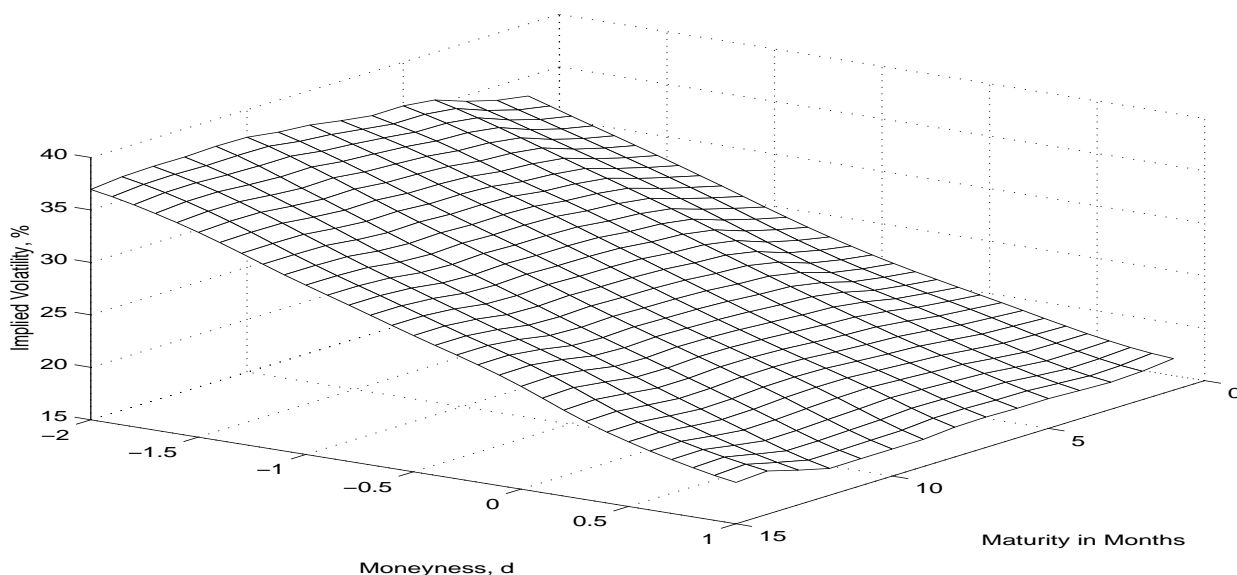
Mark Kac  
*Statistical Independence in Probability Analysis and Number Theory*  
Chapter 3, The Normal Law (p. 52)

# Overview

- For S&P500 index options, the implied volatility smirk does not flatten out as maturity increases.
- To capture this behavior, we price options using the “Finite Moment Log Stable Process” (FMLS)
  - with unbounded variance, skewness, kurtosis of returns
  - yet finite expectation of price, price<sup>2</sup>, price<sup>3</sup> etc.
- Calibration of our FMLS model to the data suggests that:
  - it outperforms other stationary jump (-diffusion) models
  - by capturing the stability of the volatility smirk across maturity.

# Volatility Smirks at Different Maturities

- Nonparametrically smoothed implied volatility surface for S&P 500 index options from April 4th, 1999 to May 31st, 2000:



- S&P500 index option vols do NOT “wipe that smirk off your face”.
- Note that the horizontal axis uses “moneyness” defined as:

$$\text{Moneyness } d = \frac{\ln K/F}{\sigma\sqrt{\tau}}$$

which is approximately the number of standard deviations that the strike is away from the forward (in the Black Scholes model).

- The smirk DOES flatten if the horizontal axis is instead defined as  $K$ ,  $\frac{K}{F}$  or  $\ln\left(\frac{K}{F}\right)$ .

