An Alternative Approach for Valuing Continuous Cash Flows

Abstract

We consider the problem of replicating the payoffs from variable annuities with a continuous cash flow given by a function of some traded asset's price. The standard approaches involve either dynamic trading in this underlying asset or a static position in a continuum of options of all strikes and maturities. We present an alternative approach which combines dynamic trading in the underlying asset with a static position in options of a single maturity. In many instances, our approach yields explicit valuation formulas and hedging strategies when the volatility of the underlying is an arbitrary function of its price.

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Introduction

Many valuation problems arising in modern finance involve variable cash flows paid frequently over time. These cash flows are often a known function of the price of some traded asset. For example, projects modelled as real options may involve revenue and/or cost streams which are a function of the price of a commodity. Alternatively, we may consider credit-sensitive bonds which have a coupon rate which explicitly depends on the issuer's credit rating. If this rating is modelled as a known function of the firm's asset value or stock price, then the interest payments depend on a traded asset's price as required. Similarly, if the equity of a firm is modelled as a call option on the firm's assets, then over long horizons, the dividends on the stock may be approximated as occuring continuously over time and will likely depend on the firm's asset value or stock price. In particular, if either variable drops below some threshold, then dividend payments are likely to be suspended.

There are numerous other examples in which the cash flows of the annuity vanish when the underlying's price is in some range. For example, the discount to apply to a corporate bond or preferred stock due to suspension of payments when the firm is in jeopardy can be modelled as the value of the cash flows lost when the asset value or stock price is below a boundary. Similarly, the early exercise premium of an American put capitalizes the excess of an interest flow over a dividend flow when the underlying is below the early exercise boundary. Conversely, exchange rate restrictions may induce firms to repatriate funds on a daily basis, so long as the rate exceeds a floor. Finally, corridor notes sold over-the-counter pay a fixed amount for each day an underlying asset spends in a corridor.

The classical Black Scholes[2]/Merton[17] approach for hedging such path-dependent securities involves dynamic trading in the underlying asset. Ross[20] and Breeden & Litzenberger[4] initiated an alternative literature which considers static positions in standard options as a second approach for valuing derivatives. When feasible, static hedging has the advantage of generating explicit valuation formulas in terms of option prices and does not require specifying the stochastic process for the underlying. However, to replicate a variable annuity with continuous cash flows up to some fixed maturity, static hedging requires using options of all strikes and of all maturities up to that of the annuity.

The purpose of this paper is to explore a third approach for hedging contingent claims with continuous cash flows. We term this approach *enhanced delta hedging* (henceforth EDH), because standard dynamic trading in the underlying asset is enhanced by static positions in options. When compared with static hedging, EDH requires options of only a single maturity, but also requires specifying the stochastic process of the underlying. When compared with dynamic hedging, EDH has the advantage of permitting explicit valuation formulas and hedging strategies for fairly general specifications on the volatility and carrying costs of the underlying. More specifically, we show that when the payout on the variable annuity is deferred without interest to maturity, then explicit solutions are possible whenever volatility is a known positive function of the underlying price[1]. When payouts are not deferred, then cash flows linked to the spot price can still be valued explicitly for arbitrary volatility functions, so long as dividends are modelled as constant over time. In contrast, non-deferred payouts linked to the futures price cannot be valued explicitly in closed form for an arbitrary volatility function. However, many commonly used volatility specifications do yield closed form solutions.

The structure of this paper is as follows. The next section presents the model setup and reviews standard approaches to hedging the payoffs of claims with continuous cash flows. The following section develops the general theory of EDH. The subsequent section specializes the analysis to cash flows deferred without interest to maturity, while the following section focusses on non-deferred payouts. The final section concludes and suggests further research.

I Model Setup

We assume frictionless markets and consider a claim maturing at T with continuous cash flows, which are a given function $g(\cdot)$ of the price of some underlying asset. This price may be a spot price or it may be a futures or forward price for delivery at date $T' \geq T$. To deal with all 3 cases jointly, we let U_t

[1]For any given drift coefficient, the volatility function is also assumed to satisfy Lipschitz and growth conditions which guarantee the existence of a well-defined diffusion process.

denote the price of the underlying asset. We assume that the riskfree rate is constant at r and thus claims linked to forward prices are identical to the corresponding claim linked to futures prices. If the cash flows are paid out continuously and invested at r, then the random cumulative value at the maturity date T is $\int_0^T e^{r(T-t)} g(U_t)dt$. Since it is costly in practice to pay out cash flows continuously, we allow for the possibility that the cash flows on the contract are paid out in a single lump sum at the maturity date T. In this case, we let r_c denote the constant contracted rate at which cash flows earn interest between the time they are realized and the single payout date T. Thus, the only payment on these contracts is $\int_0^T e^{r_c(T-t)} g(U_t)dt$, which is paid at the maturity date T. Note that by setting $r_c = r$, we capture the possibility that cash flows are paid out continuously. On the other hand, since many contracts such as corridor notes defer the payout to maturity without interest, we set $r_c = 0$ to cover this possibility.

We assume that the cost of carrying the underlying asset over time is a function $r_u(\cdot)$ of the underlying asset price U. If the cash flows are a function of a futures price F_t, then $r_u(F) = 0$, since futures contracts are costless. If the cash flows are a function of the spot price S_t, and if dividends on the stock are paid continuously, then the cost of carrying the underlying is the riskfree rate less the dividend yield, where the latter can be an arbitrary function $\delta(S)$ of the spot price.

Suppose that the price process of the underlying is given by the following stochastic differential equation[2] (s.d.e.):

$$\frac{dU_t}{U_t} = \alpha(t, U_.)dt + \sigma(t, U_.)dW_t, \qquad t \in [0, T], \tag{1}$$

where W is a standard Brownian motion on the line. Thus, under the original probability measure, the expected growth rate α and volatility σ can depend on the time t and on the price path $U_.$ up to this time. If the origin is attainable and regular, we impose an absorbing boundary condition there.

[2]Letting $b(t, x_.) \equiv x\alpha(t, x_.)$ and $a(t, x_.) \equiv x\sigma(t, x_.)$, the existence of a strong solution requires that b and a be Lipshitz, i.e.: there exists some $K < \infty$ such that:

$$\begin{aligned} |b(t, x_.) - b(t, y_.)| &\leq K|x - y|_t^* \\ |a(t, x_.) - a(t, y_.)| &\leq K|x - y|_t^*, \end{aligned}$$

for every $t \in [0, T]$ and $x \in \Re, y \in \Re$, where $f_t^* \equiv sup\{|f(s)| : s \leq t\}$. It also requires that for each constant $T > 0$, there is some C_T such that $|a(s, 0)| + |b(s, 0)| \leq C_T$ for all $s \leq T$ (see Rogers and Williams[22] pg. 136 for an excellent exposition.).

