Simulating Bermudan Interest Rate Derivatives

Abstract

We use simulation to develop a Markov chain approximation for the value of caplets and Bermudan interest rate derivatives in the Market Model developed by Brace, Gatarek, and Musiela (1995) and Jamshidian (1996a,b). One and two factor versions of the Market Model were numerically studied. Our approach yields numerical values for caplets which are in close agreement with analytic solutions. We also provide numerical solutions for several Bermudan swaptions.

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1. Introduction

Term structure modeling is one of the most challenging problems in asset pricing theory. The modern approach has focused on the restrictions imposed by the absence of arbitrage and emanates from the seminal paper of Merton (1973). Significant contributions to the valuation and hedging of interest rate derivatives such as bond options were made by Black (1976), Vasicek (1977), Brennan and Schwartz (1982), Courtadon (1982), Ball and Torous (1983), Cox, Ingersoll, and Ross (CIR, 1985) , Ho and Lee (1986), Schaefer and Schwartz (1987), Black, Derman, and Toy (BDT, 1990), Black and Karasinski (BK,1991), Heath, Jarrow, and Morton (HJM,1988), Jamshidian (1989), Hull and White (1990), and more recently by Brace, Gatarek, and Musiela (BGM,1995), Jamshidian (1996a, b), and Flesaker and Hughston (1996).

The papers by Black (1976), Ball and Torous (1983) and by Schaefer and Schwartz (1987) all mimicked the original Black-Scholes stock option theory in specifying the underlying bond price dynamics directly. The other papers prior to Ho and Lee took off from Merton (1973) and instead specified the dynamics of the spot rate and possibly another factor such as the long-term rate. In these models, the bond price dynamics arise as a consequence of the absence of arbitrage and a specification of the market price of interest rate risk. While these models provide valuable insights into the determinants of bond prices and into the relationship between bond prices of different maturities, they have difficulty in matching the initial term structure exactly. An "inversion of the term structure" is required to eliminate the market prices of risk from interest rate derivative values.

Ho and Lee (1986) pioneered a new approach to term structure modeling. They took the initial discount curve as given and modelled the arbitrage-free movement of the entire discount curve in a binomial setting.. Inspired by the work of Ho and Lee (1986) and the martingale methods of Harrison and Pliska (1981), Heath, Jarrow, and Morton (1988) developed an elegant mathematical framework for the term structure of interest rates. The HJM approach takes the initial forward rate curve as given and develops a drift restriction for the stochastic evolution of the forward rate curve under the equivalent martingale measure. In common with Ho and Lee, this approach has the advantage of providing arbitrage-free interest rate derivative prices that do not explicitly depend on the "market price of risk". The HJM framework is very general and most of the work before 1988 can be viewed as special cases of it. Research on interest rate derivatives pricing after HJM has focused on specializing the framework to tractable models so that it can be efficiently calibrated to market prices of observable instruments and can efficiently price other interest rate derivatives.

Despite the efforts of many academics and practitioners, there is still no universal model for pricing interest rate derivatives. The challenges of interest rate derivative pricing are more computational than methodological. In order to obtain tractable sub-cases of the HJM model, the volatility structure has to be restricted. For realistic volatility structures, the HJM model is very expensive to calibrate. Furthermore, the valuation of Bermudan and American interest rate derivatives is computationally intensive. For such options, the computation time for even a single factor model is too slow to meet industry requirements. Although there are published results on certain versions of HJM, the issues of convergence, accuracy and efficiency have never been precisely documented. Realizing the theoretical advantages of the HJM approach, researchers have looked for an "ideal" special case of the general HJM model that possesses the following properties:

a. Arbitrage-free:

There must be no arbitrage in the price dynamics of bonds and other interest rate derivatives. This is the most fundamental requirement for any term structure model. As mentioned before, the HJM model solves the above problem and most of the current term structure models do enforce the no arbitrage condition in the above sense.

b. Versatility:

A good model should be consistent with a broad range of possible term structure shapes, volatility profiles, and correlations among yields of different maturity. The empirical studies of Canabarro (1995) demonstrate the inability of certain one factor models to account for observed yield correlations. In contrast, multifactor models such as HJM are consistent with arbitrary yield correlations.

c. Positive Interest Rates:

Although rare cases may require negative interest rates, it is widely accepted that a model that guarantees positive interest rates will be preferred over ones that do not, ceteris paribus. Despite the claims that the likelihood of negative interest rates is small by Gaussian modelers, Rogers (1995a,b) demonstrated that under certain circumstances, large negative interests can arise and cause non-negligible distortions of derivative prices. The positive term structure models in wide use are CIR, BDT, BK, nearly proportional HJM, BGM, and Flesaker and Hughston.

d. Computational Efficiency

Whether for model calibration, deal pricing, book revaluation, dynamic hedging, or firmwide risk management, the pricing model will be used many times during a day. Consequently, it is very critical that a model be able to evaluate derivatives efficiently. Closed form solutions for realistic problems are highly desired, although extremely rare in practice. Consequently, efficient numerical schemes are necessary for a valuation model to make the transition from theory to practice. In particular, one of the most computationally demanding problems is the pricing of Bermudan interest rate derivatives in a non-Markovian multi-factor model.

In general, the goal of term structure modeling is to find an “ideal” model that possesses the four features above. Recently, a special case of the HJM framework has emerged which satisfies most of the above criteria. This approach, termed the Market Model by its developers BGM (1995) and Jamshidian (1996a,b), is arbitrage-free, provides strictly positive interest rates, and generates closed form solutions for caps or European swaptions. Since models are usually calibrated to these instruments, this feature speeds up the calibration process considerably. Unfortunately, for Bermudan derivatives, extant applications of the Market Model are computationally demanding.

In this paper, we use simulation to develop a Markov Chain Approximation (MCA) to value Bermudan derivatives in the Market Model. We show that this approach allows such derivatives to be evaluated more efficiently than existing approaches such as non-recombining trees.

The structure of this paper is as follows. The next section reviews the Market Model, while the following section reviews our MCA approach. The fourth section applies MCA to the Market Model, while the following section presents our numerical implementation. The final section concludes.

