SIAM J. FINANCIAL MATH.
Vol. 1, pp. 2–15

Local Volatility Enhanced by a Jump to Default*

Peter Carr† and Dilip B. Madan‡

Abstract. A local volatility model is enhanced by the possibility of a single jump to default. The jump has a hazard rate that is the product of the stock price raised to a prespecified negative power and a deterministic function of time. The empirical work uses a power of -1.5. It is shown how one may simultaneously recover from the prices of credit default swap contracts and equity option prices both the deterministic component of the hazard rate function and revised local volatility. The procedure is implemented on prices of credit default swaps and equity options for General Motors and the Ford Motor Company over the period October 2004 to September 2007.

Key words. recovering default free option prices, truncated power prices, Weibull distribution, default adjusted drifts

AMS subject classifications. 60G99, 60J60, 60J75

DOI. 10.1137/090750731

1. Introduction.

The development of liquid markets in credit default swaps (CDSs) has made it clear that some of the value of deep downside put options must be due to the probability of default rendering equity worthless. Stock price models that assume a strictly positive price for equity potentially overstate downside volatilities to compensate for the assumed absence of the default event in the model. To rectify this situation one needs to recognize the default possibility in the stock price model. This could be done in a variety of parametric models [9], [2], [1], [14], [3], [4], and one could then seek to infer the parameters of the model from traded option prices. Having done so, one would extract in principle a risk neutral default probability from option prices that could then be compared to a similar probability obtained from the prices of CDSs.

Alternatively one could recognize that option prices constitute an indirect assessment of default, while the CDS is clearly more directly focused on this event. This suggests that we jointly employ data on CDSs and equity options to simultaneously infer the risk neutral stock dynamics in the presence of default as a likely event. Such an approach is called for all the more in the context of local volatility stock price dynamics [12], [11] that introduce a two dimensional local volatility surface describing the instantaneous volatility of the innovations in the stock price as a deterministic function of the stock price and calendar time. All the equity option prices are then needed to infer the local volatility surface, and one needs other traded assets to infer the parameters related to the default likelihood. Given the popularity of

*Received by the editors February 24, 2009; accepted for publication (in revised form) August 10, 2009; published electronically January 21, 2010.
http://www.siam.org/journals/sifin/1/75073.html
†Bloomberg LP and Courant Institute, New York University, New York, NY 10022 (pcarr4@bloomberg.com).
‡Robert H. Smith School of Business, University of Maryland, College Park, MD 20742 (dbm@rhsmith.umd.edu).
This author's work was partially supported by the Financial Integrity Research Network, an ARC network, during his term in Australia as a FIRN visitor.

local volatility in assessing equity risks, we consider enhancing this model to include a default event. For other enhancements that have recently been considered we refer the reader to Ren, Madan, and Qian [17].

One could seek to perform such an enhancement by allowing the stock to diffuse to zero and to then be absorbed there. There are such models referenced above that lack a local volatility structure, and one may seek to enhance these with a local volatility formulation. However, it has long been recognized that a purely diffusive formulation has difficulties with credit spreads especially at the shorter maturities. Hence the popularity of reduced form credit models [13] that incorporate jumps in the asset price. In the interests of parsimony we introduce just a single jump to default with a hazard rate or instantaneous default probability that is just a function of the stock price and time. Unlike local volatility with its eye on the two dimensional surface of implied volatilities, we recognize that our additional assets are at best the one dimensional continuum of CDS quotes and so we model the hazard rate as a product of two functions, one a deterministic function of calendar time, while the other is just given by the stock price raised to a negative power. The latter recognizes the intuition that default gets more likely as equity values drop.

With this enhancement we show how one may use CDS and equity option prices jointly to infer the deterministic function of calendar time in the hazard rate and the local volatility surface of this default extended local volatility model for a prespecified level of the elasticity of hazard rates to the stock price. The procedures developed are implemented on CDS and equity option prices for the Ford Motor Company and General Motors (GM) covering the period October 2004 to September 2007.

It is observed that on average the elasticity of the proportion of traditional local volatility attributed to the default component declines with respect to strike and is $-.4768$ and $-.3967$ for GM and Ford, respectively. This proportion is positively related to maturity with elasticities of .0646 and .0504 for GM and Ford, respectively.

Additionally the deterministic component of the hazard function is generally increasing with maturity and has semielasticities and elasticities of .1306 and .2163 for GM and .0975 and .2641 for Ford.

The outline of the rest of the paper is as follows. Section 2 presents the formal model for the stock price evolution. In section 3 we develop the equations for the deterministic component of the hazard rate and the default enhanced local volatility surface. Section 4 presents the methods employed to evaluate truncated power prices in the embedded default free dynamics that are needed for the local volatility construction. Section 5 presents the application of the methods to Ford and GM and summarizes the results. Section 6 concludes the paper.

