

On the Nature of Options

Peter Carr and Keith Lewis
Banc of America Securities
Dilip Madan
University of Maryland

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Introduction

- The standard theory of contingent claim pricing through dynamic replication gives no special role to options.
- Thus, it is not at all obvious why markets have organized to offer these simple hockey-stick type payoffs, when other collections of functions such as polynomials, circular functions, or wavelets might offer greater advantages.
- While the complete answer to this question is undoubtedly highly complex, the purpose of our paper is to analyze the particular role which options play when markets for their underlying are frictionless, and when the underlying stock has arbitrarily stochastic volatility and jumps with stochastic arrival rates.
- In this context, we show how options can be regarded as vehicles for enhancing dynamic trading in their underlying assets by allowing implicit trading in the underlying at one or more price levels, even if the market jumps across the level.
- Our analysis has implications for both
 1. risk management: we derive a general expression which shows the sources of hedging error which arise when a portfolio of options is incorrectly hedged by assuming that the volatility and/or arrival rate (as given by the Lévy measure) is constant, when either or both quantities is stochastic.
 2. product innovation: we show how such errors can be turned into opportunities by presenting several special cases of hedging error which might be sold off into the market as an attractive contingent claim. In contrast to most derivative securities, the payoff to these claims can be perfectly replicated, even though markets are incomplete due to arbitrarily stochastic volatility and jump sizes.

Overview of Presentation

- We consider the role of options when markets in its underlying asset are frictionless and when this underlying has a volatility process and jump arrival rates which are arbitrarily stochastic.
- By combining a static option position with a particular dynamic hedging strategy, we characterize the option's time value as the (risk-neutral) expected benefit from being able to buy or sell one share of the underlying at the option's strike whenever the strike price is crossed. The buy/sell decision can be conditioned on the post jump price, so that a rational investor buys on rises and sells on drops.
- Thus, an option provides liquidity at its strike even when the market doesn't.
- We next present two methods for extending this local liquidity to every price between the pre and post jump level:
 1. hold a continuum of options of all strikes
 2. hold one option and adjust the dynamic hedging strategy.
- We discuss the advantages and disadvantages of each approach and consider the benefits of combining them.

A Characterization of Option Time Value

- Consider a strategy consisting of:
 1. being long one European call, and
 2. a binary delta hedging strategy consisting of being short one forward contract with delivery price K when the forward price $F > K$ and not hedging otherwise.

Theorem 1: *Assuming frictionless markets in a call's underlying asset and a semi-martingale process for the forward price, the terminal P&L from the above strategy is:*

$$P\&L_T = -[C_0 e^{rT} - (F_{0T} - K)^+] + \frac{1}{2} L_T^F(K) + \sum_{0 < t \leq T} [1(F_{t-} \leq K)(F_t - K)^+ + 1(F_{t-} > K)(K - F_t)^+],$$

where $L_T^F(K)$ is the terminal local time of the forward price at the strike.

Breakout Option

- The cash flows generated by the strategy described in the last overhead can be packaged into an attractive product for investors.
- The “breakout option” pays the straight sum of all the “overshoots” of a fixed level during its life.
- It is straightforward to have discrete (eg. daily) monitoring, in which case the hedge involves trading in the underlying at most daily.
- By purchasing a standard call struck at the overshoot level and with the same life as the product, and undertaking the binary delta-hedging strategy, the payoff can be replicated perfectly.
- This hedge is perfect even though markets may be grossly incomplete due to arbitrarily stochastic volatility and arrival rates.
- Thus, the arbitrage-free price of this breakout option is the initial time value of a standard option.
- By adding the call to the binary strategy, the hedger inherits a localized liquidity which is unavailable to other market participants.
- This access to markets allows the investor to perfectly replicate the payoffs to the contract.

Positioning in a Continuum of Strikes

Theorem 2: *Assume that we have frictionless markets in the asset underlying a continuum of options of maturity T . Also assume a semimartingale process for the forward price. Consider the terminal P&L from combining dynamic trading in forward contracts with a static position in bonds, forward contracts, and options which pays off $f(S_T)$ at T . Suppose that the static position is purchased for V_0^f initially, and that the number of forwards held over $[0, T)$ is the negative of the left derivative of $f(F_t)$ w.r.t. F_t . Then the terminal P&L is:*

$$\begin{aligned}
 P\&L_T = & - \left[\frac{V_0^f}{B_0} - f(F_0) \right] + \frac{1}{2} \int_0^\infty L_T^F(K) m(dK) \\
 & + \int_0^T \int_{-\infty}^\infty \int_{F_{t-}}^{F_t} f''(K) (F_t - K) dK \mu(dx, dt),
 \end{aligned}$$

where m is a signed measure giving the second derivative of f in the generalized function sense, and $\mu(dx, dt)$ is the integer valued random measure which counts the number of jumps in any region of space-time.