2. The Market Model

A significant advance in the search for an “ideal” term structure model was made by Brace, Gatarek, and Musiela (1995) and Jamshidian(1996a,1996b). Conscious of structures trading in the market, they developed the so-called Market Model, which takes forward Libor rates or swap rates as inputs and directly models their evolution.. Although the Market Model can be viewed as a special case of the HJM model, it differs from traditional approaches in its use of the so-called “numeraire induced measure”, instead of the conventional risk-neutral measure. Assuming a deterministic forward rate volatility, the Market Model prices cap or European swaptions by the standard Black formula. Besides being consistent with the industry standard, the assumption of deterministic volatility guarantees positive interest rates.

Since we will price Bermudan derivatives using the Market Model, we now give a brief introduction to the model following Jamshidian’s (1996a) approach. We will not go into technical details such as how to change measure etc. but instead refer the reader to Jamshidian (1996a, 1996b) for the mathematical foundations of the Market Model.

We denote the sequence of Libor payment dates by T_n and the sequence of spot Libor reset dates by t_n. Assuming that the reset dates match the payment dates ($t_n = T_n$), we have a tenor structure:

$$0 < T_1 < T_2 < \ldots < T_{N+1} = T^* .$$

The day count factors are defined as:

$$\delta_n = T_{n+1} - T_n, \quad (1 \le n \le N).$$

For later use, we define $n(t)$ as one plus the number of payments as of date t:

$$n(t) = \{m : T_{m-1} < t \le T_m\} .$$

Let $B_n(t)$ be the price at t of a zero-coupon bond maturing at T_n for $n \ge 1$ and $0 \le t \le T_n$. Let $L_n(t)$ be the forward Libor rate for accrual period $[T_n, T_{n+1}]$ and let $S_{n,N}(t)$ the forward swap rate for a swap starting at T_n and maturing at T_{N+1}. Then we have:

$$L_n(t) = \delta_n^{-1}\left(\frac{B_n(t)}{B_{n+1}(t)} - 1\right) ,$$

$$B_{n+1}(t) = B_{n(t)}(t) \prod_{i=n(t)}^{n} \frac{1}{1 + \delta_i L_i(t)} ,$$

$$S_{n,N}(t) = \frac{B_n(t) - B_{N+1}(t)}{B_{n,N}(t)} ,$$

$$B_{n,N}(t) = \sum_{i=n}^{N} \delta_i B_{i+1}(t), \quad (t \le T_{n+1})$$

The Market Model can be generically represented by:

$$dK_n(t) = D_{num}(t, K_n(t))\, dt + \sum_{i=1}^{L} V_i(t, K_n(t))\, dz_{num}^i(t) .$$

Here, $K_n(t)$ can be either the Libor rate $L_n(t)$ or the swap rate $S_{n,N}(t)$, $D_{num}(t, K_n(t))$ is the drift term, $V^i(t, K_n(t))$ is the absolute volatility, and z_{num}^i is the Brownian motion associated with the measure induced by the numeraire $B_{num}(t)$. By specifying the rate

$K_n(t)$ and the numeraire $B_{num}(t)$, we can get the following three versions of the Market Model:

(1) Libor market model in spot Libor measure:

Introducing a "rolling zero-coupon bond"

$$B(t) = \frac{B_{n(t)}(t)}{B_1(0)} \prod_{n=1}^{n(t)-1} (1 + \delta_n L_n(T_n)) = B_{n(t)}(t) \prod_{n=2}^{n(t)} \frac{1}{B_n(T_{n-1})}$$

and taking $K_n(t) = L_n(t)$ and $B_{num}(t) = B(t)$, we have the following general equations governing the stochastic evolution of the forward Libor rate:

$$dL_n(t) = \sum_{i=n(t)}^{n} \frac{\delta_i \beta_i(t) \beta_i(t)^t}{1 + \delta_i L_i(t)} dt + \beta_n(t) dz_B(t).$$

The value of any Libor rate derivative satisfies:

$$C(t) = B(t) E_t^B [\frac{C(T)}{B(T)}] \qquad (t \le T).$$

Here, E^B is the expectation under the measure P^B induced by $B(t)$ (the spot Libor measure), and $z_B(t)$ is the associated Brownian motion (the spot Libor Brownian motion).

If $\lambda_n(t) = \frac{\beta_n(t)}{L_n(t)}$ is a bounded deterministic function, then the general equation reduces to the following "Libor market model in spot Libor measure":

$$\frac{dL_n(t)}{L_n(t)} = \sum_{i=n(t)}^{n} \frac{\delta_i \lambda_i(t) \lambda_n(t) L_i(t)}{1 + \delta_i L_i(t)} dt + \lambda_n(t) dz_B(t).$$

There exists a unique positive solution for $L_n(t)$ and caplets are priced by the Black formula.

(2) Libor market model in terminal measure:

Taking $K_n(t) = L_n(t)$ and $B_{num}(t) = B_n(t)$, we have:

$$dL_n(t) = \beta_n(t) dz_{n+1}(t) \quad (t \le T_n)$$

and

$$C(t) = B_n(t) E_t^n [\frac{C(T)}{B_n(T)}] \qquad (t \le T \le T_n).$$

Fixing $z_*(t) = z_{N+1}(t)$, we have

$$dL_n(t) = -\sum_{i=n+1}^{N} \frac{\delta_i \, \beta_i(t) \beta_i(t)^t}{1+\delta_i L_i(t)} dt + \beta_n(t) \, dz_*(t)$$

and

$$C(t) = B_*(t) E_t^* [\frac{C(T)}{B_*(T)}] \qquad (t \le T \le T_*).$$

Under the lognormal volatility assumption, we have

$$\frac{dL_n(t)}{L_n(t)} = -\sum_{i=n+1}^{N} \frac{\delta_i \, \lambda_i(t) \lambda_n(t) L_i(t)}{1+\delta_i L_i(t)} dt + \lambda_n(t) \, dz_*(t).$$

This is the Libor market model in terminal measure which BMG (1995) developed. There exists a unique positive solution and caplets are priced by the Black formula.