2. Local volatility enhanced by a single jump to default.

We enhance a local volatility formulation for the risk neutral evolution of the stock price by including a single jump to default. The jump occurs at random time given by the first time a counting process $N(t)$ jumps by unity. Thereafter we set the arrival rate of jumps to zero, and the process $N(t)$ remains frozen at unity. Prior to the jump in the process $N(t)$ it has an instantaneous arrival rate $\lambda(t)$ of a jump given by the product of a deterministic function of time $f(t)$ and the stock price raised to the power p, which will be a negative number in our applications:

$$\lambda(t) = (1 - N(t_{-}))\, f(t) S(t_{-})^{p}.$$

This model for the default intensity in the context of a constant volatility was previously considered and solved in closed form by Linetsky [14]. An extension to the context of a constant elasticity of variance for the volatility specification was solved by Carr and Linetsky [8]. We generalize here to the local volatility context.

Let $\sigma(S,t)$ denote the volatility in the stock price when the stock price is at level S at time t and default has not yet occurred. The stock price dynamics may then be written as

$$\begin{aligned} dS &= (r(t)-q(t))S(t_{-})dt + \sigma(S(t_{-}),t)S(t_{-})dW(t) \\ &\quad - S(t_{-})\left[dN(t) - (1-N(t_{-}))f(t)S(t_{-})^{p}dt\right], \end{aligned} \tag{1}$$

where $W(t)$ is a Brownian motion and $r(t), q(t)$ are deterministic continuously compounded interest rates and dividend yields, respectively.

Equation (1) may be explicitly solved in terms of the product of two martingales, one continuous, $M(t)$, and the other, $J(t)$, that is continuous until it jumps to zero and then stays there.

The stock price may also be written as

$$\begin{aligned} S(t) &= A(t)M(t)J(t), \\ A(t) &= S(0)\exp\left(\int_0^t (r(u)-q(u))du\right), \\ M(t) &= \exp\left(\int_0^t \sigma(S(u_{-},u))S(u_{-})dW(u) - \frac{1}{2}\int_0^t \sigma^2(S(u_{-},u))S(u_{-})^2 du\right), \\ J(t) &= \exp\left(\int_0^t (1-N(u_{-}))f(u)S(u_{-})^{p}du\right)(1-N(t)). \end{aligned}$$

The only path to default or a zero equity value is the jump in the process $N(t)$ to the level 1. The dynamics are risk neutral, and the stock growth rate is the interest rate less the dividend yield.

Embedded in the stock evolution subject to the possibility of default is the default free stock price model, or the law of the stock price conditioned on no default or, equivalently, on being positive. This is a useful process, and we shall use it in reconstructing the local volatility functions, in particular from quoted option prices. Hence we now proceed to describe this process to which we henceforth refer as the default free process.

We begin by noting that the probability of surviving t units of time is given by

$$P(\text{no default to } t) = V(t) = E\left[\exp\left(-\int_0^t f(u)(1-N(u_{-}))S(u_{-})^{p}du\right)\right]. \tag{2}$$

Consider now any path dependent claim paying at t: $F(S(u), u \le t)$ if there is no default until time t, and zero otherwise. The time t forward price, w, of this claim is

$$w = E\left[\exp\left(-\int_0^t \lambda(u)du\right)F(S(u), u \le t)\right]. \tag{3}$$

Let $\widetilde{Q}$ be the default free law or the law of the stock conditional on no default to time t. It follows by definition that

$$w = V(t)E^{\widetilde{Q}}\left[F(S(u), u \le t)\right]. \tag{4}$$

Combining (3) and (4) we observe that

$$\frac{d\widetilde{Q}}{dQ} = \frac{\exp\left(-\int_0^t \lambda(u)du\right)}{E\left[\exp\left(-\int_0^t \lambda(u)du\right)\right]}. \tag{5}$$

Define by $p(S,t)$ the density of the stock at time t under the measure $\widetilde{Q}$. This density can in principle be recovered from call option prices conditioned on no default via the methods of Breeden and Litzenberger [5]. The prices conditional on default may be recovered from market prices and the CDS curve using (7) and (8) as explained later. By construction this density integrates to unity over the positive half-line, and it may be extracted from data on default free prices or option prices under the law $\widetilde{Q}$.