# Maturity Pattern of the Volatility Smirk

**Key Observation:** When graphed against moneyness, the volatility smirk does not flatten out with maturity.

- The volatility smirk is a reflection of non-normality in the risk-neutral distribution:  
slope  $\Leftrightarrow$  skewness; curvature  $\Leftrightarrow$  kurtosis.
- The maturity pattern of the volatility smirk reflects the extent of the non-normality at different levels of time aggregation.
- If it holds, the Central Limit Theorem implies that the return distribution converges to normal as maturity  $\tau$  increases. Under certain moment conditions, the CLT holds if:
  - return innovations are IID (Lévy processes)
    - \* skewness drops as  $1/\sqrt{\tau}$  and kurtosis falls as  $1/\tau$ ,
  - or
  - volatility is stochastic:
    - \* return distribution still converges to normality, although slower, so long as the volatility process is stationary.
- Question: *Is the CLT holding for S&P 500 Index Options?*

# A Statistical Test on the Maturity Pattern

At each day:

- and for each maturity  $j$ , regress implied volatility on moneyness:

$$IV_j = a_j + b_j d_j + e_j$$

where  $b_j$  is the slope of the implied volatility in moneyness at maturity  $j$ .

- A downward sloping smile implies a negative estimate for  $b_j$
  - We restrict moneyness to  $d = [-2, 0]$ , which is the (quasi) linear part.
- We then regress the smirk slope on maturity:

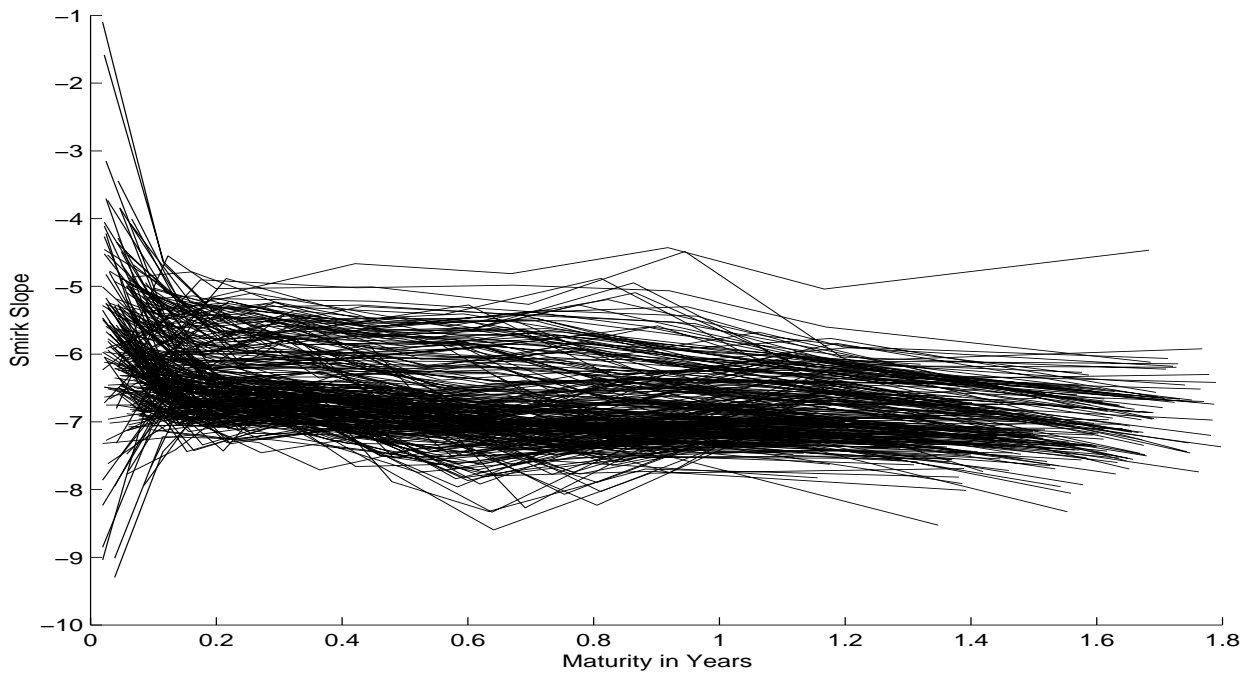
$$b_j = \alpha + \beta \tau_j + \varepsilon,$$

where  $\beta$  captures how the slope of the smirk changes with maturity.

- $\beta$  is a cross partial  $\frac{\partial^2 IV}{\partial d \partial \tau}$ :  $\beta > 0$  means that the smirk becomes flatter as maturity increases.
- Results:  $\beta$  is negative for most days tested  
 $\Rightarrow$  smirk becomes steeper as maturity increases!