The continuity of the s.d.e. permits the claim described above to be perfectly hedged by dynamically trading in the underlying. In an effort to obtain closed form solutions for the theoretical value which this strategy engenders, we now impose the restrictive assumption that the volatility at time t depends only on the contemporaneous price:

$$\sigma(t, U_.) = \sigma(U_t).$$

Then by a minor generalization of the results in Merton[18] and Black[1], the claim value $V(U,t)$ solves the following partial differential equation (p.d.e.):

$$\frac{\sigma^2(U)U^2}{2}\frac{\partial^2 V}{\partial U^2}(U,t) + r_u(U)U\frac{\partial V}{\partial U}(U,t) - rV(U,t) + \frac{\partial V}{\partial t}(U,t) + e^{(r_c-r)(T-t)}g(U) = 0, \quad U > 0, t \in [0,T],$$

subject to the following terminal condition:

$$V(U,T) = 0.$$

It is also well-known that by the Feynman-Kac Theorem, the solution[3] has the following risk-neutral representation:

$$V(U,t) = e^{-r(T-t)} \int_t^T e^{r_c(T-u)} E\left[g(U_u)\middle| U_t = U\right] du, \tag{2}$$

where expectations are evaluated under the risk-neutral process:

$$\frac{dU_t}{U_t} = r_u(U_t)dt + \sigma(U_t)dW_t, \qquad t \in [0,T].$$

Unfortunately, there is no general solution for this expectation for an arbitrary carrying cost $r_u(U)$ or volatility function $\sigma(U)$. Solutions are known only for certain cases, eg. constant carrying cost $r_u(U) = b$ and Constant Elasticity of Variance (CEV) volatility $\sigma(U) = \sigma U^p$ (see Cox[9] for spot prices and Choi & Longstaff[8] for futures prices).

One solution to this problem is to assume the existence of a complete term and strike structure of European options. It is well known (see Breeden & Litzenberger[4], Green & Jarrow[12], Nachman[19],

[3] We thank a referee for pointing out that this solution can also be expressed in terms of the resolvent operator G_{r_c} associated with the risk-neutral diffusion process for U. See Rogers and Williams [21] chapter III for a superb introduction to resolvents.

and Ross[20]) that a complete strike structure of options maturing at t allows one to statically replicate a continuous payoff $g(U_t)$ occuring at t. In particular, Carr and Madan[6] show that any C^2 payoff[4] decomposes into the payoffs from static positions in bonds and options maturing at t:

$$g(U_t) = g(\kappa_t) + g'(\kappa_t)[(U_t - \kappa_t)^+ - (\kappa_t - U_t)^+] + \int_0^{\kappa_t -} g''(K)(K - U_t)^+ dK + \int_{\kappa_t +}^{\infty} g''(K)(U_t - K)^+ dK, \quad (3)$$

where κ_t is an arbitrary deterministic process. In particular, the investor holds $g(\kappa_t)$ bonds, $g'(\kappa_t)$ synthetic forwards, $g''(K)dK$ puts for all strikes $K < \kappa_t$, and $g''(K)dK$ calls for all strikes $K > \kappa_t$.

It follows that if investors can also take positions in all option maturities from 0 to T, then they can statically create the payoff $\int_0^T e^{r_c(T-t)} g(U_t)dt$ occuring at T. In particular, multiplying (3) by $e^{r_c(T-t)}$ and integrating over t gives:

$$\begin{aligned}\int_0^T e^{r_c(T-t)} g(U_t)dt &= \int_0^T e^{r_c(T-t)} g(\kappa_t)dt + \int_0^T e^{r_c(T-t)} g'(\kappa_t)[(U_t - \kappa_t)^+ - (\kappa_t - U_t)^+]dt \\ &+ \int_0^T e^{r_c(T-t)} \left[\int_0^{\kappa_t -} g''(K)(K - U_t)^+ dK + \int_{\kappa_t +}^{\infty} g''(K)(U_t - K)^+ dK\right] dt.\end{aligned}$$

Recognizing that the options mature at t rather than T, one can value the cash flow by:

$$\begin{aligned}V_0 &= e^{-rT} \int_0^T e^{r_c(T-t)} g(\kappa_t)dt + \int_0^T e^{(r_c - r)(T-t)} g'(\kappa_t)[C_0(\kappa_t, t) - P_0(\kappa_t, t)]dt \\ &+ \int_0^T \int_0^{\kappa_t -} e^{(r_c - r)(T-t)} g''(K) P_0(K, t) dK dt + \int_0^T \int_{\kappa_t +}^{\infty} e^{(r_c - r)(T-t)} g''(K) C_0(K, t) dK dt, \quad (4)\end{aligned}$$

where $C_0(K, t)$ and $P_0(K, t)$ denote the resepective initial prices of European calls and puts of strike K and maturity t. In contrast to the results for dynamic replication, this valuation formula is always explicit (in terms of option prices), and holds for an arbitrary underlying price process.

Unfortunately, the term structure in listed options markets is far from continuous. The next section develops an EDH strategy which also permits explicit valuation formulas, but only assumes that options of maturity T are available. To obtain our results, we assume the continuous price process described by (1) and use dynamic trading in the underlying to supplement the strike structure of the single maturity T.

[4] The smoothness requirements on g in (3) can be weakened by using generalized functions.

II Enhanced Delta Hedging

To develop the enhanced delta hedge of a claim with the single payoff $\int_0^T e^{r_c(T-t)} g(U_t)dt$ occuring at T, apply Itô's lemma to the product of a twice differentiable function $f(U_t)$ and the function of time $e^{r_c(T-t)}$:

$$f(U_T) = f(U_0)e^{r_c T} + \int_0^T e^{r_c(T-t)} f'(U_t)dU_t + \int_0^T e^{r_c(T-t)} \left[f''(U_t)\frac{\sigma^2(U_t)U_t^2}{2} - r_c f(U_t) \right] dt. \qquad (5)$$

Now subtract the carrying cost $r_u(U_t)U_t dt$ from the stochastic integrator:

$$\begin{aligned} f(U_T) &= f(U_0)e^{r_c T} + \int_0^T e^{r_c(T-t)} f'(U_t)[dU_t - r_u(U_t)U_t dt] \\ &+ \int_0^T e^{r_c(T-t)} \left\{ \frac{\sigma^2 U_t^2}{2} f''(U_t) + r_u(U_t)U_t f'(U_t) - r_c f(U_t) \right\} dt. \end{aligned} \qquad (6)$$

Suppose we choose $f(S)$ to solve the following *ordinary* differential equation (o.d.e.):

$$\frac{\sigma^2(U)U^2}{2} f''(U) + r_u(U)Uf'(U) - r_c f(U) = g(U). \qquad (7)$$

Then, substituting (7) in (6) and re-arranging gives:

$$\int_0^T e^{r_c(T-t)} g(U_t)dt = f(U_T) - f(U_0)e^{r_c T} - \int_0^T e^{r_c(T-t)} f'(U_t)[dU_t - r_u(U_t)U_t dt]. \qquad (8)$$

Recall the static decomposition (3) applied to the function $f(U)$:

$$f(U_T) = f(\kappa) + f'(\kappa)[(\kappa - U_T)^+ - (\kappa - U_T)^+] + \int_0^{\kappa-} f''(K)(K - U_T)^+ dK + \int_{\kappa+}^{\infty} f''(K)(U_T - K)^+ dK,$$

where $\kappa \equiv \kappa_T$. Suppose that we also require that f have zero value and slope at κ:

$$f(\kappa) = f'(\kappa) = 0. \qquad (9)$$

Then the decomposition simplifies into the following static position in options:

$$f(U_T) = \int_0^{\kappa-} f''(K)(K - U_T)^+ dK + \int_{\kappa+}^{\infty} f''(K)(U_T - K)^+ dK. \qquad (10)$$

