Financial Interpretations of Probabilistic Concepts

or:

A Banker Reads a Book

Presentation for Midwest Probability Conference

Peter Carr
Morgan Stanley
1585 Broadway, 16th floor
New York, NY 10036
(212) 761-7340
carrp@ms.com

Initial version: October 21, 1998
File reference: finprob.tex

Contents

Part I

Introduction

A Double Play: Accounting to Probability to Finance

- In Luca Pacciolo's masterwork *Summa de arithmetic, geometria et proportionalità* published in 1494, the monk posed the following "problem of the points":

  > A and B are playing a fair game of *balla*. They agree to continue until one has won six rounds. The game actually stops when A has won five and B three. How should the stakes be divided?

  In 1654, the inveterate gambler Chevalier de Méré brought the problem of the points to the attention of Pascal, who initiated a correspondence with Fermat. Pascal solved this problem with the aid of his famous triangle, putting probability theory on a theoretical foundation for the first time.

- In 1994, during the heyday of the derivatives debacle, the cover story of Time magazine read:

  > [T]his fantastic system of side bets is not based on old-fashioned human hunches but on calculations designed and monitored by computer wizards using abstruse mathematical formulas ... developed by so-called quants, short for quantitative analysts.

A Banker Reads a Book

- All of the probabilistic concepts come from Continuous Martingales and Brownian Motion, Second Edition, by Daniel Revuz and Marc Yor, Springer Verlag 1994.
- I will refer to this text as RY.
- Two good sources for financial concepts are:
  1. Introduction to Mathematical Finance, by Stanley Pliska, Blackwell Publishers, 1997
  2. Dynamic Asset Pricing Theory, Second Edition, by Darrell Duffie, Princeton University Press, 1996.
- These overheads can be downloaded from my web site: www.math.nyu.edu/research/carrp/papers

Part II

Expected Value and No Arbitrage Value

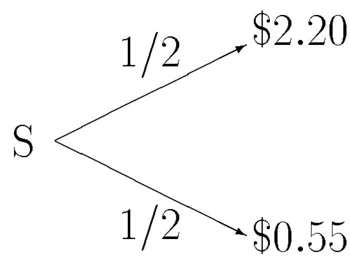
Definitions

- RY don't define expected value for obvious reasons.
- Fortunately, finance people define expected value in the same way as probabilists.
- Unfortunately, the word "value" in expected value is interpreted by some practitioners a little too literally.

Practitioners and Academics

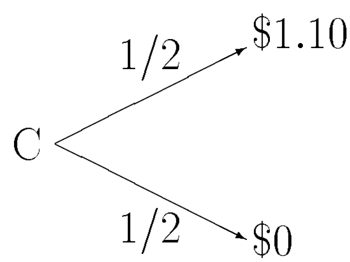
- Practitioners frequently exhort that the value of an asset is just its "expected discounted value". In the following examples, the interest rate is 10% per year.

  Example 1: Stock



$$\Longrightarrow S = \$1.25$$

  Example 2: Call Struck at $1.10

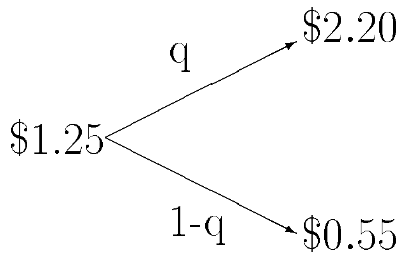


$$\Longrightarrow C = \$0.50$$

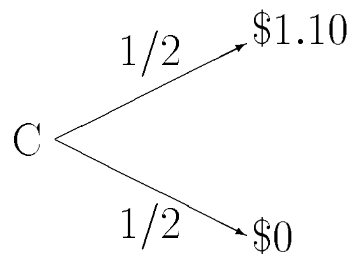
- Academics frequently retort that the value of an asset is actually just "the expected value of the ratio of the payoff to a money market account, where expectations are calculated under an appropriate equivalent martingale measure".