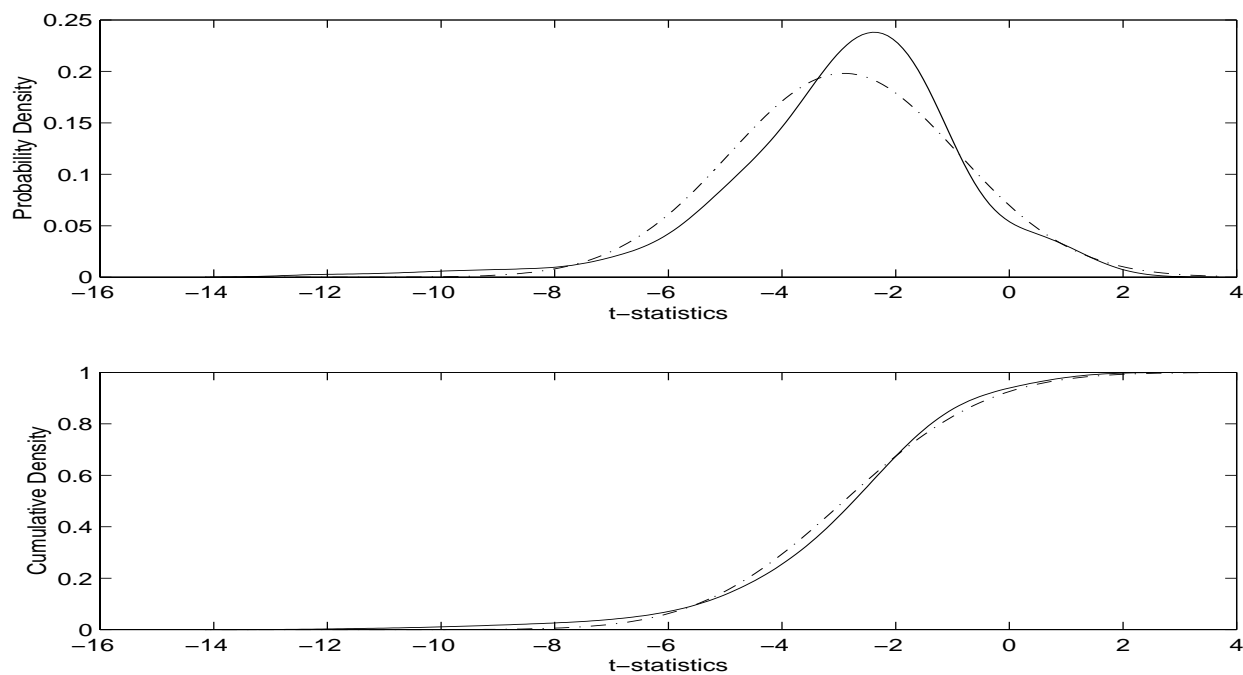
# A Statistical Test on the Maturity Pattern

The slope of the smirk becomes more negative as maturity increases.



# A Statistical Test on the Maturity Pattern

$t$ -statistics: The probability that  $\beta > 0$  (i.e. smirk flattens out) is less than 1%.



# A Summary of the Evidence

- The CLT implies that the return distribution should converge to normality
- and that the volatility smirk should flatten out as maturity increases.
- But the S&P 500 index option implied “break the law”.
- This is a modeling challenge as the CLT applies to all models where log price relatives:
  - have finite moments (eg. skewness and kurtosis)
  - and are stationary
- None of the standard models work for S&P 500 index options
- A common approach is to make them non-stationary (i.e. add term structure)
- Can we work with infinite moments instead?

# Stable Processes

## Characteristics of Stable Processes

- *Self-similarity*: the tail behavior is invariant to time aggregation.
- *Parsimonious*: Four parameters govern mean  $\mu \in \mathfrak{R}$ , dispersion  $\sigma > 0$ , skewness  $\beta \in [-1, 1]$ , and tail behavior  $\alpha \in (0, 2]$ .
- *Infinite Return Moments*: Return moments of order higher than  $\alpha$  do not exist:
  - $\Rightarrow$  the usual CLT does not apply
  - Does a martingale measure exist?