(3) Swap market model:

Taking $K_n(t) = S_{n,N}(t)$ and $B_{num}(t) = B_{n,N}(t)$, we have:

$$dS_{n,N}(t) = \phi_n(t) dz_{n,N}(t) \quad (t \le T_n) \text{ and}$$

$$C(t) = B_{n,N}(t) E_t^{n,N} [\frac{C(T)}{B_{n,N}(T)}] \qquad (t \le T \le T_n)$$

Under the lognormal volatility assumption, we have the following swap market model:

$$\frac{dS_{n,N}(t)}{S_{n,N}(t)} = \theta_n(t) dz_{n,N}(t) \quad (t \le T_n)$$

There exists a unique positive solution and now European swaptions are priced by the Black formula.

As with the general HJM model, the numeraire (the money market account in HJM) plays a very important role. Jamshidian(1996b) demonstrates that under appropriate measurability assumptions, payoffs which are a function of the path of Libor rates or swap rates can be attained by a self-financing trading strategy involving only the finite number of zero-coupon bonds that define the rates, even when the market is incomplete. This means:

$$C(t) = E_t^{num} (\sum_{i=1}^{N+1} N_{i_i}(T) B_i(T) / B_{num}(T)) \; B_{num}(t)$$

Here $B_{num}(t)$ is the numeraire, which can be either the spot Libor numeraire $B(t)$ or any of the zero-coupon bonds $B_i(t)$, or the forward swap rate numeraire $B_{n,N}(t)$.

In the special case of the pure discount bond, we get:

$$B_i(T) = E_T^{num}(1 / B_{num}(T_i))\, B_{num}(T) .$$

Thus, if one knows the dynamics of the numeraire $B_{num}(T)$ in the appropriate measure, then the value of the derivative, $C(t)$, is determined. In fact, in the Market Model, one needs to know the dynamics of the numeraire only at reset dates. This is fortunate because in the Market Model, the dynamics of the numeraire are not known inbetween reset dates.

The above property of the numeraire demonstrates a very general principle: the dynamics of the numeraire in the appropriate measure are all we need to price interest rate derivatives whose payoff can be attained by a self-financing trading strategy in the appropriate bonds. The focus on the risk-neutral dynamics of the spot rate when the money market account is the numeraire can be attributed to the above principle. We will see that this principle is the key to pricing Libor derivatives efficiently.

As mentioned previously, the Market Model is efficient in pricing caps or swaptions. There also exist other closed form solutions for some simple derivatives. However, for Bermudan derivatives, the market models face the same difficulties as the general HJM model. Although Bermudan derivatives can be evaluated by a non-recombining tree, the computational inefficiency of the method makes it almost useless in practice. For this reason, the next section discusses Markov Chain Approximation which will be used to efficiently price Bermudan derivatives.

3. Markov Chain Approximation (MCA)

MCA is a method for approximating a continuous time stochastic process. A graphical representation of the MCA used in this paper is given in figure 1. MCA includes, recombining (binomial, trinomial, and multinomial) lattices, non-recombining (binomial, trinomial, and multinomial) trees and the finite difference (explicit and implicit) schemes, spectral methods, and finite elements as special cases. Derivatives are valued in a MCA by the usual backward induction method.

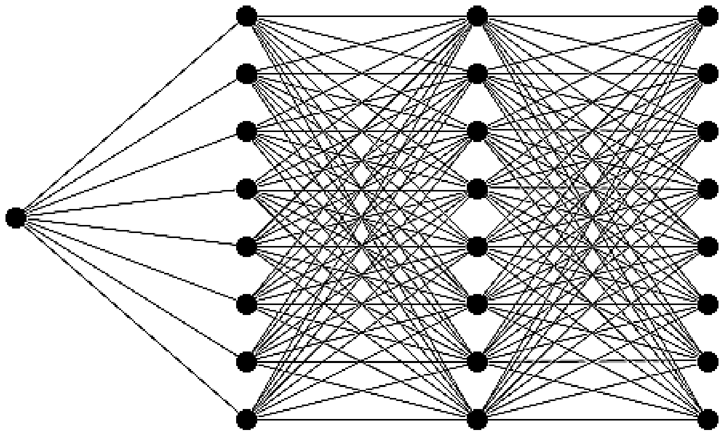

Figure 1. A Markov Chain Approximation (MCA)

A MCA can be fully described by the node values of the underlying and the transition probability matrix. For $t \in \{0, t_1, t_2, \ldots, t_{n-1}, t_n = T\}$, we represent the values of the underlying variable by

$\vec{r}_{i(t)}(t)$ (or more briefly $\vec{r}_i(t)$)

and the transition probability matrix by

$p_{i(t),j(t+1)}(t)$ (or more briefly $p_{i,j}(t)$) with $\sum_{j=0}^{I(t+1)} p_{i,j}(t) = 1$,

where $i(t) = 0,1,\ldots, I(t)$, $j(t+1) = 0,1,\ldots, I(t+1)$, $I(t) \geq 0$,

$\vec{r}_i(t) = (r_i^l(t) \in R)$, $l = 0,1,\ldots, L(i,t)$, $L(i,t) \geq 0$ and
$p_{i,j}(t) \geq 0$.

In its most general form, a MCA with n steps has $\sum_{k=1}^{n} \sum_{l=0}^{I(k)} L(l, t_k) + \sum_{k=0}^{n-1} (I(k)+1) * (I(k+1))$ degrees of freedom and the degrees of freedom go to infinity as n goes to infinity. In practice, we usually set $I(t)$ and $L(i,t)$ to constant or simple linear functions.

The conditional expectation of a function, $f(t, \vec{r})$, in terms of a MCA is represented by:

$$E(f(t+\Delta t, \vec{r}(t+\Delta t)) | \vec{r} = \vec{r}_i(t)) = \sum_{j=0}^{I(t+\Delta t)} f(t+\Delta t, \vec{r}_j) p_{i,j}(t)$$

Traditional numerical methods such as non-recombining trees have the problem of exponential growth in the number of nodes. The advantage of our MCA method lies in the fact that by having a transition matrix which is not sparse, we can control the growth of the number of nodes. The ability to control the growth of the number of nodes results in significant savings in computation time. While MCA also has a cost in terms of the extra computational effort required to get the transition matrix, this cost is very small compared to the resulting reduction in computational effort.