Probabilities from Prices

- Rightly or wrongly, practitioners obtain the probabilities needed for valuation by backing them out of market prices:



$\Longrightarrow q = 1/2$

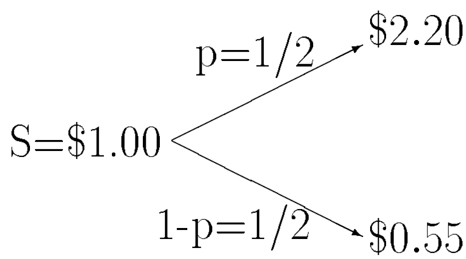


$\Longrightarrow C = \$0.50$

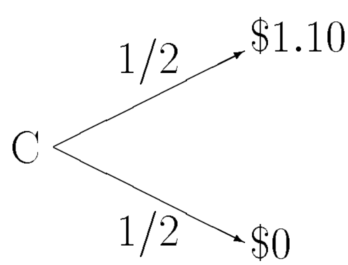
- As long as probabilities are implied from prices, practitioners get the right answer. These implied probabilities are referred to as "risk-neutral" probabilities.

Type A Error

- Type A error (A for arbitrage) arises when actual probabilities (p,1-p) are used in place of risk-neutral probabilities (q,1-q). For example:

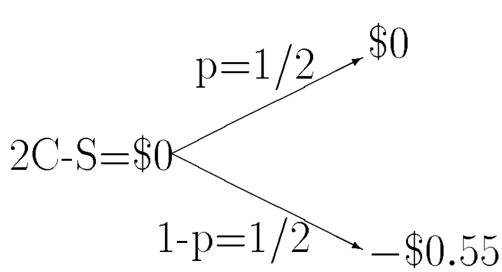


- Risk aversion (fear) causes the stock to be priced below expected discounted value. Suppose a call struck at \$1.10 is valued as:



$$\Longrightarrow C = \$0.50$$

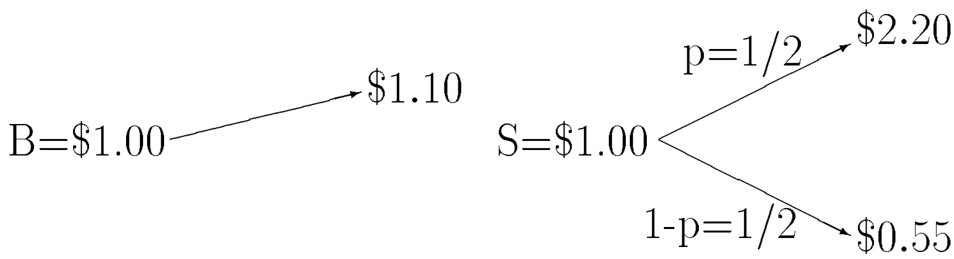
- If we make markets in the stock for \$1 and in the call for 50¢, then greed will induce hedge funds to sell 2 calls to us and buy a share from us:



- We sold the hedge fund a free at-the-money put.

Bonds and Stocks and Calls and Puts

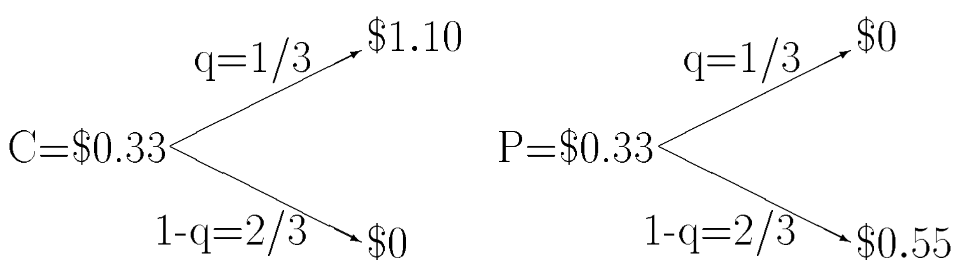
- Suppose bonds and stocks are priced as:



- Faced with these prices, practitioners and academics agree that the risk-neutral up probability is 1/3 since:

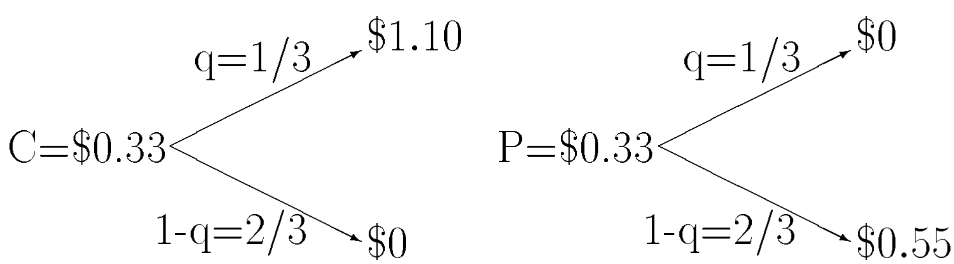
$$1/3 \times \$2 + 2/3 \times \$0.5 = \$1.$$