# Finite Moment LogStable Process

## The Model

- Suppose (risk-neutral) stock returns are driven by a stable motion:

$$dS_t/S_t = (r - q)dt + \sigma dL_t^{\alpha, \beta}$$

- $L_t^{\alpha, \beta}$  denotes a Lévy  $\alpha$ -stable motion, distributed  $\alpha$ -stable with mean 0, dispersion  $t^{1/\alpha}$ , tail index  $\alpha \in (0, 2]$ , and skew index  $\beta \in [-1, 1]$ . If  $\alpha = 2$ , get Black Scholes model.

- We set  $\beta = -1$ , so that the return is maximally negatively skewed.

- When  $\beta = -1$ :

- all return moments of order  $> \alpha$  are unbounded (including variance, skewness, and kurtosis)

- yet all price moments are finite:

$$E \left[ e^{n\sigma L_t^{\alpha, -1}} \right] = \exp \left( -tn^\alpha \sigma^\alpha \sec \frac{\pi\alpha}{2} \right) \quad \text{is finite for all } n > 0$$

- so a martingale measure exists.

- **Intuition:** The exponential map from returns to prices fattens the right tail, so we need to start from a right tail which is not too fat.

- For  $\beta = -1$  (maximum negative skewness), the initial right tail is as thin as can be. If  $\beta > -1$ , the right tail is too fat: expected stock and call prices are infinite. Forcing  $\beta = -1$  is the key departure from other  $\alpha$ -stable option pricing models.

# The FMLS Model

The log price relative over maturity  $\tau$  is:

$$s_\tau \equiv \ln S_{t+\tau}/S_t = \left( r - q + \sigma^\alpha \sec \frac{\pi\alpha}{2} \right) \tau + \sigma L_\tau^{\alpha, -1}$$

- The log price relative has a finite drift only when  $\beta = -1$ .

- The characteristic function of the return  $s_\tau$  is:

$$\phi_s(u) \equiv E_t [e^{i u s_\tau}] = \exp \left( i u \left( r - q + \sigma^\alpha \sec \frac{\pi\alpha}{2} \right) \tau - \tau (i u \sigma)^\alpha \sec \frac{\pi\alpha}{2} \right)$$

- The  $n$ th conditional moment of  $S_T$  is

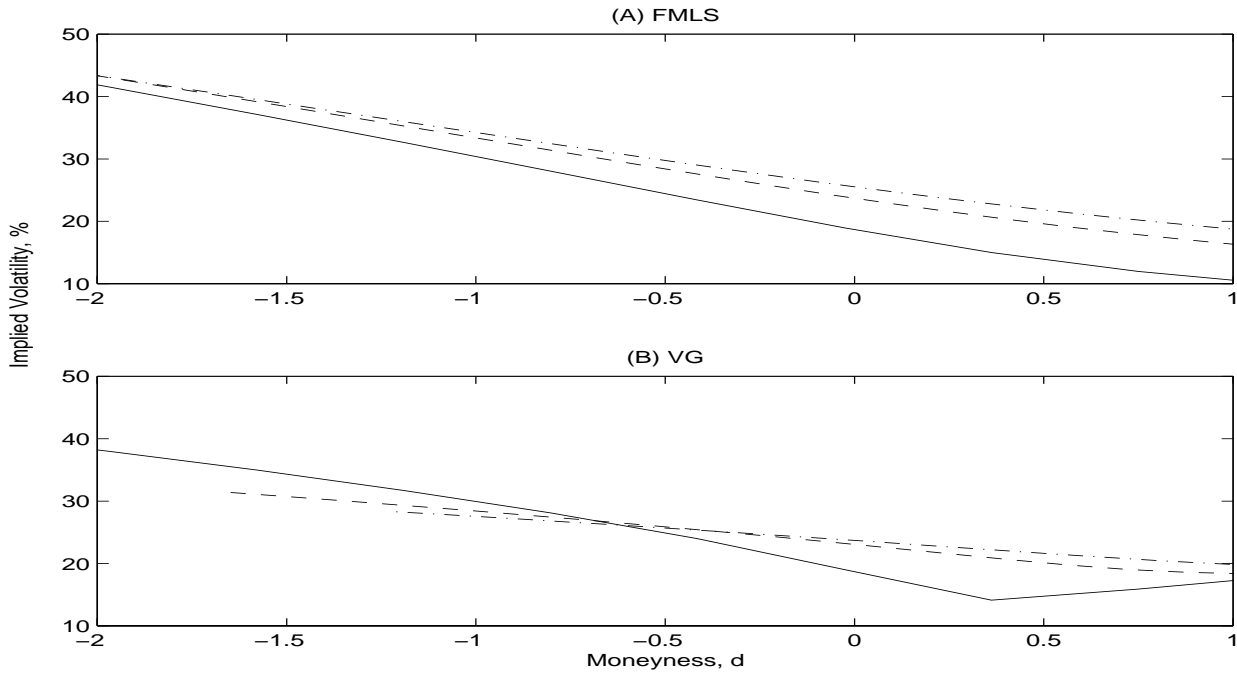
$$E_t [S_T^n] \equiv S_t^n E_t [e^{n s_\tau}] = S_t^n \left( n \left( r - q + \sigma^\alpha \sec \frac{\pi\alpha}{2} \right) \tau - \tau (n \sigma)^\alpha \sec \frac{\pi\alpha}{2} \right)$$