Thus, the purchase of options with a continuum of strikes provides liquidity at every strike price.

Price Variance Swap

- Theorem 2 can be used as the basis for synthesizing another interesting financial product.
- By Taylor series:

$$F_T^2 = F_0^2 + 2F_0(F_T - F_0) + \int_{F_0+}^{\infty} 2(F_T - K)^+ dK + \int_0^{F_0-} 2(K - F_T)^+ dK.$$

- The initial cost of creating this payoff is:

$$V_0^{S^2} = F_0^2 B_0 + \int_{F_0+}^{\infty} 2C_0(K) dK + \int_0^{F_0-} 2P_0(K) dK.$$

- Theorem 2 implies that the terminal P&L from layering on a dynamic position in $-2F_t$ forward contracts is:

$$P\&L_T = F_0^2 - \frac{V_0^{S^2}}{B_0} + \int_0^{\infty} L_T^F(K) dK + \int_0^T \int_{-\infty}^{F_t} \int_{F_{t-}}^{F_t} 2(F_t - K) dK \mu(dx, dt),$$

which simplifies to:

$$\begin{aligned} P\&L_T = & - \int_{F_0+}^{\infty} 2 \frac{C_0(K)}{B_0} dK - \int_0^{F_0-} 2 \frac{P_0(K)}{B_0} dK \\ & + [F, F]_T^c + \int_0^T \int_{-\infty}^{\infty} (F_t - F_{t-})^2 \mu(dx, dt), \end{aligned}$$

where $[F, F]_T^c$ is the terminal quadratic variation from the continuous component of the forward price process.

- Consider a nonnegative payoff at maturity equal to the last two terms, which are the sum of the squared price changes.
- If the premium for this payoff is paid at maturity, then the fair fixed price to charge for this random payoff is $\int_{F_0+}^{\infty} 2 \frac{C_0(K)}{B_0} dK$.

Achieving a Desired Continuous Payoff

- We have seen that the continuously paid payoffs from a breakout option and a price variance swap can both be replicated by combining options of all strikes with dynamic trading in the underlying.
- Allowing static positions in options of multiple maturities increases the set of continuously paid payoffs which can be perfectly replicated.
- Suppose an investor wishes to have the term:

$$\int_0^\tau \int_{-\infty}^{\infty} g(F_t; F_{t-}, t) \mu(dx, dt)$$

paid to him at some fixed time $\tau > 0$, where the function $g(F; F_-, t)$ maps the post jump price F , the pre-jump price F_- , and the time t into the continuous payoff realized at t and paid at τ .

Theorem 3: *Assume that we have frictionless markets in the asset underlying a continuum of options of strikes $K > 0$ and maturities $T \in [0, \tau]$. Also assume a semi-martingale process for the forward price. Consider a continuous payout function $g(F; F_-, t)$ which satisfies:*

$$\begin{aligned} g(F_-; F_-, t) &= 0 \quad t \in [0, \tau], F_- > 0, \\ g_1(F_-; F_-, t) &= 0, \quad t \in [0, \tau], F_- > 0. \\ g_{11}(F; F_-, t) &\text{ is independent of } F_-. \end{aligned}$$

Then the continuously paid payoff can be replicated perfectly by combining a static position in options with a dynamic trading strategy in the underlying.

- The paper gives the details of the replicating strategy.

Lévy Hedging

- The assumption of a continuum of strikes and maturities may be quite unrealistic in certain option markets. For this reason, we return to an options market where the initial liquidity is only in a single call with fixed strike K and fixed maturity T .
- Lévy Hedging also extends the local liquidity of a static position in a call to a continuum of price levels.
- In order to include the standard Black model in our analysis, we now assume that $X_t \equiv \ln F_t$ is a semi-martingale under the statistical probability measure P .
- Since the call is assumed to not trade between time 0 and T , the option buyer uses *value functions* to determine intermediate P&L and to hedge. For simplicity, we restrict attention to marking and hedging value functions $C(F, t)$ which are Markov in the forward price F and time t . In order to retain unrestricted interest rates, we also work with *forward value functions*:
 1. The pricing valuation function $C^p(F, t) \equiv E^p[S_T - K]^+ | F_t = F$, based on a pricing measure Q^p , which the investor uses to mark the option position.
 2. The hedging valuation function $C^h(F, t) \equiv E^h[S_T - K]^+ | F_t = F$, based on a hedging measure Q^h , which the investor uses to hedge the option position.

Terminal P&L

- Assume that the market spot option price C_0 is arbitrage-free and that the *initial* forward pricing function matches the market forward price of the call, i.e. $C^p(F_0, 0) = \frac{C_0}{B_0}$.
- For tractability reasons, we assume that X is a Lévy process under both the pricing measure Q^p and the hedging measure Q^h .