Of particular interest is the growth rate of the stock under the law $\widetilde{Q}$. This is determined by evaluating the expectation of $S(t)$ under $\widetilde{Q}$. This is given by

$$\begin{aligned}
E^{\widetilde{Q}}[S(t)] &= E^Q\left[\frac{\exp(-\int_0^t \lambda(u)du)}{E\left[\exp(-\int_0^t \lambda(u)du)\right]}S(t)\right] \\
&= S(0)E^Q\left[\frac{\exp(-\int_0^t \lambda(u)du)}{E\left[\exp(-\int_0^t \lambda(u)du)\right]}\exp\left(\int_0^t (r(u)-q(u))du\right)\right. \\
&\qquad \times \exp\left(\int_0^t \sigma(S(u_-,u))S(u_-)dW(u) - \frac{1}{2}\int_0^1 \sigma^2(S(u_-,u))S(u_-)^2 du\right) \\
&\qquad \left.\times \exp\left(\int_0^t \lambda(u)du\right)\right] \\
&= S(0)\frac{E^Q\left[\exp(\int_0^t (r(u)-q(u))du)\right]}{V(t)}
\end{aligned}$$

as under $\widetilde{Q}$, $N(t)=0$.

Let us now write

$$V(t) = \exp\left(-\int_0^t h(u)du\right),$$

where by construction

$$h(t) = -\frac{\partial \log V(t)}{\partial t}.$$

We then obtain that

$$E^{\widetilde{Q}}[S(t)] = S(0)\exp\left(\int_0^t (r(u)-(q(u)-h(u)))du\right).$$

We then see that under the $\widetilde{Q}$ measure we have a dividend yield adjustment to

$$q_a(t) = q(t) - h(t). \tag{6}$$

The exposure to the default hazard appears in the default free model as a negative dividend yield of $h(t)$. This is quite intuitive, as an operational way to get default free is to buy insurance against it at the premium flow $h(t)$ that is an expense to be paid out of the dividend stream. Hence the reduced dividend flow.

Default free option prices or prices of options under the measure $\widetilde{Q}$ may be constructed by a simple transformation. Suppose we have estimated $V(t)$, the survival probability curve, from CDS prices. We can then construct prices of default free call and put options by

$$\widetilde{C}(K,t) = \frac{C(K,t)}{V(t)}, \tag{7}$$

$$\widetilde{P}(K,t) = \frac{P(K,t) - Ke^{-\int_0^t r(u)du}(1 - V(t))}{V(t)}. \tag{8}$$

Equation (7) reflects the computation that the defaultable call price is the probability of no default times the conditional expectation of the call payoff given no default. For (8) we must recognize in addition that the defaultable put price also includes the receipt of the strike in default.

We also have by definition of $p(S,t)$ that these prices satisfy the equations

$$\widetilde{C}(K,t) = e^{-\int_0^t r(u)du} \int_K^\infty (S-K)p(S,t)dS,$$
$$\widetilde{P}(K,t) = e^{-\int_0^t r(u)du} \int_0^K (K-S)\, p(S,t)dS.$$

We may evaluate the put call parity condition for these prices and observe that

$$\widetilde{C}(K,t) - \widetilde{P}(K,t) = \frac{C}{V(t)} - \frac{P - Ke^{-\int_0^t r(u)du}(1 - V(t))}{V(t)}$$
$$= \frac{S(0)e^{-\int_0^t q(u)du}}{V(t)} - Ke^{-\int_0^t r(u)du}.$$

We see again that for these default free prices the value of the forward stock is as shown earlier,

$$\widetilde{E}[S(t)] = \frac{S(0)e^{-\int_0^t q(u)du}}{V(t)}.$$

We may define term dividend yields and hazard rates by

$$\widetilde{q}(t) = \frac{\int_0^t q(u)du}{t},$$
$$\eta(t) = \frac{\int_0^t h(u)du}{t}$$

and write

$$\widetilde{E}[S(t)] = S(0)e^{-(\widetilde{q}(t) - \eta(t))t}. \tag{9}$$

We may now apply any standard default free model to the prices $\widetilde{C}(K,t), \widetilde{P}(K,t)$ with the dividend yield adjusted to $q_a(t) = \widetilde{q}(t) - \eta(t)$ to recover the default free densities $p(S,t)$, and we shall shortly show how we use this density to build the deterministic component of the hazard rate $f(t)$ and the local volatility function.

We note here that the actual default free law given by the density (5) has a path dependent hazard rate $\lambda(t)$ that depends on the path of the stock price. The pricing of path dependent claims under $\widetilde{Q}$ would require the use of this hazard rate. However, for mere functions of the final stock price we may proceed from the prices (7), (8) to extract the densities $p(S,t)$, and for this we make the appropriate adjustment to the forward price given by the default free forward price (9). As these are all the functions we need, we do not work with the more involved path dependent hazard rates.

3. Recovering hazard and volatility functions from CDS and option markets. In this section we describe how to recover the deterministic component of the hazard rate and the local volatility function from prices of CDSs and equity options. A similar analysis was conducted in Carr and Javaheri [7] in a slightly different context. For completeness and the specificity of our context we provide a comparable derivation.