4. Approximating Market Model Values by MCA

In this section, we show how Monte Carlo simulation can be used to generate a MCA of derivative security values in the Market Model. Our focus will be on Bermudan derivatives since these are not efficiently valued by other techniques. The main difficulty in implementing the Market Model for Bermudan derivatives is that the model has a very high dimension, which is equal to the number of rates being modeled. In order to compute Bermudan prices efficiently, we collapse the dimensionality of the state space down to a single variable plus time. The single variable chosen is the numeraire, since as mentioned in section 2, one only needs the dynamics of this variable in the appropriate measure to price any interest rate derivative. In collapsing the dimensionality, the joint distribution of the rates and the numeraire is replaced with just the marginal distribution of just the numeraire. This marginal distribution is obtained by Monte Carlo simulation as detailed below. Since the exercise strategy is only allowed to depend on the level of the numeraire, our approximation is a lower bound for the true value of the Bermudan derivative. Thus, our approach is similar in spirit to the approach taken by Barraquand an Martineau (1995) for the valuation of multivariate American equity derivatives. In common with Barraquand and Martineau, it is difficult to know whether our lower bounds are tight. However, our lower bounds can still be used as a conservative estimate of value.
Furthermore, they serve as a benchmark for other methods which generate lower bounds on Bermudan values.

We first approximate the (forward or swap) rate diffusions using an explicit Euler scheme:

$$\Delta K_n(t) = D_{num}(t, K_n(t))\Delta t + \sum_{i=1}^{L} V_i(t, K_n(t))\Delta z_{num}^i(t)$$

with $\Delta K_n(t) = K_n(t+\Delta t) - K_n(t)$, $\Delta z_{num}^i(t) = z_{num}^i(t+\Delta t) - z_{num}^i(t)$ and $\forall t \in \{0, \Delta t, \ldots, T\}$.

Next, we discretize the numeraire space. More specifically, $\forall t \in \{0, T_1, \ldots, T_{N+1}\}$ choose a finite partition, $R(t) = (R_1(t), \ldots, R_{k(t)}(t))$, of the numeraire state space R , i.e. a set of $k(t)$ subsets of R satisfying:

$$\bigcup_{i \in [1,k(t)]} R_i(t) = R \quad \text{and} \quad \forall (i,j) \in [1,k(t)]^2, i \neq j, R_i(t) \cap R_j(t) = \emptyset$$

The time grid for the MCA differs from that for the simulated rate equations since for Bermudan derivatives, we only need a MCA at the reset dates. We assume that the partition $R(0)$ has only two cells:

$R_1(0) = \{B_{num}(0)\}$, and $R_2(0) = R \setminus \{B_{num}(0)\}$

Once the M sample paths $k^1(t),\ldots,k^M(t)$ are computed, $a_i(t)$, the number of samples entering $R_i(t)$, $b_{i,j}(t)$, the number of samples moving from $R_i(t)$ to $R_j(t+\Delta t)$, and $B_{num}^{sum}(t)$, the sums of sample numeraire values within the cell $R_i(t)$ are easily computed. The entries in the transition probability matrix can be calculated as:

$$p_{i,j}(t) = \frac{b_{i,j}(t)}{a_i(t)}$$

the average value of the numeraire within $R_i(t)$, $B_{num}^i(t)$, is

$$B_{num}^i(t) = \frac{B_{num}^{sum}(t)}{a_i(t)}$$

With the calculated values of $p_{i,j}(t)$ and $B_{num}^i(t)$, a well defined MCA is generated. The conditional expectation in terms of a MCA is represented by:

$$E^{num}(X(t+\Delta t, B_{num}(t+\Delta t)) | B_{num} = B_{num}^i(t)) = \sum_{j=0}^{I(t+\Delta t)} X(t+\Delta t, B_{num}^j) p_{i,j}(t)$$

Contingent claims can be evaluated from the following formula:

$$C(t, B_{num}^i(t)) = B_{num}(t) * E^{num}\left(\frac{C(t+\Delta t, B_{num}(t+\Delta t))}{B_{num}(t+\Delta t)} | B_{num} = B_{num}^i(t)\right)$$

for European securities, and

$$C(t, B_{num}^i(t)) = \max\left(C(t)_{ex}, B_{num}(t) * E^{num}\left(\frac{C(t+\Delta t, B_{num}(t+\Delta t))}{B_{num}(t+\Delta t)} | B_{num} = B_{num}^i(t)\right)\right)$$

for American securities, where $C(t)_{ex}$ is the exercise value of the American security.

We use one of two approaches for valuing a derivative depending on the nature of the claim. If the derivative security's payoff is insensitive to the variation in term structures, then we obtain the average yield curve conditional on the numeraire being in a specified cell by averaging over all term structures that happen to have the numeraire value in the given cell. The yield curve corresponding to $B_{num}^i(t)$ can be obtained by:

$$f_i^n(t) = \frac{f_{sum}^n(t; B_{num}^i(t))}{a_i(t)},$$

where $f_{sum}^n(t; B_{num}^i(t))$ is the sum of all term structures with the numeraire in cell $B_{num}^i(t)$.

The above method for constructing the yield curve is efficient in obtaining the average yield curve and in pricing several derivatives using one simulation. However, for derivatives that are sensitive to variation in the term structure, the above method can cause errors. For one factor models and for derivatives on rates which are highly correlated with the numeraire, our numerical results show that averaging over term structures is very accurate. However, for claims whose payoff is tied to a specified rate quite different from those which drive the numeraire, we use a different approach. Specifically, for each cell describing a small range in value of the numeraire, we average the payoff of the security over all the term structures whose corresponding numeraire was in the specified cell. We benchmarked our numerical results by comparing them with the traditional Monte Carlo method for European derivatives which are sensitive to variation in the shape of the term structure, eg. swaptions and caps in a 2 factor model.

5 Numerical Results

We applied our MCA method to price caps and swaptions in both a single factor and a two factor setting. The payer swaptions we value are (1) European swaptions, (2) Bermudan swaptions, (3) Fixed tail Bermudan swaptions, (4) Constant maturity Bermudan swaptions, and (5) Bermudan forward swaptions. Since the terminology in (2) to (5) may not be standard, we now give a precise definition of each swaption.