- Thus, the no arbitrage value of an at-the-money call and put are:



Time and State Value of Money

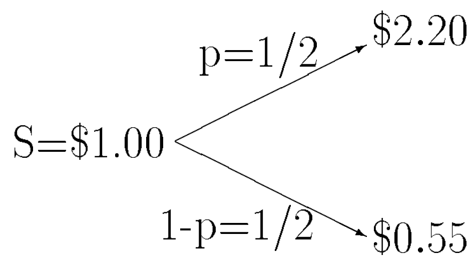
- The well known principle called "Time Value of Money" says that \$1 paid now is worth more than \$1 paid in a year because by depositing \$1 in the bank today, one gets more than a dollar (say \$1.10) in a year.
- Similarly, the "State Value of Money" principle says a bad state dollar is worth more than a good state dollar because by trading today, one gets more than one good state dollar in a year. Recall the at-the-money call and put values:



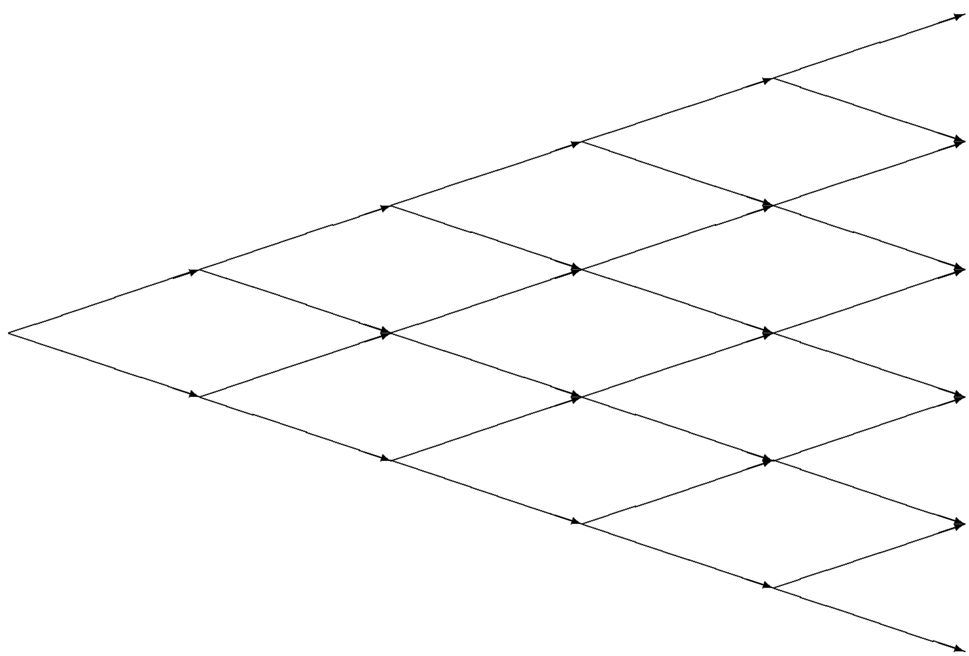
- The owner of an at-the-money put gets .55 bad state dollars. By selling the put and buying the call, a trader can instead get 1.10 good state dollars.
- Whether this is a good trade or not depends on the risk aversion of the trader or his firm.

Type B Error

- Type B error (B for bankruptcy) arises when risk-neutral probabilities (q,1-q) are used in place of actual probabilities (p,1-p). Recall the single period stock price process:



- Suppose that this evolution repeats indefinitely:



- Consider betting everything you own on the number of up moves in one trillion trials. In particular, would you bet that the proportion of up moves is closer to one half or one third?