The first step is to recover the survival function $V(t)$ from CDS quotes, and for this we employ the Weibull model for the life curve and the methods described in Madan, Konikov, and Marinescu [15].

We next observe on differentiating $\ln V(t)$ with respect to t that

$$\frac{1}{V(t)}\frac{\partial V(t)}{\partial t} = -f(t)E^{\widetilde{Q}}\left[S(t)^p\right]. \tag{10}$$

Hence the function $f(t)$ may be recovered from $V(t)$ and the prices of options under $\widetilde{Q}$. We see immediately how we will use the law of $S(t)$ under $\widetilde{Q}$ or the density $p(S,t)$ to recover the function deterministic component of hazard rates or the function $f(t)$. In fact the characteristic function of $\ln S(t)$ under $\widetilde{Q}$ evaluated at $-ip$ gives us directly the default free power price $E^{\widetilde{Q}}\left[S(t)^p\right]$ from which we may construct $f(t)$ in accordance with (10).

For the recovery of the local volatility function with deterministic interest rates we proceed as follows. By definition market call prices are given by

$$C(K,t) = \exp\left(-\int_0^t r(u)du\right) E\left[\ \exp\left(-\int_0^t \lambda(u)du\right)(S(t)-K)^+\right].$$

Differentiation with respect to K yields

$$C_K = -e^{-\int_0^t r(u)du} E\left[\ \exp\left(-\int_0^t \lambda(u)du\right)\mathbf{1}_{S(t)>K}\right],$$

$$C_{KK} = C_{KK} = e^{-\int_0^t r(u)du} E\left[\exp\left(-\int_0^t \lambda(u)du\right)\mathbf{1}_{S(t_-)=K}\right],$$

$$C - KC_K = E\left[\exp\left(-\int_0^t \lambda(u)du\right)S(t)\mathbf{1}_{S(t)>K}\right].$$

Applying the Meyer–Tanaka formula [16], [10], [18] to the call price payoff yields

$$\begin{aligned}(S(t)-K)^+ = (S(0)-K)^+ &+ \int_0^t (1-N(u_-))\mathbf{1}_{S(u_-)>K}dS(u)\\ &+\frac{1}{2}\int_0^t (1-N(u_-))\ \mathbf{1}_{S(u-)=K}\ \sigma^2(S(u),u)S(u)^2du\\ &+K\int_0^t \mathbf{1}_{S(u_-)>K}(1-N(u_-))dN(u).\end{aligned}$$

Taking expectations on the left and expectations of time u conditional expectations of the integrands on the right, we get

$$\begin{aligned}e^{\int_0^t r(u)du}C(K,t) = (S(0)-K)^+ &+ \int_0^t E\left[\exp\left(-\int_0^u \lambda(v)dv\right)S(u_-)(r(u)-q(u))\right]du\\ &+\frac{1}{2}\int_0^t E\left[\exp\left(-\int_0^u \lambda(v)dv\right)\mathbf{1}_{S(u_-)=K}\right]\sigma^2(K,u)K^2du\\ &+K\int_0^t f(u)E\left[\exp\left(-\int_0^u \lambda(v)\right)S(u_-)^p\mathbf{1}_{S(u_-)>K}\right]du.\end{aligned}$$

Differentiating with respect to t and multiplying by the discount factor, we get that

$$\begin{aligned}r(t)C(K,t)+C_t = e^{-\int_0^t r(u)du}\ &E\left[\exp\left(-\int_0^t \lambda(u)dv\right)S(t_-)(r(t)-q(t))\mathbf{1}_{S(t_-)>K}\right]\\ &+\frac{1}{2}e^{-\int_0^t r(u)du}E\left[\exp\left(-\int_0^t \lambda(u)du\right)\mathbf{1}_{S(t_-)=K}\right]\sigma^2(K,t)K^2\\ &+Ke^{-\int_0^t r(u)du}f(t)E\left[\exp\left(-\int_0^t \lambda(u)du\right)S(t_-)^p\mathbf{1}_{S(t_-)>K}\right].\end{aligned}$$

It follows that

$$\begin{aligned}r(t)C(K,t)+C_t = (r(t)-q(t))\left(C-KC_K\right)&+\frac{1}{2}K^2C_{KK}\sigma^2(K,t)\\ &+Kf(t)V(t)e^{-\int_0^t r(u)du}E^{\widetilde{Q}}\left[S(t_-)^p\mathbf{1}_{S(t_-)>K}\right].\end{aligned}$$

We define the truncated power price under $\widetilde{Q}$ as $\zeta(K,t)$:

$$\zeta(K,t) = e^{-\int_0^t r(u)du}E^{\widetilde{Q}}\left[S(t_-)^p\mathbf{1}_{S(t_-)>K}\right]. \tag{11}$$

Hence we construct

$$\sigma^2(K,t) = 2\frac{C_t+q(t)C+(r(t)-q(t))KC_K-Kf(t)V(t)\zeta(K,t)}{K^2C_{KK}}. \tag{12}$$

Both the functions $f(t)$ and $\sigma(K,t)$ may then be recovered from the prices of options under $\widetilde{Q}$. We consider in the next section the procedures for constructing the truncated power prices (11) under $\widetilde{Q}$.