Bermudan Swaptions:
A Bermudan swaption is an option on a swap that can be exercised on every reset date of the underlying swap up to and including the option maturity. When exercised, the swap one gets is one that starts at the exercise date and matures at a fixed date which is independent of the time of exercise. A Bermudan swaption is equivalent to a Bermudan option on a coupon bond with strike that is the par value of the bond. As an option on a coupon bond, a Bermudan swaption has positive probability of early exercise.

Fixed tail Bermudan Swaptions:
A fixed tail Bermudan swaption is a Bermudan swaption with a maturity date equal to the last reset date of the underlying swap and which has an initial lockout period during which exercise is prohibited. A fixed tail Bermudan swaption is equivalent to a Bermudan option on a coupon bond with strike that is the par value of the bond. As an option on a coupon bond, a Bermudan Fixed tail swaption has positive probability of early exercise.

Constant maturity Bermudan Swaptions:

A constant maturity swaption is an option on a swap that can be exercised on every reset date from now up to and including the swaption maturity. When exercised, the swap one gets is one that starts at the exercise date and has a time to maturity which is independent of the time of exercise. A constant maturity Bermudan swaption is equivalent to a Bermudan option on a constant maturity coupon bond with a strike that is the par value of the bond. Since the underlying bond pays coupons, there is positive probability of early exercise.

Bermudan forward Swaptions:
A Bermudan forward Swaption is an option on a swap that can be exercised on every reset date up to and including the swaption maturity. When exercised, the swap one gets is a forward swap that starts at the swaption maturity date and ends a specified period of time later. A Bermudan forward swaption is equivalent to an option to exchange a forward floating rate bond for a forward fixed rate bond. Since the underlying pays no cash flows between initiation and the swaption maturity, a Bermudan forward swaption should not be exercised before maturity.

The confidence intervals of our results are calculated using the central limit theorem. Based on the theorem, the error must be less than four times the observed standard deviation in order to obtain a 99.95 percent confidence interval. The observed standard deviations are obtained by doing 100 runs of the option price calculations with each run having a sample size of 100,000.

Our numerical results for caps are compared with closed form solutions. For European swaptions, our results are very close to the Black formula and to BMG's approximation formula in the one factor case. For the two factor model, we can only compare our European swaption values to those obtained by traditional Monte Carlo simulation. For Bermudan swaptions, even traditional Monte Carlo simulation cannot be used to check the validity of our calculations. However, we compared our results with traditional Monte Carlo simulation of European swaption valuations and found that Bermudan swaption valuations are never smaller then European values as expected..

The first test is for a one factor model. The initial term structure is flat at 10%. The volatility structure is also flat at $\lambda_1(t) = 20\%$. This is the same one factor model used by BMG(1995) to test their approximation formula. The numerical results for caps are shown in Table 1. As we can see, our MCA results agree with the closed form solution to 4-5 digits depending on the cap maturity. This close agreement with the closed solution is a strong indication that the MCA can approximate the Market Model accurately.

Table 2 shows our results for European swaptions. As shown in BMG (1995) , the Black formula is a very accurate approximation for European swaptions in this one factor model. Our MCA results are in good agreement with both the Black formula and the BMG(1995) approximation formula.

The results for Bermudan forward swaptions are given in Table 3. As we stated in the description of Bermudan forward swaptions, the value of a Bermudan forward swaption should be the same as that of the corresponding European swaption. Our numerical results confirmed this prediction. The differences between values of the European swaptions and Bermudan forward swaptions are within the Monte Carlo simulation errors.

The Bermudan constant maturity swaption calculations are listed in Table 4. As we expected, constant maturity Bermudan swaptions have larger value than European swaptions. The early exercise premium for constant maturity Bermudan swaptions increases with the swaption maturity, but decreases with the strike rate. For the values tested, the maturity of the underlying swap does not seem to have a significant effect on the exercise premium.

The Bermudan swaption calculations are given in Table 5. As we expected, Bermudan swaptions have larger value than European swaptions. The early exercise premium for Bermudan swaptions increases with the swaption maturity and decreases with the strike rate and with the maturity of the underlying swap. The early exercise premium is about 55% of the corresponding European swaption for a 3 year at-the-money Bermudan swaption on a 3 year swap. For the values tested, we found that a Bermudan swaption always has greater value than a constant maturity Bermudan swaption with the same swaption maturity date.

Table 6 shows our results for Bermudan fixed tail swaptions. As expected, Bermudan fixed tail swaptions have greater value than European swaptions maturing at the end of the lockout period. We term this difference the later exercise premium of a Bermudan fixed tail swaption. This premium increases with the strike rate and the underlying swap maturity. This is in contrast to Bermudan swaption whose early exercise premium decreases with the strike rate and swap maturity. The later exercise premium for a one year Bermudan fixed tail swaption on a 10 year swap struck at 12% is about 325% of the corresponding European swaption value.

As we have seen, our numerical results for cap and European swaptions are very accurate. The Bermudan type swaptions have no data to compare with, but our calculations are consistent with known bounds. These results indicate that our MCA is a viable technique in the one factor Market Model.

Cap Price (One factor)				
Option Maturity	Strike	MC price		Black price
		value	4*stdv	
1	8%	0.004684	0.000002	0.004681
	10%	0.001762	0.000008	0.001760
	12%	0.000473	0.000004	0.000474
2	8%	0.004625	0.000004	0.004621
	10%	0.002253	0.000009	0.002251
	12%	0.000967	0.000004	0.000967
3	8%	0.004506	0.000006	0.004501
	10%	0.002497	0.000011	0.002494
	12%	0.001297	0.000005	0.001296
4	8%	0.004342	0.000009	0.004336
	10%	0.002609	0.000012	0.002604
	12%	0.001512	0.000007	0.001510
5	8%	0.004149	0.000010	0.004142
	10%	0.002640	0.000014	0.002634
	12%	0.001645	0.000009	0.001641
6	8%	0.003941	0.000012	0.003930
	10%	0.002618	0.000015	0.002609
	12%	0.001721	0.000011	0.001714
7	8%	0.003726	0.000011	0.003711
	10%	0.002562	0.000015	0.002549
	12%	0.001753	0.000012	0.001743
8	8%	0.003510	0.000010	0.003490
	10%	0.002482	0.000014	0.002465
	12%	0.001755	0.000015	0.001740
9	8%	0.003297	0.000012	0.003271
	10%	0.002388	0.000015	0.002365
	12%	0.001734	0.000014	0.001714