Part III

Itô's Lemma and Integration by Parts vs. Gains Process

Probabilistic Concept

- RY page 139 prove the following version of Itô's lemma:

  If X is a continuous semimartingale and A is a continuous process of bounded variation and if $\frac{\partial^2 F}{\partial x^2}$ and $\frac{\partial F}{\partial y}$ exist and are continuous, then:

$$\begin{aligned} F(X_t, A_t) &= F(X_0, A_0) + \int_0^t \frac{\partial F}{\partial x}(X_s, A_s) dX_s + \int_0^t \frac{\partial F}{\partial y}(X_s, A_s) dA_s \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(X_s, A_s) d\langle X, X \rangle_s \end{aligned}$$

- Our primary interest will be in the special case when $A_t = t$. In differential form, we have:

$$dF(X_t, t) = \frac{\partial F}{\partial x}(X_t, t) dX_t + \frac{\partial F}{\partial t}(X_t, t) dt + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(X_t, t) d\langle X, X \rangle_t$$

- RY page 138 prove the following version of the integration by parts formula:

  If X and Y are two continuous semimartingales, then:

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t$$

- In differential form, we have:

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + d\langle X, Y \rangle_t.$$

- Our primary interest will be when X is the stock price and Y is the number of shares held.

Textbook Derivation of Black Scholes PDE

- Assume frictionless markets, no arbitrage, constant interest rate r, no dividends, and that the stock price process $\{S_t, t \in [0,T]\}$ is a diffusion with constant volatility σ:
$$\frac{dS_t}{S_t} = \mu_t dt + \sigma dW_t, \quad t \in [0,T].$$
- Letting $\{C_t, t \in [0,T]\}$ denote the European call price process, further assume $C_t = C(S_t,t), t \in [0,T]$ for some $C^{2,1}$ function $C(S,t)$ mapping $\Re^+ \times [0,T]$ into $\Re^+$. Then by Itô's lemma:
$$dC_t = \left[\frac{\partial C}{\partial t}(S_t,t) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 C}{\partial S^2}(S_t,t)\right]dt + \frac{\partial C}{\partial S}(S_t,t)dS_t, \quad t \in [0,T].$$
- Consider a hedge portfolio consisting of long one call and short $\frac{\partial C}{\partial S}(S,t)$ shares:
$$H_t \equiv C_t - \frac{\partial C}{\partial S}S_t, \quad t \in [0,T].$$
- The textbook argument is that:
$$\begin{aligned} dH_t &= dC_t - \frac{\partial C}{\partial S}dS_t \\ &= \left[\frac{\partial C}{\partial t}(S_t,t) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 C}{\partial S^2}(S_t,t)\right]dt \\ &= r\left[C(S_t,t) - \frac{\partial C}{\partial S}(S_t,t)S_t\right]dt, \end{aligned}$$
by the absence of arbitrage. Equating coefficients on dt yields the Black Scholes PDE:
$$\frac{\partial C}{\partial t}(S,t) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}(S,t) + r\frac{\partial C}{\partial S}(S,t)S - rC(S_t,t) = 0, t \in (0,T), S > 0.$$
- Why do the textbooks ignore integration by parts?

Gains Process

- Recall the textbook argument that if:
$$H_t \equiv C_t - \frac{\partial C}{\partial S} S_t, \qquad t \in [0,T]$$
then:
$$dH_t = dC_t - \frac{\partial C}{\partial S} dS_t, \qquad t \in [0,T].$$

- Integration by parts requires that the second equation should instead be:
$$dH_t = dC_t - \frac{\partial C}{\partial S} dS_t - d(\frac{\partial C}{\partial S}) S_t - d\langle \frac{\partial C}{\partial S}, S \rangle_t, \quad t \in [0,T].$$

- The additional two terms represent the additional investment needed to maintain the strategy. The third term involves the differential of a process of unbounded variation, so one cannot argue that dH_t is riskless.

- The fastest fixup is to replace the total derivative of H in the second equation by the *gain* on the hedge portfoio:
$$gH_t \equiv dC_t - \frac{\partial C}{\partial S} dS_t, \quad t \in [0,T].$$
If $dC_t = dC(S_t, t)$, then the argument to the PDE is the same as on the previous page.

- When a process has the form $X_t = N_t S_t$, then the gains process is $G_t \equiv \int_0^t N_t dS_t$.

- We distinguish gains from P&L, which also takes into account the cost of financing the purchase:
$$d\pi_t \equiv dC_t - \frac{\partial C}{\partial S}(S_t, t) dS_t - r[C_t - \frac{\partial C}{\partial S}(S_t, t) S_t] dt, \quad t \in [0,T].$$

- If $dC_t = dC(S_t, t)$, then $d\pi_t = 0$ by the absence of arbitrage.