We may usefully rewrite (12) in the form

$$K^2\sigma^2(K,t) + \frac{Kf(t)V(t)\zeta(K,t)}{C_{KK}} = 2\frac{C_t + q(t)C + (r(t)-q(t))KC_K}{C_{KK}} = \sigma^2_{LV}(K,T), \tag{13}$$

where the right-hand side of (13) is the traditional local dollar variance formulation of Dupire [12] and Derman and Kani [11]. We then see that the new local variance is reduced by the survival probability times the expected hazard in the region $S(t) > K$ from where we can jump to zero and earn K dollars on the call, relativized by the density at K. We shall report on the relative magnitudes of the diffusion component and the jump component in the partitioning of Dupire local variance.

4. Truncated power prices.

We use CDS quotes to construct Weibull density parameters for the survival function $V(t)$. We then transform market prices to default free prices using (7), (8). The dividend yields are adjusted using (6). One may now estimate on this data with these adjusted dividend yields any default free model of option prices. We employ for this purpose the VGSSD model reported in Carr et al. [6]. We thereby estimate the characteristic function of the logarithm of $S(t)$ for each t under the law $\widetilde{Q}$. Our use of this model here is merely in its capacity as an interpolator permitting smooth access to prices of all strikes and maturities as synthesized in the relevant marginal distribution. We subsequently use these prices in (12) for the recovery of the local volatility dynamics.

We now describe the explicit construction of truncated power prices of (11) from the estimated VGSSD default free parameters. We seek the value of

$$W(K) = \exp\left(-\int_0^t r(u)du\right) E^{\widetilde{Q}}\left[S(t)^p \mathbf{1}_{S(t)>K}\right].$$

We have the characteristic function of

$$x = \ln(S(t)),$$

which we denote by $\phi_x(u)$. We are interested in

$$w(k) = e^{-rt}\int_k^\infty (e^{xp} - e^k)f(x)dx,$$

where r is now the term discount rate.

Consider the Fourier transform

$$\begin{aligned}\gamma(u) &= e^{-rt}\int_{-\infty}^\infty e^{(\alpha+iu)k}\int_k^\infty e^{xp}f(x)dxdk \\ &= e^{-rt}\int_{-\infty}^\infty dx\ f(x)\int_{-\infty}^x e^{xp+(\alpha+iu)k}dk \\ &= e^{-rt}\int_{-\infty}^\infty dx\ f(x)\frac{e^{(\alpha+p+iu)x}}{\alpha+iu} \\ &= e^{-rt}\frac{\phi_x(u-i(\alpha+p))}{\alpha+iu}. \qquad (14)\end{aligned}$$

Table 1
Weibull parameters from CDS curves.

	GM		Ford	
	c	a	c	a
mean	8.2788	1.2610	8.5832	1.2889
std	3.0418	0.1962	3.0655	0.1633
max	15.6194	1.5579	16.2712	1.5469
min	3.2041	0.7798	5.0914	0.8949

We obtain the truncated power prices by Fourier inversion as

$$w(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk}\gamma(u)du. \tag{15}$$

These may then be substituted into (12) to obtain the local volatility surface. We compute these at a grid of strikes and maturities for which we seek the local volatilities $\sigma(K,T)$.

Once we have estimated the parameters for the no default process we may compute $\widetilde{C}(K,T)$ at a grid of strikes and maturities. We then use our estimated survival function to define

$$C(K,T) = V(T)\widetilde{C}(K,T).$$

These prices are then used in the expression (12) along with the no default truncated power prices computed as per section 4 to find the local volatility surface $\sigma(K,T)$. For the function $f(t)$ we use (10) and the characteristic function for log prices evaluated at $-ip$.

5. Sample computations for GM and Ford. We now illustrate these procedures for data on GM and Ford from October 2004 to September 2007. We estimated the Weibull parameters for 795 and 784 days, respectively, out of 1085 and 1078 days for GM and Ford, respectively. The results are summarized in Table 1.

The price of a CDS is analytically expressed in terms of the survival function and hence in terms of the parameters c, a. A least squares minimization between market and model CDS prices across all maturities results in the estimates for c, a for each company on each day.

We then used the implied Weibull survival functions to adjust market prices to default free prices along with adjusting dividend yields to get the right default free forwards. This is done in line with (7), (8), and (6) with $V(t)$ being analytically specified in the Weibull form given c, a.