Table 1 Cap Price

European Swaption Price (one factor)					
Option maturity xSwap Length	Strike	MC price		Black price	BMG price
		value	4*stdv		
.25x1	8%	0.018391	0.000004	0.018388	0.018388
	10%	0.003660	0.000017	0.003659	0.003659
	12%	0.000131	0.000009	0.000135	0.000135
1x2	8%	0.034426	0.000018	0.034405	0.034405
	10%	0.012946	0.000062	0.012936	0.012935
	12%	0.003477	0.000029	0.003487	0.003487
1x5	8%	0.074844	0.000043	0.074802	0.074797
	10%	0.028138	0.000140	0.028124	0.028114
	12%	0.007553	0.000058	0.007582	0.007573
1x10	8%	0.120499	0.000078	0.120452	0.120419
	10%	0.045258	0.000227	0.045288	0.045220
	12%	0.012120	0.000085	0.012208	0.012160
3x3	8%	0.047377	0.000073	0.047329	0.047321
	10%	0.026246	0.000130	0.026220	0.026209
	12%	0.013629	0.000075	0.013627	0.013617

Table 2 European Swaption Price

Bermudan Forward Swaption Price (one factor)						
Option maturity	Strike	Bermudan Forward Price			European Price	
xSwap Length		value	normalized value	4*stdv	value	4*stdv
.25x1	8%	0.018391	1.000000	0.000004	0.018391	0.000004
	10%	0.003660	1.000000	0.000017	0.003660	0.000017
	12%	0.000131	1.000000	0.000009	0.000131	0.000009
1x2	8%	0.034631	1.005963	0.000080	0.034426	0.000018
	10%	0.013033	1.006699	0.000085	0.012946	0.000062
	12%	0.003509	1.009205	0.000035	0.003477	0.000029
1x5	8%	0.075274	1.005742	0.000172	0.074844	0.000043
	10%	0.028319	1.006439	0.000183	0.028138	0.000140
	12%	0.007619	1.008696	0.000074	0.007553	0.000058
1x10	8%	0.121158	1.005474	0.000276	0.120499	0.000078
	10%	0.045536	1.006142	0.000283	0.045258	0.000227
	12%	0.012218	1.008107	0.000113	0.012120	0.000085
3x3	8%	0.047786	1.008631	0.000098	0.047377	0.000073
	10%	0.026513	1.010182	0.000166	0.026246	0.000130
	12%	0.013797	1.012311	0.000117	0.013629	0.000075

Table 3 Bermudan Forward Swaption Price

Bermudan Constant Maturity Swaption Price (one factor)						
Option maturity	Strike	Bermudan Constant Maturity Price			European Price	
xSwap Length		value	normalized value	4*stdv	value	4*stdv
.25x1	8%	0.018810	1.022767	0.000000	0.018391	0.000004
	10%	0.003660	1.000000	0.000017	0.003660	0.000017
	12%	0.000131	1.000000	0.000009	0.000131	0.000009
1x2	8%	0.036330	1.055312	0.000093	0.034426	0.000018
	10%	0.013291	1.026647	0.000094	0.012946	0.000062
	12%	0.003542	1.018631	0.000039	0.003477	0.000029
1x5	8%	0.079003	1.055572	0.000198	0.074844	0.000043
	10%	0.028893	1.026856	0.000198	0.028138	0.000140
	12%	0.007691	1.018296	0.000084	0.007553	0.000058
1x10	8%	0.127244	1.055982	0.000306	0.120499	0.000078
	10%	0.046491	1.027234	0.000301	0.045258	0.000227
	12%	0.012337	1.017953	0.000132	0.012120	0.000085
3x3	8%	0.056640	1.195515	0.000137	0.047377	0.000073
	10%	0.029765	1.134058	0.000172	0.026246	0.000130
	12%	0.015022	1.102187	0.000164	0.013629	0.000075

Table 4 Constant Maturity Bermudan Swaption

Bermudan Swaption Price (one factor)						
Option maturity	Strike	Bermudan Price			European Price	
xSwap Length		value	normalized value	4*stdv	value	4*stdv
.25x1	8%	0.023229	1.263061	0.000000	0.018391	0.000004
	10%	0.003660	1.000000	0.000017	0.003660	0.000017
	12%	0.000131	1.000000	0.000009	0.000131	0.000009
1x2	8%	0.051289	1.489841	0.000000	0.034426	0.000018
	10%	0.015257	1.178543	0.000088	0.012946	0.000062
	12%	0.003753	1.079286	0.000049	0.003477	0.000029
1x5	8%	0.089425	1.194818	0.000000	0.074844	0.000043
	10%	0.030104	1.069877	0.000189	0.028138	0.000140
	12%	0.007812	1.034346	0.000095	0.007553	0.000058
1x10	8%	0.132519	1.099758	0.000000	0.120499	0.000078
	10%	0.047133	1.041425	0.000297	0.045258	0.000227
	12%	0.012399	1.023020	0.000139	0.012120	0.000085
3x3	8%	0.089425	1.887509	0.000000	0.047377	0.000073
	10%	0.040717	1.551336	0.000186	0.026246	0.000130
	12%	0.018742	1.375147	0.000174	0.013629	0.000075

Table 5 Bermudan Swpation

Fixed Tail Bermudan Swaption Price (one factor)						
Lockout Period	Strike	Bermudan Fixed Tail Price			European Price	
xSwap Length		value	normalized value	4*stdv	value	4*stdv
.25x1	8%	0.018471	1.004364	0.000008	0.018391	0.000004
	10%	0.004918	1.343553	0.000019	0.003660	0.000017
	12%	0.000874	6.657649	0.000012	0.000131	0.000009
1x2	8%	0.035589	1.033785	0.000043	0.034426	0.000018
	10%	0.015750	1.216607	0.000055	0.012946	0.000062
	12%	0.006181	1.777506	0.000062	0.003477	0.000029
1x5	8%	0.081121	1.083870	0.000185	0.074844	0.000043
	10%	0.041987	1.492176	0.000170	0.028138	0.000140
	12%	0.021581	2.857282	0.000187	0.007553	0.000058
1x10	8%	0.139524	1.157886	0.000325	0.120499	0.000078
	10%	0.083011	1.834168	0.000285	0.045258	0.000227
	12%	0.051185	4.223304	0.000358	0.012120	0.000085
3x3	8%	0.049547	1.045802	0.000106	0.047377	0.000073
	10%	0.029514	1.124512	0.000100	0.026246	0.000130
	12%	0.017110	1.255417	0.000106	0.013629	0.000075