Substituting (10) in (8) gives the following representation of the desired payoff:

$$\begin{aligned} \int_0^T e^{r_c(T-t)} g(U_t)dt &= -f(U_0)e^{r_c T} + \int_0^{\kappa-} f''(K)(K - U_T)^+ dK + \int_{\kappa+}^{\infty} f''(K)(U_T - K)^+ dK \\ &- \int_0^T e^{r_c(T-t)} f'(U_t)[dU_t - r_u(U_t)U_t dt]. \end{aligned} \qquad (11)$$

The first term on the RHS of (11) is the payment required to cover a short position in $f(U_0)e^{r_c T}$ bonds, each paying a dollar at T. The next two terms on the RHS sum to the payoff from being long $f''(K)dK$ puts for all strikes $K < \kappa$ and long $f''(K)dK$ calls for all strikes $K > \kappa$, with all options maturing at T. Finally, the last term is the cumulative gains from a dynamic strategy in $e^{(r_c - r)(T-t)}f'(U_t)$ units of the underlying asset at each $t \in [0,T)$, where all purchases are financed by borrowing at the riskfree rate r and all sales earn interest at r. Thus, the desired payoff is the sum of a static position in bonds and options of all strikes and a dynamic strategy in the underlying asset. Since the cost of financing the dynamic strategy has already been accounted for, the initial value of the payoff $\int_0^T e^{r_c(T-t)}g(U_t)dt$ at T is given by:

$$V_0 = -f(U_0)e^{(r_c - r)T} + \int_0^{\kappa-} f''(K)P_0(K,T)dK + \int_{\kappa+}^{\infty} f''(K)C_0(K,T)dK, \tag{12}$$

where f solves the o.d.e. (7).

It is interesting to compare the theoretical value obtained by the above EDH strategy with those obtained by classical delta-hedging and pure static hedging. If the model for the underlying dynamics is correct and if the options used in the hedge are priced according to this model, then the theoretical values from all three strategies are identical. However, if either the model is incorrect or if any option used in the hedge is mispriced relative to the model, then the costs can differ. The replication cost can also differ when one accounts for the transactions costs *in options*. The EDH strategy is obviously more exposed to this aspect of the replication cost than pure delta hedging, but it is less exposed than pure static hedging. Assuming no transactions costs and that the model dynamics are correct, it is possible that EDH is a more profitable hedging strategy than either of the two alternatives, and so it should at least be examined for this possibility.

If we allow for the practical certainty that the model is incorrect, then EDH offers a different set of model risks than the other two alternatives. In particular, if one is concerned that the model for the underlying dynamics is incorrect, then EDH is more exposed to this source of model error than pure static hedging, but less exposed to this error than pure dynamic hedging. Similarly, when transactions costs of the underlying are considered, EDH is more exposed to this source of model error than pure static hedging, but less exposed to this error than pure dynamic hedging. The reason for this lesser exposure is that by setting $\kappa = U_0$, (9) and (11) imply that the EDH strategy can be chosen to have no position in the underlying initially. Furthermore, for contracts which have zero payout in certain regions (eg. corridor notes), no dynamic trading is required in the EDH strategy whenever the underlying is in these regions. In general, these considerations suggest that the choice of an optimal hedging strategy is likely to be both difficult and situation-specific.

The options literature sometimes resolves these difficulties by formally introducing new risks into the model such as jumps, stochastic volatility, stochastic interest rates, and future transactions costs. When these risks are introduced parametrically (eg. a lognormal process for volatility), then one can exactly quantify the effect of ignoring a risk such as stochastic volatility in the simpler model. However, it should be pointed out that this procedure can introduce further model risks (eg. that volatility of volatility is not constant). When a simple model such as ours produces multiple equivalent hedges, it is often possible to select one by comparing them in a non-parametric generalization of the simple model. To illustrate in the current context, suppose that one must choose between pure dynamic hedging and EDH, since we assume that not all options of maturity less than T trade. If the two strategies generate the same model value and if the EDH involves only long positions in options, then EDH *dominates* delta-hedging when one extends the model to allow for the possibility of an initial jump in price of arbitrary size. Conversely, EDH is dominated by delta hedging if the former involves only short option positions.

The decomposition (12) underlying EDH has importance, even if options are not used in the hedge. The decomposition can be used to transfer intuition on options to the variable annuity and to reduce the valuation problem for the annuity down to determining the expected value of a terminal payoff. Thus, our decomposition makes the annuity valuation problem equivalent to the problem of valuing a standard option, which is in turn equivalent to solving for the transition density. For many valuation environments, the valuation of these claims is quite likely to have been already implemented. These benefits accrue because the decomposition in (12) eliminates the time integral which arises in the valuation formulas (2) and (4), derived via dynamic and static replication respectively. Thus, a problem which *a priori* appears to require an integration across both time and space reduces to a single integration across space. Hence, this result is the financial equivalent of Green's theorem, which analogously reduces a surface integral to a line integral.

III Deferred Payouts

Our valuation equation (12) only becomes explicit once we solve the o.d.e. (7) for f. This section shows that this o.d.e. can always be solved whenever the payout on the annuity is deferred without interest to maturity, i.e. $r_c = 0$. The next section presents the corresponding analysis for non-deferred cash flows, i.e. $r_c = r$.

III-A Deferred Payout on Futures

Suppose that the cash flow at t is a given function $g(\cdot)$ of the futures price F_t for delivery at $T' \geq T$. Thus we assume the diffusion process (1) for the futures price and that there exists an entire strike structure of European futures options maturing at T. Since listed futures options are usually American-style, the latter assumption is a potential obstacle to EDH. One approach is to ignore the American feature, since futures options are typically only optimally exercised near maturity. A better approach available to the issuer of the variable annuity is to set $T' = T$ in the definition of the variable annuity and hedge with listed European options on spot instead. Another alternative is to link the cash flows to the spot price as is done in the next subsection. Since futures contracts are costless, the cost of carrying the underlying is $r_u(F) = 0$. Since this section assumes that the cash flows are deferred to maturity without interest, we also set $r_c = 0$. Replacing U_t with F_t and substituting $r_u = r_c = 0$ in (11) and (7) gives:

$$\int_0^T g(F_t)dt = -f(F_0) + \int_0^{\kappa-} f''(K)(K - F_T)^+ dK + \int_{\kappa+}^{\infty} f''(K)(F_T - K)^+ dK - \int_0^T f'(F_t)dF_t, \qquad (13)$$

where $f(F)$ solves the 1 dimensional Poisson equation:

$$f''(F) = \frac{2g(F)}{\sigma^2(F)F^2}. \qquad (14)$$

Integrating once gives the first derivative:

$$f'(F) = \int_{\kappa}^{F} \frac{2g(x)}{\sigma^2(x)x^2}dx, \tag{15}$$

where for $b > a$, $\int_b^a h(x)dx \equiv -\int_a^b h(x)dx$. The lower limit of the integral has been chosen to be consistent with (9). Similarly, integrating once more gives:

$$f(F) = \int_{\kappa}^{F}\int_{\kappa}^{y} \frac{2g(x)}{\sigma^2(x)x^2}dxdy. \tag{16}$$

Substituting (14) to (16) in (13) gives an explicit representation of the payoff:

$$\begin{aligned}\int_0^T g(F_t)dt \;&=\; -\int_{\kappa}^{F_0}\int_{\kappa}^{y} \frac{2g(x)}{\sigma^2(x)x^2}dxdy + \int_0^{\kappa-} \frac{2g(K)}{\sigma^2(K)K^2}(K-F_T)^+dK + \int_{\kappa+}^{\infty} \frac{2g(K)}{\sigma^2(K)K^2}(F_T-K)^+dK \\ &\quad - \int_0^T \int_{\kappa}^{F_t} \frac{2g(x)}{\sigma^2(x)x^2}dxdF_t, \end{aligned} \tag{17}$$

Although this decomposition appears complicated, we note that the integrals determining f and f' in (16) and (15) can often be found in closed form, since the payoff rate g is likely to be a simple function. An alternative proof of (17) with $\kappa = F_0$ is presented in the appendix for readers familiar with the concept of local time (see Carr and Jarrow[5]).