Part IV

Martingale Problem and Equivalent Martingale Measure

Probabilistic Concept

- On page 281, RY define for each time s, the second order differential operator
$$L_s = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(s,\cdot)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(s,\cdot)\frac{\partial}{\partial x_i},$$
where for each s, a and b denote a matrix field and a vector field on $\Re^d$ subject to the conditions:
  1. the maps $x \to a(x)$ and $x \to b(x)$ are Borel measurable and locally bounded,
  2. for each x, the matrix $a(x)$ is symmetric and non-negative i.e. for any $\lambda \in \Re^d, \Sigma_{i,j}\, a_{ij}(x)\lambda_i\lambda_j \geq 0$.

- RY page 283 define the *Martingale problem* as follows:

  A probability measure $Q^{\$}$ is a solution to the Martingale problem $\pi(x, a, b)$ if:

  1. $Q^{\$}[X_0 = x] = 1$
  2. for any $f \in C_K^{\infty}$, the process:
  $$M_t^f = f(X_t) - f(X_0) - \int_0^t L_s f(X_s) ds$$
  is a $Q^{\$}$-martingale w.r.t. the filtration $(\sigma(X_s, s \leq t)) = \mathcal{F}_t^0)$.

- We will be primarily interested in the single dimension case $d = 1$.

Dynamic Hedging

- We focus attention on derivative securities which have a specified final dollar payout $f(S_T)$ paid at a fixed time T, and which also have a specified intermediate dollar payout $i(S_t, t)$ paid at every $t \in [0, T]$.
- Consider the problem of dynamically hedging the sale of such a claim under the following assumptions:
  1. Frictionless markets
  2. No arbitrage
  3. Constant interest rate r
  4. Underlying pays a constant proportional dividend continuously over time:
  $$\frac{\$ \text{ amount of dividend over } [t, t+dt]}{dt} = \delta S_t,$$
  where δ is a non-negative constant.
  5. Continuous spot price process under P:
  $$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t, \quad t \in [0, T],$$
  where the mean growth rate process μ_t is adapted and the volatility process σ_t is a function of S_t and t only, i.e., there is a function σ such that:
  $$\sigma_t = \sigma(S_t, t).$$
- We also assume that m_t and $\sigma(S, t)$ are chosen so as to prevent negative prices and explosions.

Representing the Payoffs

- Itô's lemma applied to the function $V(S_t,t)e^{r(T-t)}$ gives:

$$V(S_T,T) \;=\; V(S_0,0)e^{rT} + \int_0^T e^{r(T-t)}\frac{\partial V}{\partial S}(S_t,t)dS_t$$
$$+\int_0^T e^{r(T-t)}\left[\frac{\sigma^2(S_t,t)S_t^2}{2}\frac{\partial^2 V}{\partial S^2}(S_t,t) - rV(S_t,t) + \frac{\partial V}{\partial t}(S_t,t)\right]dt.$$

- The 2nd term accumulates gains on $\frac{\partial V}{\partial S}(S_t,t)$ shares held at each $t\in[0,T]$. To get P&L instead, subtract the carrying cost from the gains as follows:

$$V(S_T,T) \;=\; V(S_0,0)e^{r(T-t)} + \int_0^T e^{r(T-t)}\frac{\partial V}{\partial S}(S_t,t)\left[dS_t - (r-\delta)S_t dt\right]$$
$$+\int_0^T e^{r(T-t)}\left[\frac{\sigma^2(S_t,t)S_t^2}{2}\frac{\partial^2 V}{\partial S^2}(S_t,t) + (r-\delta)S_t\frac{\partial V}{\partial S}(S_t,t) - rV(S_t,t)\right.$$
$$\left.+\frac{\partial V}{\partial t}(S_t,t)\right]dt.$$

- Now, by choosing $V(S,t)$ to solve the following PDE:

$$\frac{\sigma^2(S,t)S^2}{2}\frac{\partial^2 V}{\partial S^2}(S,t)+(r-\delta)S\frac{\partial V}{\partial S}(S,t)-rV(S,t)+\frac{\partial V}{\partial t}(S,t) \;=\; -i(S_t,t),$$

with:

$$V(S,T) \;=\; f(S),$$

we get:

$$f(S_T) + \int_0^T e^{r(T-t)} i(S_t,t)dt$$
$$=\; V(S_0,0)e^{rT} + \int_0^T e^{r(T-t)}\frac{\partial V}{\partial S}(S_t,t)\left[dS_t - (r-\delta)S_t dt\right].$$