The resulting candidates for default free prices and dividend yields are employed to fit the interpolating VGSSD model to get the parameters for a smooth version of the default free option prices. These are summarized in Tables 2 and 3 for GM and Ford, respectively.

The next step is the computation of truncated power prices with the no default dynamics and the adjusted dividend yields. These are constructed for an expanding strike range as we raise maturity from 0.05 to 1. We used the strike range

$$kd(t) = p/1.1 - .2 * p * sqrt(t - t_{\min}),$$
$$ku(t) = p * 1.1 + .2 * p * sqrt(t - t_{\min}),$$

Table 2
GM default free surface.

	σ	ν	θ	γ
mean	0.2796	0.3471	−0.3579	0.3628
std	0.0743	0.3764	0.3584	0.1014
max	0.5036	2.2549	0.6492	0.5266
min	0.0035	0	−2.5097	0.0339

Table 3
Ford default free surface.

	σ	ν	θ	γ
mean	0.2821	0.3155	−0.1732	0.4101
std	0.0837	0.3601	0.2949	0.0717
max	0.5381	2.4358	3.1131	0.6925
min	0.0099	0	−1.2926	0.1827

where $kd(t)$ is the lowest strike in the grid at maturity t, while $ku(t)$ is the highest strike in the grid at maturity t, and p here is the initial spot price.

On this grid we construct truncated power prices under the $\widetilde{Q}$ measure by the Fourier inversion described in (15) and (14). We use a power of -1.5. We now have the power price under $\widetilde{Q}$ as well as the truncated power prices and we can use the Weibull c.d.f. and (10) for $f(t)$ to build the function $f(t)$.

In the next step we construct $\widetilde{Q}$ call prices using the adjusted dividend yields (6) and the smooth interpolation, VGSSD parameters. These prices are then transformed back to market call prices that are defaultable prices via

$$C(K,t) = V(t)\widetilde{C}(K,t),$$

where $V(t)$ comes from the Weibull model. We now have all the ingredients to implement the revised local volatility construction of (12). The final output consists of a local volatility function and the function $f(t)$.

These are graphed in Figure 1 for GM and Ford over four to five subsets partitioning levels of the parameter c in the survival function and the level of aggregate volatility of the default free surface as measured by $\sigma^2 + \theta^2\nu$. We observe that when c is high and the mean lifetime is large the deterministic component of the hazard rate is fairly flat. For low mean lifetimes and hence c, the deterministic component rises sharply when default free volatility is low but rises more slowly for a high default free volatility. Given a fixed observed volatility in the defaultable market volatilities, there is a trade-off in how volatility splits between the hazard rate and the implied default free structure. The higher the volatility in the hazard rate, the lower it is in the default free options. This trade-off is observed in comparing the local volatility graphs and the hazard rate graphs. The kinks occurring in the tails are numerical consequences of being deep out of the money when the volatility has dropped to a low level.

For Ford we had four subsets, while for GM we had five subsets.

With a view toward summarizing the results we computed the proportion of the traditional