The decomposition (17) once again indicates that the desired payoff is the sum of a static position in bonds and options of all strikes combined with a dynamic strategy in the underlying futures. Assuming that marking-to-market occurs continuously, the dynamic strategy involves holding $-e^{-r(T-t)}\int_{\kappa}^{F_t}\frac{2g(x)}{\sigma^2(x)x^2}dx$ futures contracts at each $t \in [0,T]$. Since futures contracts are costless, the initial value of the payoff $\int_0^T g(F_t)dt$ at T is explicitly given by:

$$V_0 = -e^{-rT}\int_{\kappa}^{F_0}\int_{\kappa}^{y} \frac{2g(x)}{\sigma^2(x)x^2}dxdy + \int_0^{\kappa-} \frac{2g(K)}{\sigma^2(K)K^2}P_0(K,T)dK + \int_{\kappa+}^{\infty} \frac{2g(K)}{\sigma^2(K)K^2}C_0(K,T)dK. \tag{18}$$

To illustrate further, consider valuing a corridor note paying coupons at a constant rate c for each instant the underlying futures price is inside a corridor bracketing the initial futures price i.e. $F_0 \in (L,H)$. Thus, the given payout rate is $g(F) = c1_{F\in(L,H)}$, where 1_A denotes the indicator function of the event A. In order to compare with known results, we assume a CEV process for the futures price, i.e. $\sigma(F) = \sigma F^p$, where σ and p are constants. To obtain the enhanced delta hedge for the corridor note, observe that substituting g and σ in (14) yields:

$$f_c''(F) = \frac{2c1_{F\in(L,H)}}{\sigma^2 F^{2p+2}}. \tag{19}$$

Setting $\kappa = F_0$, (18) implies that the sale of a corridor note can be hedged by initially buying $\frac{2c}{\sigma^2 K^{2p+2}}dK$ puts of all strikes K between L and F_0 and $\frac{2c}{\sigma^2 K^{2p+2}}dK$ calls of all strikes K between F_0 and H. To complete the hedge, define $\underline{\bar{F}}$ as the futures price capped at H and floored at L. Then, substituting g and σ in (15):

$$f_c'(F) = \int_{F_0}^{F} \frac{2c1_{x\in(L,H)}}{\sigma^2 x^{2p+2}}dx = \begin{cases} \frac{-2c}{\sigma^2(2p+1)}\left[F_0^{-2p-1} - (\underline{\bar{F}})^{-2p-1}\right] & \text{if } p \neq -\frac{1}{2}; \\ \frac{2c}{\sigma^2}\ln\left(\frac{\underline{\bar{F}}}{F_0}\right) & \text{if } p = -\frac{1}{2}. \end{cases} \tag{20}$$

From (11), the number of futures held at each $t \in [0,T]$ is $-e^{-r(T-t)}f_c'(F_t)$. Note that in contrast to standard delta-hedging, no dynamic trading is required to hedge the corridor note whenever $F_t < L$ or $F_t > H$. To obtain the corridor note value, simply substitute g and σ in (12):

$$V_0 = \frac{2c}{\sigma^2}\left[\int_L^{F_0-} \frac{1}{K^{2p+2}}P_0(K,T)dK + \int_{F_0+}^{H} \frac{1}{K^{2p+2}}C_0(K,T)dK\right]. \tag{21}$$

Clearly, more complicated payoff structures and volatility functions are easily handled.

As the corridor note example illustrates, only f'' and f' are needed for both hedging and valuation. Although the payoff function f is never needed, it is interesting to integrate (20) to see what payoff[5] the options are providing:

$$f(F) = \begin{cases} \frac{2c}{\sigma^2(1+2p)}\left[\frac{F}{F_0^{1+2p}} + \frac{1}{2p\underline{\bar{F}}^{2p}} - \frac{2p+1}{2pF_0^{2p}} + \frac{\underline{\bar{F}}-F)}{\underline{\bar{F}}^{2p+1}}\right] & \text{if } p \neq 0, -\frac{1}{2}; \\ \frac{2c}{\sigma^2}[F\ln\left(\frac{\underline{\bar{F}}}{F_0}\right) + F_0 - \underline{\bar{F}}] & \text{if } p = -\frac{1}{2}; \\ \frac{2c}{\sigma^2}\left[\ln\left(\frac{F_0}{\underline{\bar{F}}}\right) + F\left(\frac{1}{F_0} - \frac{1}{\underline{\bar{F}}}\right)\right] & \text{if } p = 0. \end{cases} \tag{22}$$

The dynamic component of the EDH strategy can be interpreted as a standard delta-hedging strategy for the static position in options. Under this interpretation, the dynamic hedging strategy is conducted under the false assumption that volatility is zero. The use of the wrong volatility in the delta induces a 'hedging error' of size $\frac{\sigma^2(F_t)F_t^2}{2}f''(F_t)$ at each instant t. The payoff f is chosen so that this error is exactly the desired instantaneous cash flow g.

[5]The function f in (16) has zero value and slope at $F = F_0$ as required by (9), and is continuous and differentiable at L and at H. Although it is not twice differentiable at L and H, the standard form of Itô's lemma given in (5) is still valid (see Harrison[13] pg. 70).

Recall that the last section argued that in the presence of transactions costs, the additional setup costs in the EDH strategy may be offset by a lower trading volume in the futures contracts, both initially and through the option's life. To analyze this further in the current context, note from (14) that if the claim to be hedged has a nonnegative payoff (i.e. $g \geq 0$), as is typically the case, then $f'' \geq 0$ and so from (12), the EDH stategy involves a long position in options. The positive gamma in the enhanced delta hedge portfolio reduces the transactions costs associated with trading in the underlying. It also provides ad hoc protection against stochastic volatility and jumps, although further analysis is required to optimally insure against these contingencies.

The last section also argued that even if options are not used in the hedge, our representation of the claim's payoff simplifies valuation when dynamically replicating. To illustrate this point in the current context, note that when valuing a simple corridor note under geometric Brownian motion, the standard approach would require evaluating:

$$V_0^c = Be^{-rT} E_0^Q \int_0^T c1_{F_t \in (L,H)} dt = ce^{-rT} \int_0^T Q_0\{F_t \in (L,H)\} dt,$$

under the risk-neutral process:

$$\frac{dF_t}{F_t} = \sigma dW_t, \qquad t \in [0,T].$$

While one can evaluate the integral explicitly, the answer must be the same as evaluating $e^{-rT} E_0^Q f_c(F_T)$ where for $p = 0$, f_c is given in (16) as $\frac{2c}{\sigma^2}\left[\ln\left(\frac{F_0}{\underline{F}}\right) + F\left(\frac{1}{F_0} - \frac{1}{\underline{F}}\right)\right]$. This expectation is easily expressed in terms of the standard normal distribution and density functions.