which is finite for all  $n > 0$ .

- The FMLS model has only two free parameters:  $\sigma > 0$  (dispersion),  $\alpha \in (0, 2]$  (tail)

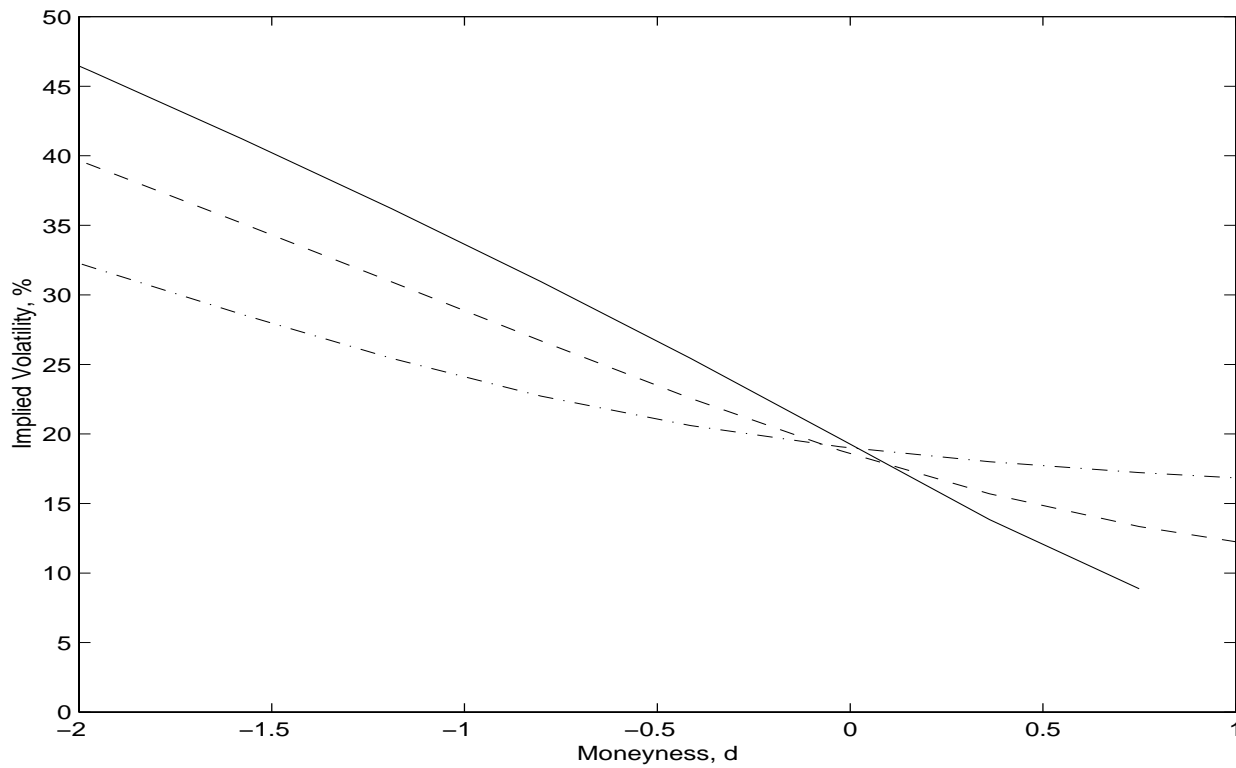
# The FMLS Model vs. VG Model

Options can be priced via the FFT method of Carr and Madan (1999).