Representing the Payoffs (con'd)

- Recall the representation of the final and intermediate payoffs:

$$
\begin{aligned}
& f(S_T) + \int_0^T e^{r(T-t)} i(S_t, t)dt \\
= & \; V(S_0, 0)e^{rT} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_t, t) \left[dS_t - (r - \delta)S_t dt \right].
\end{aligned}
$$

- Thus the final and intermediate payoffs are the sum of:
  1. the future value of the initial investment $V(S_0, 0)$ and
  2. the accumulated P&L from holding $\frac{\partial V}{\partial S}(S_t, t)$ shares, where all purchases are financed by borrowing and all sales are invested in the bank.
- It follows that the no arbitrage value of the payoff is $V(S_0, 0)$.

The Equivalent Martingale Measure

- Recall that the cost of creating $\int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_t, t) \left[dS_t - (r - \delta)S_t dt\right]$ paid at T was zero, given that V satisfied a certain PDE.
- Viewed as a process in T, the absence of arbitrage clearly requires that this stochastic integral have realizations on both sides of zero for all T (or else be zero).
- Consequently, one can define a measure $Q^{\$}$ such that the integral has zero mean under $Q^{\$}$ for all T.
- Since the integral is a $Q^{\$}$-martingale by the definition of $Q^{\$}$, and $Q^{\$}$ is equivalent to P, $Q^{\$}$ is called an *equivalent martingale measure*.
- Given our assumptions and given the initial prices of the stock and bond, this martingale measure is uniquely determined.
- Under $Q^{\$}$, the integrator $dS_t - (r - \delta)S_t dt$ has zero mean and has variance $\sigma^2(S_t, t)S_t^2 dt$. Consequently, there exists a unique $Q^{\$}-$Brownian motion $W_t^{\$}$ such that:

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma(S_t, t)dW_t^{\$}, \qquad t \in [0, T], \text{ where } S_0 = S.$$

Risk-Neutral Stock Price Process

- Recall that the absence of arbitrage has allowed us to define a unique martingale measure $Q^{\$}$ and a unique standard Brownian motion $\{W_t^{\$}; t \in [0, T]\}$ such that:

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma(S_t, t)dW_t^{\$}, \qquad t \in [0, T], \text{ where } S_0 = S.$$

- The drift of this process is simply the cost of carrying the underlying and has no greater significance. It is also the drift which would arise in equilibrium if all investors were risk-neutral, and for this reason, the process is also called the "risk-neutral" process. The martingale measure is also called the risk-neutral measure. These are terribly mis-leading terms, since we are definitely *not* assuming that investors are risk-neutral.

- The volatility of the risk-neutral process is the same as the volatility of the assumed process. Girsanov's theorem (covered in the next part) shows that this preservation of volatility arises whenever one has continuous sample paths, but it does *not* necessarily follow when prices can jump.

Risk-Neutral Valuation

- Since $f(S_T) = \int_0^\infty \delta(S_T - K) f(K) dK$ and for each t, $i(S_t, t) = \int_0^\infty \delta(S_t - K) i(K, t) dK$, the key to valuing derivatives in closed form is to find the value of a claim with payoff $\delta(S_T - K)$ at T. We call the (forward) price of this "butterfly spread" the risk-neutral transition density $q(t, S; u, K) \equiv Q^\$\{S_u \in dK | S_t = S\}$.

- Thus, the value of any derivative of the type we are examining can be written as:

  $$V(S,t) = \int_t^T e^{-r(u-t)} \int_0^\infty i(K,u) Q^\$\{S_u \in dK | S_t = S\} + e^{-r(T-t)} \int_0^\infty f(K) Q^\$\{S$$

- Note that the payouts $i(K, u)$ and $f(K)$ are unitless, while $Q^\$$ is measured in dollars.

- In general, to value a derivative given that we are at time t with the stock price at S, we first determine the forward price of each path from the measure $Q^\$$ defined by

  $$\frac{dS_u}{S_u} = (r - \delta) du + \sigma(S_u, u) dW_u^\$, \qquad u \in [t, T], \text{ where } S_t = S.$$

- For the derivative securities we are examining, all that matters about these paths is that they start at (t, S) and end at (u, K). When volatility is constant, the total measure of this path bundle is well-known to be:

  $$Q\{S_u \in dK | S_t = S\} = \frac{dK}{\sqrt{2\pi\sigma^2(u-t)}K} \exp\left\{-\frac{1}{2}\left[\frac{\ln(K/S) - \mu(u-t)}{\sigma\sqrt{u-t}}\right]^2\right\}$$

  We then determine the payout along each path bundle from the functions $i(S, u), u \in [t, T]$ and $f(S)$. The value is given by multiplying the payout along each path bundle by its price and then summing (integrating) over path bundles.