III-B Deferred Payout on Spot

The results of the last subsection required the ability to trade in futures on the underlying and in European futures options. For some assets, futures contracts may not be available, or European options may only be written on the spot price. For these reasons, this subsection considers EDH when the payoff depends on the spot price path $\{S_t, t \in [0,T]\}$ of an asset. Without loss of generality, we take this asset to be a stock. We assume that dividends are paid continuously over time and thus the cost of carrying the stock is

$r_u = r - \delta(S)$, where the dividend yield at t is assumed to be an arbitrary function $\delta(S_t)$ of the stock price. As in the last subsection, we assume that the payout is deferred without interest to maturity ($r_c = 0$), so that the final payoff at T is $\int_0^T g(S_t)dt$. The next section considers the analysis when the contractual rate is the riskfree rate ($r_c = r$). Replacing U_t with S_t and substituting $r_u = r - \delta(S)$ and $r_c = 0$ in (11) and (7) gives:

$$\int_0^T g(S_t)dt = -f(S_0) + \int_0^{\kappa-} f''(K)(K-S_T)^+ dK + \int_{\kappa+}^{\infty} f''(K)(S_T-K)^+ dK - \int_0^T f'(S_t)\{dS_t - [r-\delta(S_t)]S_t dt\}, \quad (23)$$

where $f(S)$ solves the following o.d.e.:

$$\frac{\sigma^2(S)S^2}{2} f''(S) + [r - \delta(S)]Sf'(S) = g(S), \quad (24)$$

subject to (9). The first three terms on the RHS of (23) again comprise a static position in bonds, puts, and calls respectively. The final term is now the value at T from being short $e^{-r(T-t)} f'(S_t)$ shares at each time $t \in [0, T]$. Since the integral accounts for the fact that all purchases and sales are financed with the riskless asset, there is no cost associated with initializing or maintaining this strategy. Consequently, the initial value of the payoff $\int_0^T g(S_t)dt$ at T is again given by:

$$V_0 = -f(S_0)e^{-rT} + \int_0^{\kappa-} f''(K)P_0(K,T)dK + \int_{\kappa+}^{\infty} f''(K)C_0(K,T)dK. \quad (25)$$

The dynamic component of the EDH strategy can again be interpreted as a standard delta-hedging strategy for a portfolio of options with total payoff f. Now, the dynamic hedging strategy is conducted under the false assumptions that the volatility and carrying cost of the underlying are both zero. The use of the wrong volatility and carry in the delta induces a 'hedging error' of size $\frac{\sigma^2(S_t)S_t^2}{2} f''(S_t) + [r - \delta(S_t)]S_t f'(S_t)$ at each instant t. The payoff f must be chosen so that this error is exactly the desired instantaneous cash flow g.

Thus, to complete the valuation, we must solve the o.d.e. (24). This is a first order o.d.e. in f' which must be solved subject to (9). Introducing an integrating factor yields:

$$f'(S) = e^{-\beta(S)} \int_\kappa^S e^{\beta(y)} \frac{2g(y)}{\sigma^2(y)y^2} dy, \text{ with } \beta(x) \equiv \int^x \frac{2[r - \delta(z)]z}{\sigma^2(z)z^2} dz, \quad (26)$$

where recall that for $b > a$, $\int_b^a h(y)dy \equiv -\int_a^b h(y)dy$. If we set $\kappa = S_0$, then from (9) and (25), there is no need to integrate (26) for f. There is also no need to differentiate (26) for f'', since (24) implies that f'' is a simple function of f' and g:

$$f''(S) = -\frac{2[r-\delta(S)]S}{\sigma^2(S)S^2} f'(S) + \frac{2}{\sigma^2(S)S^2} g(S). \tag{27}$$

The standard delta-hedging strategy involves solving a *partial* differential equation (p.d.e.) for the value. In contrast, the EDH strategy only requires solving a first order o.d.e for the delta of a final payoff. Solutions to p.d.e.'s are only explicitly given for certain dividend yield and volatility functions. In contrast, for arbitrary dividend yield and volatility functions of the stock price, the o.d.e can always be solved, with the solution expressed as an integral on a bounded domain. Since the payoff rate g is likely to be simple, this integral can often be found in closed form.

To illustrate this EDH strategy, again consider valuing a corridor note paying c for each instant the underlying stock price is in a corridor bracketing the initial price, i.e. $g(S) = c1_{S\in(L,H)}$ with $S_0 \in (L, H)$. In order to compare with known results, we assume constant proportional dividends i.e. $\delta(S) = \delta$ and a CEV process for the stock i.e. $\sigma(S) = \sigma S^p$, where δ, σ, and p are constants. Then from (26), $\beta(x) = -c_0 x^{-2p}$ with $c_0 \equiv \frac{r-\delta}{\sigma^2 p}$ for $p \neq 0$ and $\beta(x) = 2\frac{r-\delta}{\sigma^2}\ln x$ when $p = 0$. Substituting g and β in (26) and setting $\kappa = S_0$ gives:

$$f_c'(S) = \begin{cases} e^{c_0 S^{-2p}} \int_{S_0}^{S} e^{-c_0 y^{-2p}} \frac{2c1_{y\in(L,H)}}{\sigma^2 y^{2+2p}} dy & \text{if } p \neq 0, \\ e^{-2\frac{r-\delta}{\sigma^2}\ln S} \int_{S_0}^{S} e^{2\frac{r-\delta}{\sigma^2}\ln y} \frac{2c1_{y\in(L,H)}}{\sigma^2 y^2} dy & \text{if } p = 0. \end{cases} \tag{28}$$

Defining $\bar{\underline{S}}$ as the stock price capped at H and floored at L gives:

$$f_c'(S) = \begin{cases} e^{c_0 S^{-2p}} \frac{2c}{\sigma^2} \int_{S_0}^{\bar{\underline{S}}} e^{-c_0 y^{-2p}} y^{-2-2p} dy & \text{if } p \neq 0, \\ S^{-2\frac{r-\delta}{\sigma^2}} \frac{2c}{\sigma^2} \int_{S_0}^{\bar{\underline{S}}} y^{2\frac{r-\delta}{\sigma^2}-2} dy & \text{if } p = 0. \end{cases} \tag{29}$$

When $p \neq 0$, and for[6] $r \neq \delta$, the change of variables $t = c_0 y^{-2p}$ implies that the integral can be expressed in terms of the incomplete gamma function $\gamma(a, z) \equiv \int_0^z t^{a-1}e^{-t}dt$:

$$f_c'(S) = e^{c_0 S^{-2p}} \frac{c c_0^{-q}}{r-\delta} \left[\gamma(q, c_0 S_0^{-2p}) - \gamma(q, c_0 \bar{\underline{S}}^{-2p})\right], \tag{30}$$

where $q \equiv \frac{1}{2p} + 1$.

[6] If $r = \delta$, then the spot price process matches the forward/futures price process covered in the previous section.