**Maturity Pattern:** 1m (solid), 6m (dashed), 12m (dash-dotted)

# The Tail Index $\alpha$



Tail Index  $\alpha$ : 1.2 (solid), 1.5 (dashed), 1.8 (dashed-dotted).  
When  $\alpha = 2$ , the volatility smile is flat

# Model Calibration

- Objective: Choose the two free parameters by minimizing the sum of squared pricing errors of the OTM options:

$$\text{sse} = \min_{\Theta} \sum_{i=1}^{N_C} w_i^C (C_i - \hat{C}_i(\Theta))^2 + \sum_{i=1}^{N_P} w_i^P (P_i - \hat{P}_i(\Theta))^2 .$$

- Equal weight on OTM prices:  $w_i = 1$  for all  $i$ .
- Used 8 strikes at three maturities: 1m, 6m, 12m.
- Compared three models all allowing jumps:
  - Our finite moment logstable model (FMLS)
  - Madan, Carr, Chang (1998)'s Variance-Gamma model (VG)
  - Merton (1976)'s Poisson Jump-Diffusion model (MJD)

# Model Performance

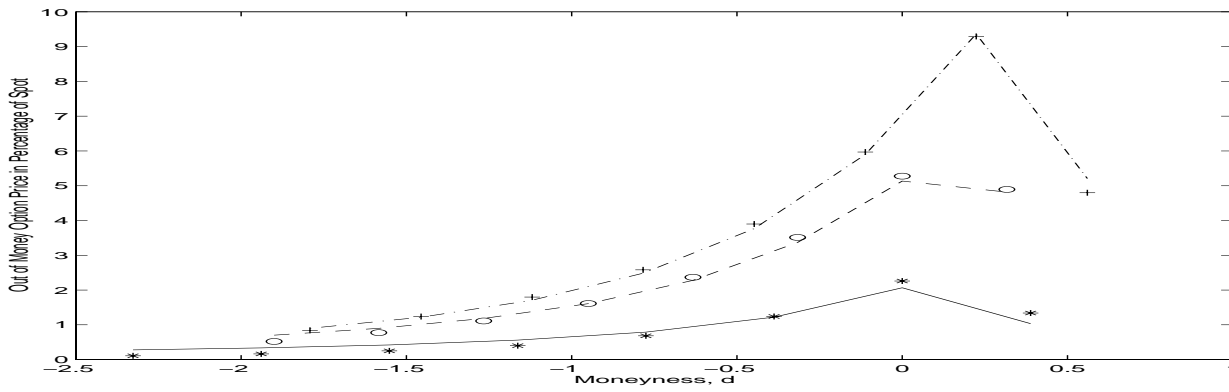
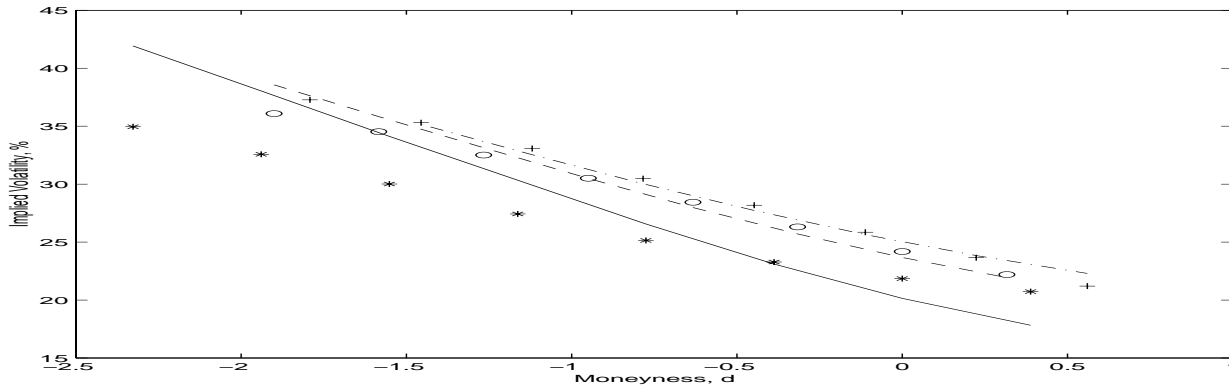
(standard deviation in parentheses)