Theorem 4: *Assuming frictionless markets in a call's underlying asset and a semi-martingale process for the forward price, consider the P&L from buying a static position in a call at time 0 which pays off $[S_T - K]^+$ at T . Suppose that the call is purchased for C_0 initially and that the number of forward contracts held over $[0, T)$ is the negative of the first partial derivative of C^h w.r.t. F , where $C^h(F, t) \equiv E^h[S_T - K]^+ | F_t = F$. Then the terminal P&L is:*

$$\begin{aligned}
 P\&L_T = & -[C^p(F_0, 0) - C^h(F_0, 0)] + \int_0^T (\sigma_{t-}^2 - \sigma_h^2) \frac{F_{t-}^2}{2} \frac{\partial^2 C^h}{\partial F^2}(F_{t-}, t) dt \\
 & + \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} \frac{\partial^2 C^h}{\partial F^2}(L, t) (F_{t-}e^x - L) dL [\mu(dx, dt) - \nu^h(dx) dt].
 \end{aligned}$$

Local Liquidity Density

- The paper shows that the final P&L can be also re-written in terms of the *local liquidity density* $p_t(L) \equiv \frac{\partial^2 C^h}{\partial F^2}(L, t)$ as:

$$\begin{aligned}
 P\&L_T = & -[C^h(F_0, 0) - C^p(F_0, 0)] + \int_0^T (\sigma_{t-}^2 - \sigma_h^2) \frac{F_{t-}^2}{2} p_t(F_{t-}) dt \\
 & + \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} p_t(L) (F_{t-}e^x - L) dL [\mu(dx, dt) - \nu^h(dx) dt].
 \end{aligned}$$

where for each t , $p_t(L)$ is a proper probability density function (PDF).

- The results of the first half arise as the special case when the hedging volatility and Lévy density are both assumed to vanish. Thus, by using a (non-degenerate) Lévy hedging strategy an investor spreads out the one forward contract traded at the option's strike into a proper density with support on all positive levels.

Comparing Approaches

- Compare the P&L from hedging the intrinsic value of a continuum of strikes:

$$P\&L_T = - \left[\frac{V_0^f}{B_0} - f(F_0) \right] + \frac{1}{2} \int_0^\infty L_T^F(K) m(dK) \\ + \int_0^T \int_{-\infty}^\infty \int_{F_{t-}}^{F_t} f''(K)(F_t - K) dK \mu(dx, dt),$$

with the P&L from holding just one call, and Lévy hedging with the hedging measure chosen so that $C^h(F_0, 0) = C^p(F_0, 0)$:

$$P\&L_T = \int_0^T (\sigma_{t-}^2 - \sigma_h^2) \frac{F_{t-}^2}{2} \frac{\partial^2 C^h}{\partial F^2}(F_{t-}, t) dt \\ + \int_0^T \int_{-\infty}^\infty \int_{F_{t-}}^{F_{t-}e^x} \frac{\partial^2 C^h}{\partial F^2}(L, t)(F_{t-}e^x - L) dL [\mu(dx, dt) - \nu^h(dx) dt].$$

- When f is convex, the first approach has a strictly positive initial cost and generates strictly nonnegative cash flows. In contrast, the second approach has zero initial cost and generates cash flows with positive probability of both losses and gains at each t . In the second approach, the liquidity and optionality based on the actual stochastic variance rate σ_t^2 and the actual stochastic arrival rate $\mu(dx, dt)$ is financed by selling the liquidity and optionality based on some pairing of a constant variance rate σ_h^2 and a constant Lévy density $\nu^h(dx)dt$ consistent with the initial price.
- Thus, the net cash flow results from the difference between the stochastic variance and arrival rates and the time decay as measured in terms of constant volatility and arrival rate. The inflow and the outflow both depend on the price path.

Combining Approaches

- We now explore the benefits of Lévy hedging a continuum of options.
- Assume that there exists initial liquidity in a continuum of strikes of a fixed maturity. Recall the two types of value functions:
 1. The pricing valuation function $V^p(F, t) \equiv E^p[f(F_T)|F_t = F]$, based on a pricing measure Q^p , which the investor uses to mark the option portfolio.
 2. The hedging valuation function $V^h(F, t) \equiv E^h[f(F_T)|F_t = F]$, based on a hedging measure Q^h , which the investor uses to determine hedges for the option portfolio.