For $p=-1$ and $p=-\frac{1}{2}$, the integral in (29) can also be expressed in terms of other special functions. When $p=-1$, the process is Gaussian[7] and the integral can be expressed in terms of the standard normal distribution function. When $p=-\frac{1}{2}$, the process is a square root process and the integral can be expressed in terms of the exponential integral $Ei(x) \equiv \int_{-\infty}^{x} \frac{e^t}{t} dt$. When $p=0$, the process is geometric Brownian motion and the integral can be solved explicitly:

$$f_c'(S) = S^{-2\frac{r-\delta}{\sigma^2}} \frac{c}{r-\delta-\sigma^2/2} \left[\underline{\bar{S}}^{2\frac{r-\delta}{\sigma^2}-1} - S_0^{2\frac{r-\delta}{\sigma^2}-1} \right]. \tag{31}$$

Recall from (23) that the replication involved a short position in $e^{-r(T-t)} f'(S_t)$ shares at each time $t \in [0,T]$. The static portion involves initially buying $f''(K)dK$ puts at all strikes below S_0 and $f''(K)dK$ calls at all strikes above S_0. The second derivative is obtained from substituting $g(S) = c1_{S\in(L,H)}$, $\delta(S)=\delta$, and $\sigma(S)=\sigma S^p$ in (27):

$$f_c''(K) = -\frac{2[r-\delta]K}{\sigma^2 K^{2p+2}} f_c'(K) + \frac{2}{\sigma^2 K^{2p+2}} 1_{K\in(L,H)}. \tag{32}$$

Substituting this and $f(S_0)=0$ in (25) gives the initial value of the corridor note.

IV Non-Deferred Payouts

Thus far, we have dealt with claims whose payout is deferred without interest to maturity. We now consider claims which pay out continuously over time i.e. pay-as-you-go claims. When the cash flows of the annuity are linked to the spot price, then the annuity can be explicitly valued for arbitrary volatility functions, provided that dividends are constant. In contrast, when the cash flows of the annuity are linked to the futures price, explicit solutions are obtainable only by restricting volatility, although dividends are now unrestricted.

[7]One should place an absorbing boundary at the origin to rule out negative stock prices.

IV-A Pay-As-You-Go on Spot

If the continuous cash flow at t is a given function g of the spot price S_t, then the total cash flow accumulates to $\int_0^T e^{r(T-t)} g(S_t)dt$ by T. Replacing U_t with S_t and substituting $r_u = r - \delta(S)$ and $r_c = r$ in (11) and (7) gives:

$$\int_0^T e^{r(T-t)} g(S_t)dt = -f(S_0)e^{rT} + \int_0^{\kappa-} f''(K)(K-S_T)^+ dK + \int_{\kappa+}^{\infty} f''(K)(S_T-K)^+ dK - \int_0^T e^{r(T-t)} f'(S_t)\{dS_t - [r-\delta(S_t)]S_t dt\}, \tag{33}$$

where $f(S)$ solves the following o.d.e.:

$$\frac{\sigma^2(S)S^2}{2} f''(S) + [r - \delta(S)]Sf'(S) - rf(S) = g(S), \tag{34}$$

subject to (9).

From (33), the desired payoff is once again the sum of a static position and a dynamic strategy. The initial cost of creating the static component of the payoff is:

$$V_0 = -f(S_0) + \int_0^{\kappa-} f''(K)P_0(K,T)dK + \int_{\kappa+}^{\infty} f''(K)C_0(K,T)dK. \tag{35}$$

The final term in (33) is now the value at T from being short $f'(S_t)$ shares at each time $t \in [0,T]$ with the initial stock position and the subsequent revisions financed with the riskless asset. Since there is no out-of-pocket cost associated with initializing or maintaining this strategy, the initial value of the payoff $\int_0^T e^{r(T-t)} g(S_t)dt$ at T is also given by (35).

The dynamic component of the EDH strategy can again be interpreted as a standard delta-hedging strategy for a portfolio of options with total payoff f. This dynamic hedging strategy is conducted under the false assumptions that the volatility and carrying cost of the underlying are both zero. The hedging strategy also wrongly assumes that the premium is received at maturity rather than initially. The cumulative effect of these mistakes is a 'hedging error' of size $\frac{\sigma^2(S_t)S_t^2}{2} f''(S_t) + [r - \delta(S_t)]S_t f'(S_t) - rf(S_t)$ at each instant t. The payoff f must be chosen so that this error is exactly the desired instantaneous cash flow g.

To obtain an alternative interpretation of the valuation equation (35), note from (10) with $U_t = S_t$ that the risk-neutral value of a payoff f with $f(\kappa) = f'(\kappa) = 0$ is:

$$E_0^Q e^{-rT} f(S_T) = \int_0^{\kappa-} f''(K) P_0(K,T) dK + \int_{\kappa+}^{\infty} f''(K) C_0(K,T) dK, \tag{36}$$

where expectations are evaluated under the process:

$$\frac{dS_t}{S_t} = [r - \delta(S)]dt + \sigma(S_t) dW_t.$$

Substituting (36) into (35) implies that the value of the cash flow $g(S_t)$ paid continuously from $t = 0$ to $t = T$ is:

$$V_0 = E_0^Q e^{-rT} f(S_T) - f(S_0) = \phi(S_0) - E_0^Q e^{-rT} \phi(S_T), \tag{37}$$

where $\phi(S) \equiv -f(S)$. Replacing f with ϕ in (34) implies that $\phi(S)$ describes the value of a perpetual claim paying the cash flow $g(S_t)$ into perpetuity. Thus, $\phi(S_0)$ on the RHS of (37) is the initial value of this perpetuity, while $E_0^Q e^{-rT} \phi(S_T)$ is the value of a deferred perpetuity, where the cash flows begin at time T. The difference between these two terms is clearly the value of the cash flow $g(S_t)$ paid continuously from $t = 0$ to $t = T$. Thus, we have generalized the usual argument determining the value of a fixed annuity from the value of a perpetuity less a deferred perpetuity. The main difference[8] is that the last term on the RHS of (37) is obtained via static replication with options, so that (35) obtains.

To make (35) an explicit valuation formula, we must solve the o.d.e. (34). It is well-known that the key to solving this second order inhomogeneous o.d.e. is to find a solution $h(S)$ to the corresponding homogeneous equation:

$$\frac{\sigma^2(S)S^2}{2} h''(S) + [r - \delta(S)] S h'(S) - r h(S) = 0. \tag{38}$$

To solve (38), we set $\delta(S) = \frac{d}{S}$, i.e. we now require that the continuous dividend flow be constant at rate d over the time interval $[0, T]$. If the underlying is a stock, then the assumption of a constant dividend payment rate is more consonant with reality than the usual assumption of constant proportional dividends, particularly over short and intermediate horizons (eg. less than 2 years). If the underlying is a currency,

[8]Furthermore, it is difficult to apply this argument to variable annuities written on futures prices or with deferred payout.

or if the contract is long term, then the payout on the variable annuity should be either deferred or written on the futures, so that the results of the other sections can be employed.