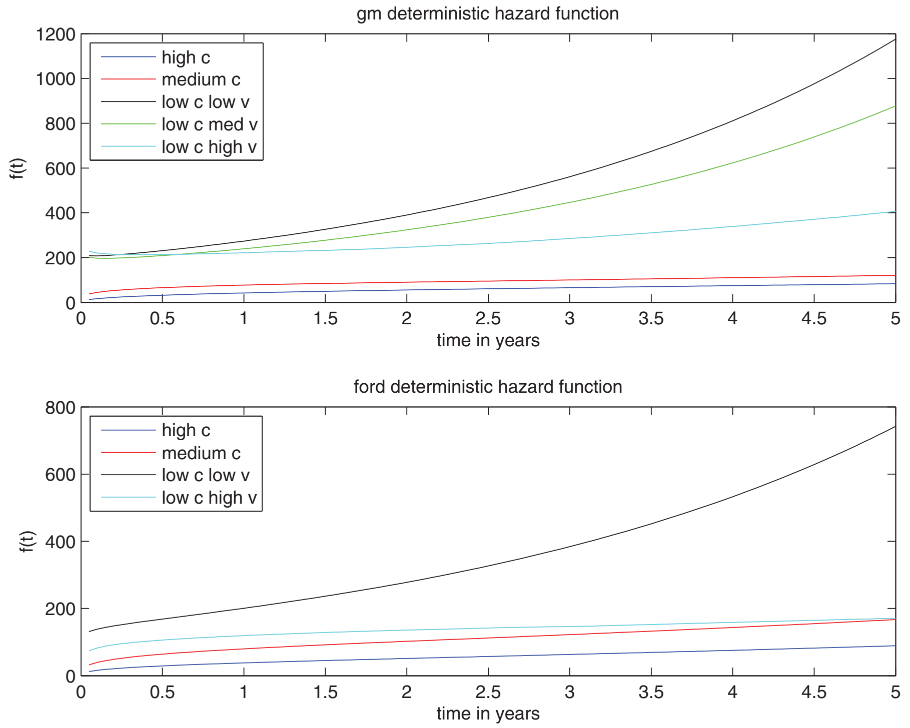

Figure 1. *Deterministic components of hazard rate functions for GM and Ford.*

local volatility that is allocated to the default component, or the fraction

$$\sigma_{def}^2(K,t) = \frac{Kf(t)V(t)\zeta(K,t)}{C_{KK}\sigma_{LV}^2(K,t)}.$$

This proportion of volatility allocated to the default component is graphed in Figure 2 for GM and Ford. It generally varies with strike and maturity, and we computed the elasticities of this proportion by regressing the logarithm of this ratio on the logarithm of the strike and maturity. Additionally we also regressed the logarithm of $f(t)$ on maturity and the logarithm of maturity. To avoid the effects of cases where these summary functional forms did not fit well, we report summary statistics of these regressions only when the R^2 exceeded 90% along with the proportion of times that this criterion was met. Table 4 provides the results for the default proportion of local volatility, while Table 5 presents the results for the deterministic hazard function $f(t)$.

We see from Table 4 that the elasticity with respect to the strike of the default proportion is -0.4768 and -0.3967 for GM and Ford, respectively. The corresponding elasticities with respect to maturity are 0.0646 and 0.0504, respectively.

We see from Table 5 that the hazard function is generally increasing with maturity with semielasticities and elasticities of 0.1306 and 0.2163 for GM and 0.0975 and 0.2641 being the corresponding values for Ford.

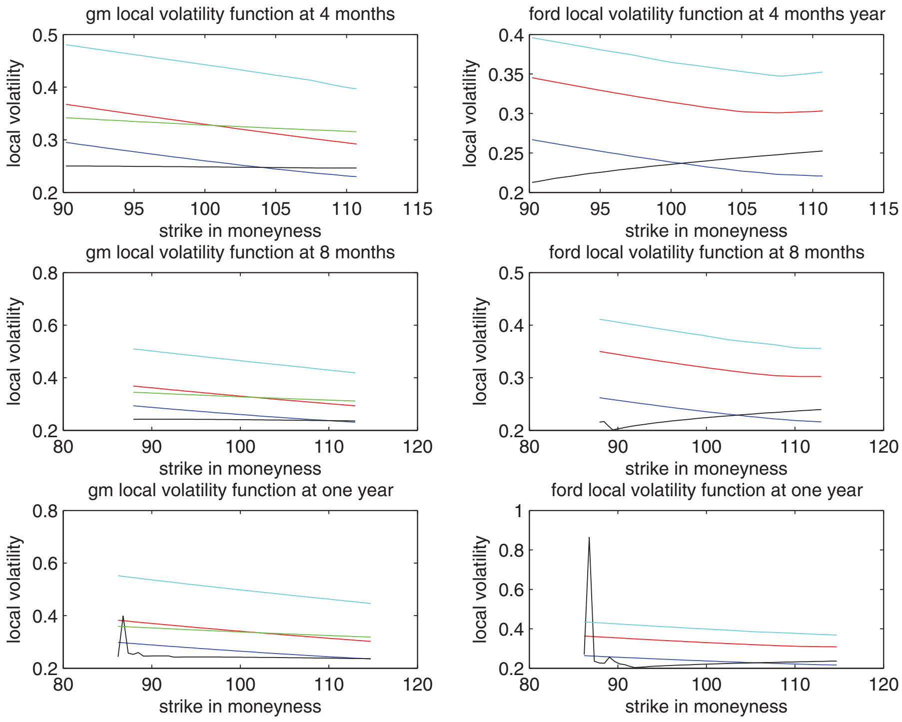

Figure 2. *Implied local volatilities for GM and Ford.*

Table 4
log *default proportion regressions.*

	constant	log(K/100)	log(t)
GM (R^2 criterion satisfaction 79.89%)			
mean	2.3585	−0.4768	0.0646
std	1.7332	0.3487	0.0537
min	0.7516	−1.7387	0.0042
max	8.5069	−0.1531	0.2555
Ford (R^2 criterion satisfaction 71.43%)			
mean	1.9454	−0.3967	0.0504
std	1.0498	0.2145	0.0275
min	0.8528	−1.5003	0.0097
max	7.3322	−0.1731	0.1937

6. Conclusion. We enhance a local volatility model by the addition of the possibility of a single jump to default with a hazard rate that is a deterministic function of time scaled by the stock price raised to a prespecified negative power. Our empirical work uses the prespecified power of −1.5. We show in this context how one may simultaneously recover from prices of CDS contracts and the equity option prices both the deterministic component of the hazard rate function and revised local volatility. The procedure requires one to construct, after estimating the survival probabilities to various maturities, the prices of default free options to which one fits a standard default free model with revised dividend yields to account for the

Table 5
$\log f(t)$.