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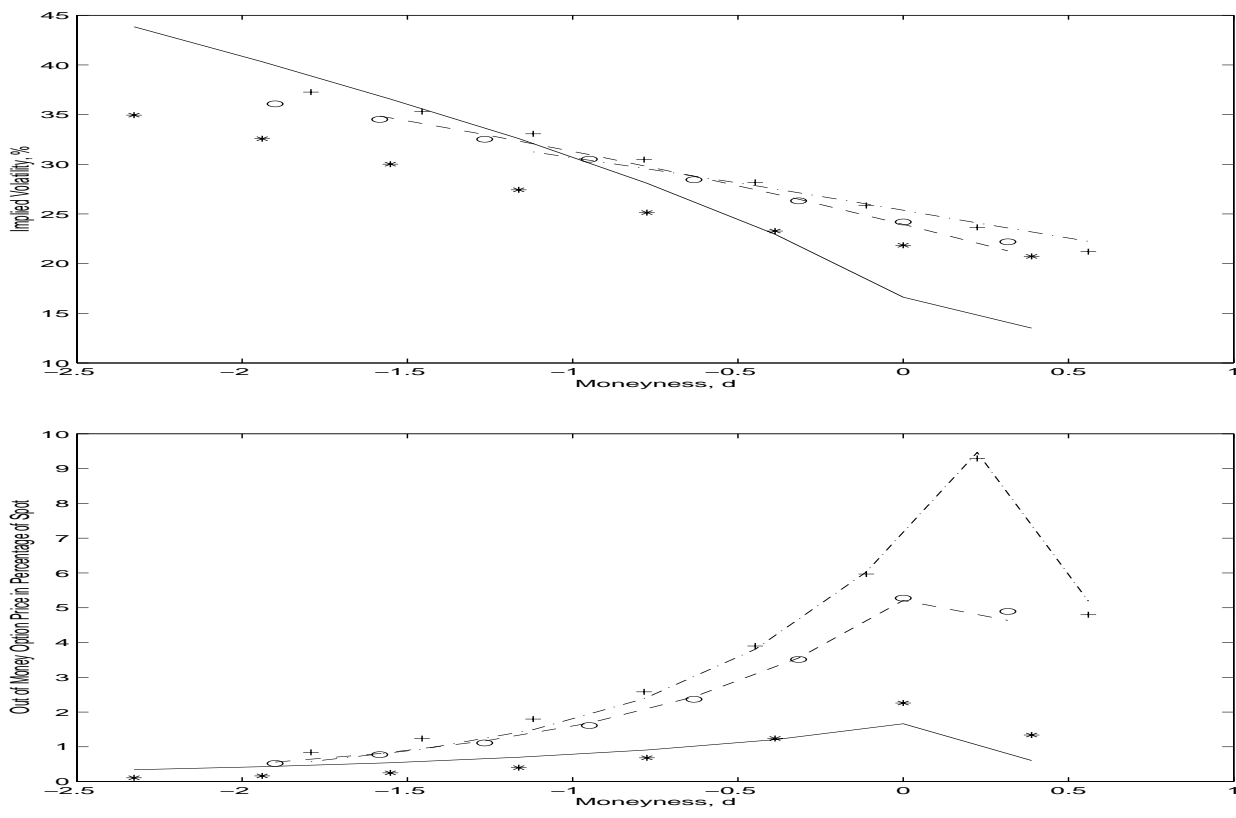
Parameters	FMLS	VG	MJD
$\alpha$	1.6145 (0.0110)	0.7221 (0.1384)	1.8604 (0.3523)
$\sigma$	0.1486 (0.0010)	0.1642 (0.0168)	0.1003 (0.0079)
$\omega$	— —	-0.1834 (0.0527)	-0.0930 (0.0117)
$\eta$	— —	— —	0.1271 (0.0196)
sse( $\times 10^5$ )	5.8004	18.2319	11.5350