Table 6 Bermudan Fixed Tail Swaption

Having successfully simulated the one factor case, we now extend our MCA method to the two factor Market Model. The initial term structure is still flat at 10%. The two factor volatility structure we tested is defined by:

$$\lambda_1(t) = .15$$

$$\lambda_2(t) = .15 - \frac{3}{\sqrt{10}}\sqrt{T-t},\ T \geq t$$

The first factor is a constant shock for all maturities, while the second factor adds a twist to the yield term structure.

The numerical results for caps are shown in Table 7. As in the one factor case, our MCA results agree with the Black formula to 4-5 digits. This close agreement with the closed form solution in the two factor case is another indication that the MCA can approximate the Market Model accurately. The values of caps in the two factor case are all smaller than that of the one factor case. This indicates that the overall volatility of the two factor case is smaller.

Table 8 shows our results for European swaptions. Unlike the one factor case, the Black formula is not a good approximation for European swaptions. This is consistent with Jamshidian’s conclusion that we can not generally have a lognormal Libor rate and a lognormal swap rate at the same time. Our numerical results for European swaptions show that the Black formula overprices European swaptions. As a consistency check, we found that our MCA results for European swaptions are the same as the traditional Monte

Carlo results (not reported here). As was true for caps, European swaption values in the two factor case are all smaller than in the one factor case.

The Bermudan forward swaption results are given in Table 9. As in the one factor case, the value of Bermudan forward swaptions should be the same as that of the European swaptions. Our numerical results confirmed this result. The differences between values of the European swaptions and Bermudan forward swaptions are within the Monte Carlo simulation errors.

Constant maturity Bermudan swaption calculations are listed in Table 10. As expected, the values of constant maturity Bermudan swaptions are larger than that of European swaptions. The early exercise premium for constant maturity Bermudan swaptions increases with the option maturity but decreases with the strike rate. The early exercise premium in the two factor case is more sensitive to the strike rate than in the one factor case and is generally larger especially when the option maturity is longer.

Bermudan swaption calculations are given in Table 11. As in the one factor case, Bermudan swaptions have larger values than European swaptions. The early exercise premium for Bermudan swaptions increases with the option maturity and decreases with the strike rate and the swap maturity. The early exercise premium is about 51% of the corresponding European swaption value for a 3 year at-the-money Bermudan swaption on 3 year swap. A Bermudan swaption has greater value than a constant maturity Bermudan swaption. Again, the early exercise premium in the two factor case is more sensitive to the strike rate. For our values, Bermudan swaptions were all exercised at a strike rate of 8% or more in both the one and two factor case.

Table 12 shows the results for Bermudan fixed tail swaptions. As expected, Bermudan fixed tail swaptions have greater value than European swaptions maturing at the lockout date. The later exercise premium of Bermudan fixed tail swaptions increases with the strike rate and swap maturity. This is in contrast to Bermudan swaptions whose early exercise premium decreases with the strike rate. The later exercise premium for a one year Bermudan fixed tail swaption on a 10 year swap struck at 12% is about 170.3% of the corresponding European swaption. As with other Bermudan type swaptions, Bermudan fixed tail swaptions in the two factor case are also more sensitive to the strike rate than in the one factor case. The later exercise premium in the two factor case can be more than twice as big as in the one factor case for a strike rate of 12%.

Cap Price (Two factor)				
Option Maturity	Strike	MC price		Black price
		value	4*stdv	
1	8%	0.004588	0.000002	0.004584
	10%	0.001529	0.000006	0.001538
	12%	0.000311	0.000003	0.000323
2	8%	0.004385	0.000003	0.004379
	10%	0.001853	0.000008	0.001857
	12%	0.000622	0.000004	0.000631
3	8%	0.004168	0.000003	0.004160
	10%	0.002000	0.000010	0.002000
	12%	0.000836	0.000004	0.000841
4	8%	0.003950	0.000005	0.003940
	10%	0.002069	0.000009	0.002066
	12%	0.000990	0.000005	0.000993
5	8%	0.003736	0.000005	0.003727
	10%	0.002096	0.000010	0.002092
	12%	0.001107	0.000005	0.001107
6	8%	0.003531	0.000007	0.003524
	10%	0.002096	0.000010	0.002092
	12%	0.001195	0.000007	0.001194
7	8%	0.003336	0.000007	0.003331
	10%	0.002078	0.000011	0.002075
	12%	0.001261	0.000007	0.001260
8	8%	0.003150	0.000008	0.003149
	10%	0.002047	0.000012	0.002046
	12%	0.001309	0.000008	0.001309
9	8%	0.002974	0.000009	0.002977
	10%	0.002005	0.000013	0.002007
	12%	0.001341	0.000010	0.001343

Table 7 Cap Price (two factor case)

European Swaption Price (two factor)				
Option maturity xSwap Length	Strike	MC price		Black price
		value	4*stdv	
.25x1	8%	0.018370	0.000002	0.018377
	10%	0.003157	0.000014	0.003502
	12%	0.000046	0.000005	0.000105
1x2	8%	0.033313	0.000016	0.033688
	10%	0.010146	0.000049	0.011305
	12%	0.001652	0.000025	0.002376
1x5	8%	0.072140	0.000040	0.073245
	10%	0.021185	0.000111	0.024579
	12%	0.003121	0.000053	0.005166
1x10	8%	0.116589	0.000066	0.117944
	10%	0.035351	0.000179	0.039579
	12%	0.005584	0.000070	0.008319
3x3	8%	0.043184	0.000030	0.043744
	10%	0.020065	0.000097	0.021029
	12%	0.007957	0.000033	0.008843