GM (R^2 criterion satisfaction 88.77%)			
	constant	maturity	log(maturity)
mean	4.3717	0.1306	0.2163
std	0.6799	0.1654	0.1999
min	3.2663	−0.4932	−0.1839
max	5.7113	0.4656	0.6203
Ford (R^2 criterion satisfaction 94.33%)			
mean	4.1925	0.0975	0.2641
std	0.5754	0.1234	0.1533
min	3.1920	−0.5760	−0.0122
max	5.4697	0.3426	0.5740

payment of premia necessary to get default free in a defaultable world. This default free model is critically used to infer the prices of powers of the stock price truncated to be above strike levels for a variety of maturities. These truncated power prices are needed in constructing the revised local volatility function from a grid of defaultable call prices that may be inferred from the default free model coupled with the survival function.

The entire procedure was implemented on prices of CDSs and equity options for GM and Ford over the period October 2004 to September 2007. We found that the revised local volatility must be reduced to accommodate the possibility of default by a proportion that is dependent on both the strike and the maturity. On average the elasticity of the default proportion of local volatility is $-.4768$ and $-.3967$ for GM and Ford, respectively. The corresponding elasticities with respect to maturity are positive at 0.0646 and 0.0504. The deterministic component of the hazard function is generally increasing with respect to maturity with semielasticities and elasticities of 0.1306 and 0.2163 for GM and 0.0975 and 0.2641 for Ford.

Acknowledgment. We acknowledge helpful discussions on the subject matter of this paper with Bruno Dupire, Alireza Javaheri, and Liuren Wu. Any remaining errors are our responsibility.

REFERENCES

[1] C. Albanese and O. Chen, *Pricing equity default swaps*, Risk, 6 (2005), pp. 83–87.
[2] L. Andersen and D. Buffum, *Calibration and implementation of convertible bond models*, J. Comput. Finance, 7 (2003/04), pp. 1–34.
[3] M. Atlan and B. Leblanc, *Hybrid equity-credit modelling*, Risk, 8 (2005), pp. 61–66.
[4] E. Ayache, P. A. Forsyth, and K. R. Vetzal, *The valuation of convertible bonds with credit risk*, J. Derivatives, 9 (2003), pp. 11–29.
[5] D. Breeden and R. L. Litzenberger, *Pricing of state-contingent claims implicit in option prices*, J. Business, 51 (1978), pp. 621–651.
[6] P. Carr, H. Geman, D. Madan, and M. Yor, *Self-decomposability and option pricing*, Math. Finance, 17 (2007), pp. 31–57.
[7] P. Carr and A. Javaheri, *The forward PDE for European options on stocks with fixed fractional jumps*, Internat. J. Theoret. Appl. Finance, 8 (2005), pp. 239–253.

[8] P. Carr and V. Linetsky, *A jump to default extended CEV model: An application of Bessel processes*, Finance Stoch., 10 (2006), pp. 303–330.
[9] M. Davis and F. Lischka, *Convertible bonds with market risk and credit risk*, in Applied Probability, AMS/IP Stud. Adv. Math. 26, AMS, Providence, RI, 2002, pp. 45-58.
[10] C. Dellacherie and P. Meyer, *Probabilités et Potentiel, Theorie des Martingales*, Hermann, Paris, 1980.
[11] E. Derman and I. Kani, *Riding on a smile*, Risk, 7 (1994), pp. 32–39.
[12] B. Dupire, *Pricing with a smile*, Risk 7 (1994), pp. 18–20.
[13] R. A. Jarrow and S. Turnbull, *Pricing options on financial securities subject to default risk*, J. Finance, 50 (1995), pp. 53–86.
[14] V. Linetsky, *Pricing equity derivatives subject to bankruptcy*, Math. Finance, 16 (2006), pp. 255–282.
[15] D. Madan, M. Konikov, and M. Marinescu, *Credit and basket default swaps*, J. Credit Risk, 2 (2006), pp. 67–87.
[16] P. Meyer, *Un Cours sur les Intégrales stochastiques*, in Séminaire de Probabilités X, Lecture Notes in Math. 511, Springer-Verlag, Berlin, 1976, pp. 245–400.
[17] Y. Ren, D. Madan, and M. Qian, *Calibrating and pricing with embedded local volatility models*, Risk, 9 (2007), pp. 138–143.
[18] M. Yor, *Rappels et Préliminaires Généraux*, in Temps Locaux, Astérique 52/53, J. Azéma and M. Yor, eds., Société Mathématique de France, Paris, 1978, pp. 17–22.