Theorem 5: *Consider the P&L from buying a static position in options at time 0 which pays off $f(S_T)$ at T . Suppose that the options are purchased for forward delivery at $V^p(F_0, 0) \equiv E^p[f(S_T)|F_t = F]$ initially and that the number of forward contracts held over $[0, T)$ is the negative of the first partial derivative of V^h w.r.t. F , where $V^h(F, t) \equiv E^h[f(S_T)|F_t = F]$. Then the terminal P&L is:*

$$\begin{aligned}
 P\&L_T = & -[V^p(F_0, 0) - V^h(F_0, 0)] + \int_0^T (\sigma_{t-}^2 - \sigma_h^2) \frac{F_{t-}^2}{2} \frac{\partial^2 V^h}{\partial F^2}(F_{t-}, t) dt \\
 & + \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} \frac{\partial^2 V^h}{\partial F^2}(K, t) (F_{t-}e^x - K) dK [\mu(dx, dt) - \nu^h(dx)dt].
 \end{aligned}$$

- Thus, the P&L arises from the difference between the amount $V^h(F_0, 0)$ received at T from the hedge strategy and the forward value of the premium paid, $V^p(F_0, 0)$, and also from the remainder term in the first order Taylor series expansion of $V^h(F, t)$.

Return Variance Swap

- At its maturity, a return variance swap pays the difference between the realized variance of returns and a fixed variance rate determined at inception.
- Perfect replication requires that $\mu = 0$ (no jumps) and the hedge sets $\nu^h(dx) = 0$ accordingly.
- By choosing the payoff to be twice the difference between the discretely compounded return and the continuously compounded return:

$$f(F) = 2 \left[\frac{F}{F_0} - 1 - \ln \left(\frac{F}{F_0} \right) \right],$$

the forward valuation function simplifies to:

$$V^h(F, t) = 2 \left[\frac{F}{F_0} - 1 - \ln \left(\frac{F}{F_0} \right) \right] + \sigma_h^2(T - t),$$

where σ_h can be chosen so that $V^h(F_0, 0) = V^p(F_0, 0)$.

- Since $\frac{\partial^2 V^h}{\partial F^2}(F, t) = \frac{2}{F^2}$, the P&L in Theorem 5 reduces to the payoff from a (continuously monitored) return variance swap:

$$P\&L_T = \int_0^T (\sigma_{t-}^2 - \sigma_h^2) dt.$$

- Since the payoff can also be represented as:

$$f(F) = \int_0^{F_0-} \frac{2}{K^2} (K - F)^+ dK + \int_{F_0+}^{\infty} \frac{2}{K^2} (F - K)^+ dK,$$

the static component of the hedge holds $\frac{2}{K^2} dK$ of each of the initially out-of-the-money options. Since $\frac{\partial V^h}{\partial F}(F, t) = 2 \left[\frac{1}{F_0} - \frac{1}{F_t} \right]$, the negative of this quantity represents the number of forwards held.

Replication Error due to Jumps

- When jumps are possible, Theorem 5 implies that the P&L from delta-hedging the option portfolio is:

$$P\&L_T = \int_0^T (\sigma_{t-}^2 - \sigma_h^2) dt + \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} \frac{2}{L^2} (F_{t-}e^x - L) dL \mu(dx, dt).$$

- The last integral simplifies to $\int_0^T \int_{-\infty}^{\infty} 2(e^x - 1 - x) \mu(dx, dt)$, so that if the investor is also short a variance swap paying $\int_0^T \left[\int_{-\infty}^{\infty} x^2 \mu(dx, dt) + \sigma_{t-}^2 - \sigma_h^2 \right] dt$ at T , the total hedging error due to jumps is $\int_0^T \int_{-\infty}^{\infty} 2 \left[e^x - 1 - x - \frac{x^2}{2} \right] \mu(dx, dt)$.
- The hedging error at each time t can be partially eliminated by projecting it on the payoff of 1 from an annuity, $e^x - 1$ from $\frac{1}{F_{t-}}$ futures, and the payoff x^2 from the variance swap.
- To leading order, the hedging error at each time t is $\frac{1}{3}x^3$, suggesting that a skewness swap payoff can be approximated when there are jumps by shorting 3 variance swaps, and hedging the continuous movements as described above.

Summary and Extensions

- We characterized an option's time value as the (risk-neutral) expected benefit from being able to buy or sell one share at the option's strike on every cross. Thus, an option provides liquidity at its strike even when the market doesn't.
- We then presented two methods for extending this local liquidity to every price between the pre and post jump level:
 1. Holding a continuum of options of all strikes and delta hedging as if the price process were deterministic.
 2. Hold one option and use a Lévy hedging strategy.
- We discussed the advantages and disadvantages of each approach and considered the benefits of combining them.
- A fairly straightforward extension to this work would involve similar analyses involving static positions in American or exotic options.
- A more challenging extension to this work would involve investigating the nature of payoffs which can be robustly hedged via dynamic strategies in options.
- Postscript/PDF files of these overheads and the paper can be downloaded from:
`www.math.columbia.edu/~pcarr/papers`