Table 8 European Swaption Price (two factor case)

Bermudan Forward Swaption Price (two factor)						
Option maturity	Strike	Bermudan Forward Price			European Price	
xSwap Length		value	normalized value	4*stdv	value	4*stdv
.25x1	8%	0.018370	1.000000	0.000002	0.018370	0.000002
	10%	0.003157	1.000000	0.000014	0.003157	0.000014
	12%	0.000046	1.000000	0.000005	0.000046	0.000005
1x2	8%	0.033510	1.005908	0.000065	0.033313	0.000016
	10%	0.010206	1.005974	0.000067	0.010146	0.000049
	12%	0.001665	1.007784	0.000026	0.001652	0.000025
1x5	8%	0.072411	1.003764	0.000128	0.072140	0.000040
	10%	0.021249	1.003004	0.000131	0.021185	0.000111
	12%	0.003135	1.004647	0.000055	0.003121	0.000053
1x10	8%	0.116909	1.002739	0.000196	0.116589	0.000066
	10%	0.035400	1.001371	0.000192	0.035351	0.000179
	12%	0.005598	1.002565	0.000075	0.005584	0.000070
3x3	8%	0.043512	1.007606	0.000096	0.043184	0.000030
	10%	0.020245	1.008935	0.000126	0.020065	0.000097
	12%	0.008051	1.011753	0.000066	0.007957	0.000033

Table 9 Bermudan Forward Swaption Price (two factor case)

Bermudan Constant Maturity Swaption Price (two factor)						
Option maturity	Strike	Bermudan Constant Maturity Price			European Price	
xSwap Length		value	normalized value	4*stdv	value	4*stdv
.25x1	8%	0.018810	1.023971	0.000000	0.018370	0.000002
	10%	0.003157	1.000000	0.000014	0.003157	0.000014
	12%	0.000046	1.000000	0.000005	0.000046	0.000005
1x2	8%	0.036027	1.081476	0.000076	0.033313	0.000016
	10%	0.010792	1.063704	0.000070	0.010146	0.000049
	12%	0.001719	1.040550	0.000032	0.001652	0.000025
1x5	8%	0.077946	1.080483	0.000000	0.072140	0.000040
	10%	0.021865	1.032077	0.000150	0.021185	0.000111
	12%	0.003169	1.015335	0.000058	0.003121	0.000053
1x10	8%	0.125514	1.076546	0.000000	0.116589	0.000066
	10%	0.035715	1.010297	0.000223	0.035351	0.000179
	12%	0.005615	1.005531	0.000079	0.005584	0.000070
3x3	8%	0.054716	1.267052	0.000119	0.043184	0.000030
	10%	0.024056	1.198881	0.000133	0.020065	0.000097
	12%	0.009135	1.147994	0.000101	0.007957	0.000033

Table 10 Bermudan Constant Maturity Swaption (two factor case)

Bermudan Swaption Price (two factor)						
Option maturity	Strike	Bermudan Price			European Price	
xSwap Length		value	normalized value	4*stdv	value	4*stdv
.25x1	8%	0.023229	1.264547	0.000000	0.018370	0.000002
	10%	0.003157	1.000000	0.000014	0.003157	0.000014
	12%	0.000046	1.000000	0.000005	0.000046	0.000005
1x2	8%	0.051289	1.539600	0.000000	0.033313	0.000016
	10%	0.012079	1.190541	0.000079	0.010146	0.000049
	12%	0.001767	1.069185	0.000036	0.001652	0.000025
1x5	8%	0.089425	1.239606	0.000000	0.072140	0.000040
	10%	0.022315	1.053328	0.000154	0.021185	0.000111
	12%	0.003174	1.017086	0.000060	0.003121	0.000053
1x10	8%	0.132519	1.136632	0.000000	0.116589	0.000066
	10%	0.035788	1.012356	0.000223	0.035351	0.000179
	12%	0.005615	1.005595	0.000079	0.005584	0.000070
3x3	8%	0.089425	2.070799	0.000000	0.043184	0.000030
	10%	0.030301	1.510138	0.000171	0.020065	0.000097
	12%	0.010303	1.294841	0.000135	0.007957	0.000033

Table 11 Bermudan Swaption (two factor case)

Fixed Tail Bermudan Swaption Price (two factor)						
Lockout Period	Strike	Bermudan Fixed Tail Price			European Price	
xSwap Length		value	normalized value	4*stdv	value	4*stdv
.25x1	8%	0.018416	1.002517	0.000007	0.018370	0.000002
	10%	0.004327	1.370468	0.000019	0.003157	0.000014
	12%	0.000550	11.930164	0.000010	0.000046	0.000005
1x2	8%	0.034097	1.023539	0.000036	0.033313	0.000016
	10%	0.012695	1.251323	0.000053	0.010146	0.000049
	12%	0.003710	2.245603	0.000045	0.001652	0.000025
1x5	8%	0.075004	1.039703	0.000115	0.072140	0.000040
	10%	0.031667	1.494727	0.000144	0.021185	0.000111
	12%	0.012792	4.098874	0.000117	0.003121	0.000053
1x10	8%	0.122095	1.047221	0.000330	0.116589	0.000066
	10%	0.060220	1.703479	0.000377	0.035351	0.000179
	12%	0.032336	5.791074	0.000305	0.005584	0.000070
3x3	8%	0.044723	1.035652	0.000074	0.043184	0.000030
	10%	0.022819	1.137231	0.000098	0.020065	0.000097
	12%	0.010765	1.352899	0.000081	0.007957	0.000033

Table 12 Bermudan Fixed Tail Swaption Price (two factor case)

6 Summary

The Market Model by BMG and Jamshidian is a widely used model in interest rate derivative markets. However, its application to Bermudan derivatives has been problematic due to the explosive growth in computation time and memory of non-recombining trees. In this paper, we applied the MCA method for pricing Bermudan interest rate derivatives using Monte Carlo simulation. One and two factor volatility structures were studied numerically. Numerical data for several types of Bermudan swaptions were presented as a benchmark for comparison with alternative implementations of the Market model. In particular, the efficacy of the MCA method for long maturity derivatives was demonstrated.

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