Under the constant dividend flow assumption $\delta(S) = \frac{d}{S}$, a solution to the homogeneous o.d.e. (38) is $h_1(S) \equiv S - \frac{d}{r}$. To interpret this solution, one could assume that dividends were constant into perpetuity, in which case $S - \frac{d}{r}$ is the risky component of the stock. In any event, using reduction of order, a second linearly independent solution to the homogeneous o.d.e. (38) is:

$$h_2(S) \equiv \left(S - \frac{d}{r}\right) \int_{d/r}^{S} \frac{e^{-\beta(z)}}{\left(z - \frac{d}{r}\right)^2} dz = \left(S - \frac{d}{r}\right) \int_{d/r}^{S} \frac{e^{-\beta(z)} - 1}{\left(z - \frac{d}{r}\right)^2} dz - 1,$$

where $\beta(z) \equiv 2 \int_{d/r}^{z} \frac{ry-d}{\sigma^2(y)y^2} dy$. Note that the rightmost representation of $h_2(S)$ implies that $h_2\left(\frac{d}{r}\right) = -1$ so that $h_2(S)$ is well-defined over all S. Armed with these two solutions to (38), perpetuities can be explicitly valued for *any* volatility function, assuming perpetually constant[9]dividends. Furthermore, using the method of variation of parameters (see eg. Boyce and DiPrima[3], pg. 123), the general solution to the inhomogeneous o.d.e. (34) with $\delta(S) = \frac{d}{S}$ and $f(\kappa) = f'(\kappa) = 0$ is:

$$f(S) = 2\left(S - \frac{d}{r}\right) \int_{\kappa}^{S} \frac{e^{\beta(x)}\left(x - \frac{d}{r}\right) g(x)}{\sigma^2(x)x^2} \int_{x}^{S} \frac{e^{-\beta(z)}}{\left(z - \frac{d}{r}\right)^2} dz dx. \tag{39}$$

IV-B Pay-As-You-Go on Futures

For some assets eg. crude oil, the futures market is more liquid than the spot market. Furthermore, if the underlying used to determine the continuous cash payout of a swap is a futures price, then no assumption on dividends is required in order to hedge the claim. For these reasons, this subsection examines the pricing and hedging of pay-as-you-go claims written on the futures price with final payoff $\int_0^T e^{r(T-t)} g(F_t) dt$ at T. Replacing U_t with F_t and setting $r_u = 0$ and $r_c = r$ in (11) and (7) leads to the following decomposition of this payoff:

$$\int_0^T e^{r(T-t)} g(F_t) dt = -f(F_0)e^{rT} + \int_0^{F_0-} f''(K)(K - F_T)^+ dK + \int_{F_0+}^{\infty} f''(K)(F_T - K)^+ dK - \int_0^T e^{r(T-t)} f'(F_t) dF_t, \tag{40}$$

[9]The randomness in the value of the perpetuity would have to arise from the random liquidation value.

where f solves the following o.d.e.:

$$\frac{\sigma^2(F)F^2}{2}f''(F) - rf(F) = g(F), \tag{41}$$

subject to (9). Thus from (40), the desired payoff is the sum of a static position in bonds and options and a costless dynamic short position in $f'(F_t)$ futures at each time $t \in [0, T]$. The arbitrage-free value of the payoff is the initial cost of creating the replicating portfolio:

$$V_0 = -f(F_0) + \int_0^{F_0-} f''(K)P_0(K,T)dK + \int_{F_0+}^{\infty} f''(K)C_0(K,T)dK. \tag{42}$$

Unfortunately, there is no general closed form solution to (41). Any linear second order o.d.e. with variable coefficients can be put in the form of (41), and so a general solution to (41) would imply a general solution to the complete set of second order linear o.d.e.'s. Nonetheless, series solutions can be used to solve (41) for an arbitrary volatility function $\sigma(F)$. Sturm-Liouville theory and Lie Group theory are also powerful solution techniques for given volatility specifications.

Fortunately, (41) has an explicit solution for a general payoff function $g(F)$ when $\sigma(F) = \sigma F^p$, i.e. for a CEV process. To obtain it, we first solve the homogeneous ODE:

$$\frac{\sigma^2 F^{2p}}{2}h''(F) - rh(F) = 0. \tag{43}$$

The two linearly independent solutions are functions of the modified Bessel functions:

$$h_1(F) = \sqrt{F} I_\nu(\wp F^{\frac{1}{2\nu}}) \tag{44}$$

$$h_2(F) = \sqrt{F} K_\nu(\wp F^{\frac{1}{2\nu}}), \tag{45}$$

where $\nu \equiv \frac{1}{2(1-p)}$ and $\wp \equiv \frac{\sqrt{2r}}{\sigma(1-p)}$. Using reduction of order, these homogeneneous solutions can be used to find the Green's function $G(F;K)$ solving the inhomogenous ODE:

$$\frac{\sigma^2 F^{2p}}{2}G''(F;K) - rG(F;K) = \delta(F-K), \tag{46}$$

and the homogeneous boundary conditions:

$$G(0;K) = 0 \tag{47}$$

$$G(\infty;K) = 0. \tag{48}$$

The function $h_1(F)$ in (44) clearly solves (47) while the function $h_2(F)$ in (45) solves (48). Accordingly, the Green's function is (after some algebra):

$$G(F;K) = -\frac{4\nu\sqrt{F}K^{-\frac{3}{2}+\frac{1}{\nu}}}{\sigma^2} I_\nu(\wp(F\wedge K)^{\frac{1}{2\nu}})K_\nu(\wp(F\vee K)^{\frac{1}{2\nu}}). \tag{49}$$

Thus, the solution of:

$$\frac{\sigma^2 F^{2p}}{2} f''(F) - rf(F) = g(F). \tag{50}$$

subject to:

$$f(0) = 0 \tag{51}$$

$$f(\infty) = 0, \tag{52}$$

is:

$$\int_0^\infty g(K)G(F;K)dK. \tag{53}$$

An important special case of the CEV process is geometric Brownian motion (p=1). In this case, the Green's function in (49) simplifies to:

$$G(F;K) = -\frac{F^{\frac{1}{2}}K^{-\frac{3}{2}}}{\sigma^2\beta}\left(\frac{F\wedge K}{F\vee K}\right)^\beta, \tag{54}$$

where $\beta \equiv \frac{\sqrt{1+\frac{8r}{\sigma^2}}}{2}$.

A second important special case of the CEV process is arithmetic Brownian motion (p=0) absorbing at the origin. In this case, the Green's function in (49) simplifies to:

$$G(F;K) = -\frac{2}{\sigma\sqrt{2r}}\sinh\left[\frac{\sqrt{2r}}{\sigma}(F\wedge K)\right]\exp\left[-\frac{\sqrt{2r}}{\sigma}(F\vee K)\right]. \tag{55}$$

V Summary and Further Extensions

We showed how EDH can be used to value and hedge contracts with continuous cash flows over time. We considered both deferred and non-deferred payouts and we considered payouts linked to both futures prices and to spot prices. In many realistic cases, by valuing relative to option prices, explicit valuation and hedging results were obtained when volatility is an arbitrary function of price.

The foregoing results can be extended in many ways. For example, one could allow for some time-dependence in the cash flows. A corridor note linked to the level of a geometric Brownian motion is easily valued if the lower and upper barriers defining the corridor grow exponentially at the same rate as the expected price. One can also extend the results to cash flows which knock out when the underlying reaches a barrier by hedging with barrier options. One can also consider EDH of claims with terminal payoffs of the form $f(U_T)\int_0^T g(U_t)dt$ by changing measure. Finally, some cash flows can be perfectly synthesized using EDH when the process is an arbitrary semi-martingale (see Carr, Lewis, and Madan[7]).