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# Model Performance: FMLS

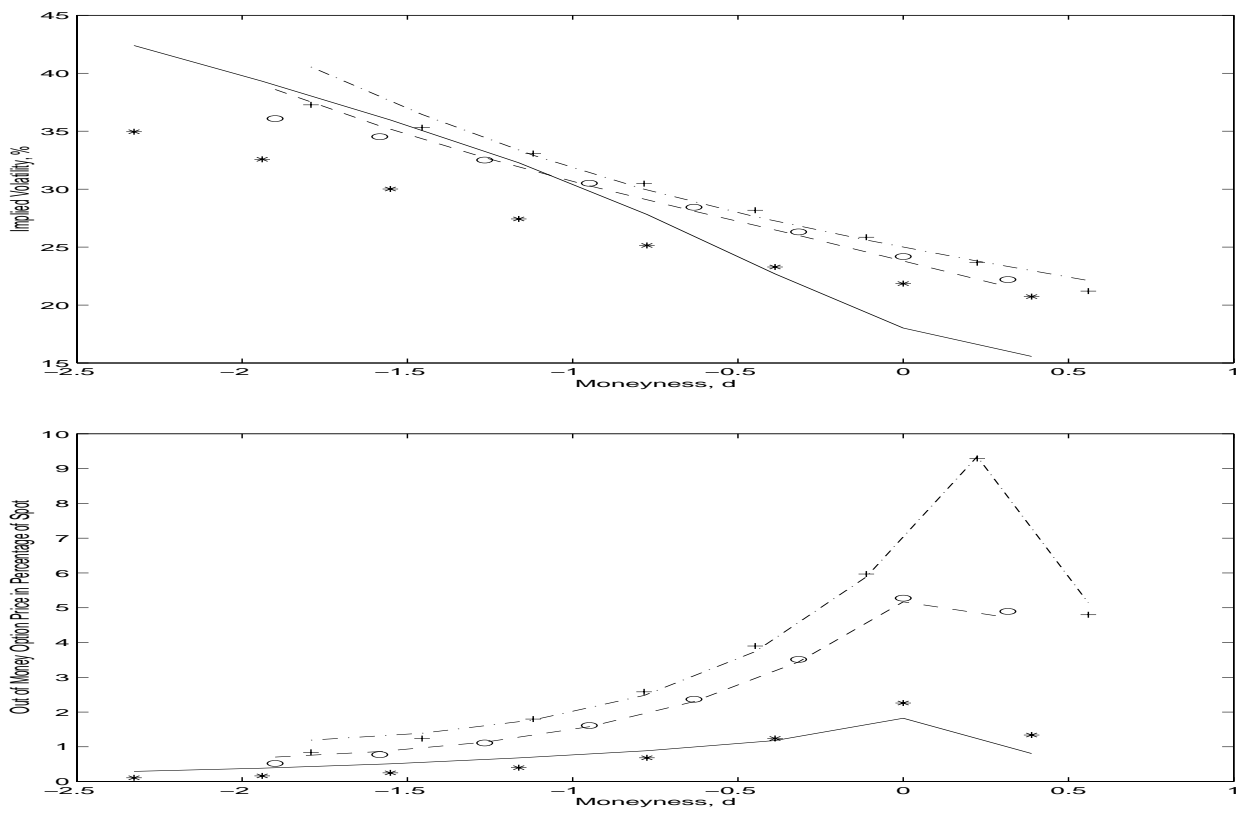


# Model Performance: VG



Key stress: maturity dimension

# Model Performance: MJD



Key stress: maturity dimension

# Summary

- **Evidence:** S&P500 Risk-neutral return distribution does NOT converge to normal as maturity increases.
- **Theory:** FMLS model
  1. Self-similarity guarantees the same tail behavior across maturities (non-flattening)
  2.  $\beta = -1$  guarantees finite stock price moments (finite option prices)
  3.  $\alpha < 2$  generates non-normality (smirk)
- **Performance:** FMLS compares favorably over other jump models in preventing smirk flattening across the maturity dimension.
- **Extensions:** can introduce additional parameters by:
  - adding stochastic dispersion by a stochastic time change
  - refining the jump structure by subordinating to a gamma process for example.
- **Download:** Postscript/PDF files of these overheads and the paper can be downloaded from:  
`www.petercarr.net` or  
`www.math.columbia.edu/~pcarr/papers`