When compared to the alternatives of pure dynamic hedging and pure static hedging, EDH holds out the possibility of extracting the best features of both alternatives. As pointed out by a referee, it is also true that EDH might suffer from the worst features of both alternatives. Unfortunately, given the constraints of time and space, determining which of these outcomes arises in a given situation will have to be left for future research.

Appendix: Local Time

This appendix re-derives (11) with $\kappa = F_0$ using the concept of local time. From Karatzas and Shreve [16], pg. 218, (7.3), for every Borel measurable $k : \Re \to [0, \infty)$:

$$\int_0^T k(F_t) d\langle F \rangle_t = 2 \int_0^\infty k(K) \Lambda_T(K) dK, \tag{56}$$

where $\langle F \rangle_t$ denotes the quadratic variation of the futures price process:

$$\frac{dF_t}{F_t} = \mu_t dt + \sigma(F_t) dW_t, \qquad t \in [0, T], \tag{57}$$

and $\Lambda_T(x)$ is the local time at T of the process F at x. For the diffusion process (57), $d\langle F \rangle_t = \sigma^2(F_t) F_t^2 dt$, so substituting this and $g(x) \equiv k(x)\sigma^2(x)x^2$ in (56) yields:

$$\int_0^T g(F_t) dt = \int_0^\infty \frac{2g(K)}{\sigma^2(K)K^2} \Lambda_T(K) dK. \tag{58}$$

From Karatzas and Shreve[16], pg. 220, the Tanaka-Meyer formulas are:

$$\begin{aligned}
(K - F_T)^+ &= (K - F_0)^+ - \int_0^T 1_{F_t < K} dF_t + \Lambda_T(K) \\
(F_T - K)^+ &= (F_0 - K)^+ + \int_0^T 1_{F_t > K} dF_t + \Lambda_T(K).
\end{aligned}$$

Substituting them in (58) yields:

$$\begin{aligned}
\int_0^T g(F_t) dt &= \int_0^{F_0} \frac{2g(K)}{\sigma^2(K)K^2} \left[(K - F_T)^+ - (K - F_0)^+ + \int_0^T 1_{F_t < K} dF_t \right] dK \\
&\quad + \int_{F_0}^\infty \frac{2g(K)}{\sigma^2(K)K^2} \left[(F_T - K)^+ - (F_0 - K)^+ - \int_0^T 1_{F_t > K} dF_t \right] dK.
\end{aligned} \tag{59}$$

For $K < F_0, (K - F_0)^+ = 0$ and for $K > F_0, (F_0 - K)^+ = 0$. Employing Fubini's theorem:

$$\begin{aligned}
\int_0^T g(F_t) dt &= \int_0^{F_0} \frac{2g(K)}{\sigma^2(K)K^2} (K - F_T)^+ dK + \int_{F_0}^\infty \frac{2g(K)}{\sigma^2(K)K^2} (F_T - K)^+ dK \\
&\quad + \int_0^T \left[\int_0^{F_0} \frac{2g(K)}{\sigma^2(K)K^2} 1_{F_t < K} dK - \int_{F_0}^\infty \frac{2g(K)}{\sigma^2(K)K^2} 1_{F_t > K} dK \right] dF_t.
\end{aligned} \tag{60}$$

If $F_t > F_0$, the stochastic integral is $-\int_{F_0}^{F_t} \frac{2g(K)}{\sigma^2(K)K^2} dF_t$, while if $F_t < F_0$, it is $\int_{F_t}^{F_0} \frac{2g(K)}{\sigma^2(K)K^2} dF_t$. Recalling that for $b > a, \int_b^a h(K) dK \equiv -\int_a^b h(K) dK$, we have:

$$\int_0^T g(F_t) dt = \int_0^{F_0} \frac{2g(K)}{\sigma^2(K)K^2} (K - F_T)^+ dK + \int_{F_0}^\infty \frac{2g(K)}{\sigma^2(K)K^2} (F_T - K)^+ dK - \int_0^T \int_{F_0}^{F_t} \frac{2g(K)}{\sigma^2(K)K^2} dK dF_t.$$

References

[1] Black, F., 1976, "The Pricing of Commodity Contracts", *Journal of Financial Economics*, **3**, 167-79.

[2] Black, F. and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities", *The Journal of Political Economy*, **81**, 637–659.

[3] Boyce and DiPrima, 1977, Elementary Differential Equations and Boundary Value Problems, third edition, Wiley, NY.

[4] Breeden, D. and R. Litzenberger, 1978, "Prices of State Contingent Claims Implicit in Option Prices," *Journal of Business*, **51**, 621-651.

[5] Carr, P., and R. Jarrow, 1989, "The Stop-Loss Start-Gain Strategy and Option Valuation: A New Decomposition into Intrinsic and Time Value", *Review of Financial Studies*, **3**, 469-492.

[6] Carr, P. and D. B. Madan, 2000, "Optimal Positioning in Derivative Securities," Working Paper, University of Maryland.

[7] Carr, P., Lewis, K., and D. B. Madan, 2000, "On the Nature of Options," Working Paper, University of Maryland.

[8] Choi, J. and F. Longstaff, 1985, "Pricing Options on Agricultural Futures: An Application of the Constant Elasticity of Variance Option Pricing Model," *Journal of Futures Markets*, **5**, Spring, 247-58.

[9] Cox, J., 1975, "Notes on Option Pricing I: Constant Elasticity of Variance Diffusions", Stanford University working paper.

[10] Cox, J. and S. Ross, 1976, "The Valuation of Options for Alternative Stochastic Processes", *Journal of Financial Economics*, **3**, 145-166.

[11] Derman, E., D. Ergener, and I. Kani, 1995, "Static Options Replication," *Journal of Derivatives*, Summer, **2**, 4, 78–95.

[12] Green, R.C. and R.A. Jarrow, 1987, "Spanning and Completeness in Markets with Contingent Claims," *Journal of Economic Theory*, **41**, 202-210.

[13] Harrison J. M., 1985, *Brownian Motion and Stochastic Flow Systems*, Wiley, NY.

[14] Harrison J. M. and D. Kreps, 1979, "Martingales and Multiperiod Securities Markets", *Journal of Economic Theory*, **20**, 381–408.

[15] Harrison J. M. and S. Pliska, 1981, "Martingales and Stochastic Integrals in the Theory of Continuous Trading", *Stochastic Processes and Their Applications*, **11**, 215–260.

[16] Karatzas I. and S. Shreve, 1988, Brownian Motion and Stochastic Calculus, Springer, NY.

[17] Merton, R., 1973, "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science*, **4**, 141–183.

[18] Merton, R., 1974, "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates", *Journal of Finance*, **29**: 449–70.

[19] Nachman, D., 1988, "Spanning and Completeness with Options," *Review of Financial Studies*, **3**, 31, 311-328.

[20] Ross, S., 1976, "Options and Efficiency," *Quarterly Journal of Economics*, **90**, 75-89.

[21] Rogers L.C.G. and D. Williams, 1984, Diffusions, Markov Processes, and Martingales, Volume 1, second edition, Wiley, NY.

[22] Rogers L.C.G. and D. Williams, 1987, Diffusions, Markov Processes, and Martingales, Volume 2, Itô Calculus, Wiley, NY.