Part V

Girsanov's Theorem and Quantoing

Probabilistic Concept

- On page 311, RY let $(\mathcal{F}_t^0), t \geq 0$ be a right continuous filtration with terminal σ-field $\mathcal{F}_\infty^0$ and they let $Q^\$$ and $Q^£$ be two probability measures on $\mathcal{F}_\infty^0$. They assume that for each $t \geq 0$, the restriction of $Q^£$ to $\mathcal{F}_t^0$ is absolutely continuous w.r.t. the restriction of $Q^\$$ to $\mathcal{F}_t^0$, which they denote by $Q^£ \triangleleft Q^\$$.
- They also let D_t be the Radon Nikodym derivative of $Q^£$ w.r.t. $Q^\$$ on $\mathcal{F}_t^0$. They show that D_t is an $(\mathcal{F}_t^0, Q^\$)$ martingale, which is strictly positive $Q^£$ almost surely. On page 314, they state that if $Q^£$ is actually equivalent to $Q^\$$, then D is also strictly positive $Q^\$$ a.s. and:

  If D is a strictly positive continuous local martingale, there exists a unique continuous local martingale L such that:

  $$D_t = \exp\{L_t - \frac{1}{2}\langle L, L\rangle_t\}.$$

- On page 315, they prove the following version of *Girsanov's Theorem*:

  If $Q^£ = \exp\{L_T - \frac{1}{2}\langle L, L\rangle_T\} \cdot Q^\$$ and $M^\$$ is a continuous $Q^\$$-local martingale, then

  $$M^£ = M^\$ - D^{-1}\langle M^\$, D\rangle = M^\$ - \langle M^\$, L\rangle$$

  is a continuous $Q^£$ local martingale.

- Note that under $Q^£$, $M^\$$ is no longer a local martingale.

Dynamic Hedging of Quanto Derivatives

- We now consider the case where the payoff currency of the derivative is different from the currency describing the price S_t of the underlying stock.
- For example, we will consider the case where the payoff currency of the derivative is in pounds, while the underlying stock is denominated in dollars.

Assumptions

1. Frictionless markets
2. No arbitrage
3. Constant interest rates $r_{£}$ and $r_{\$}$
4. Underlying stock has a constant dividend yield δ
5. Continuous underlying stock price process S_t and (spot) exchange rate process R_t (in dollars per pound):

$$\begin{aligned} \frac{dS_t}{S_t} &= m_t^s dt + \sigma_s(S_t, t) dW_{1t} \\ \frac{dR_t}{R_t} &= m_t^r dt + \sigma_r(t)[\rho(S_t, t) dW_{1t} + \sqrt{1 - \rho^2(S_t, t)} dW_{2t}], \end{aligned}$$

where W_1 and W_2 are independent standard Brownian motions.

Representing the Payoff

- Apply Itô's lemma to $V(S_t, t)e^{r_£(T-t)}$ to get:

$$
\begin{aligned}
V(S_T, T) &= V(S_0, 0)e^{r_£ T} + \int_0^T e^{r_£(T-t)} \frac{\partial V}{\partial S}(S_t, t) dS_t \\
&+ \int_0^T e^{r_£(T-t)} \left[\frac{\sigma_s^2(S_t, t) S_t^2}{2} \frac{\partial^2 V}{\partial S^2}(S_t, t) - r_£ V(S_t, t) + \frac{\partial V}{\partial t}(S_t, t) \right] dt.
\end{aligned}
$$

- Since the stock trades only in dollars, the gain in pounds from holding one share of the stock over $[t, t+dt]$ is:

$$
\begin{aligned}
\frac{dS_t}{R_{t+dt}} &= \frac{dS_t}{R_t + dR_t} \\
&= \frac{1}{R_t} \frac{1}{1 + \frac{dR_t}{R_t}} dS_t \\
&\approx \frac{1}{R_t} \left(1 - \frac{dR_t}{R_t} \right) dS_t \\
&= \frac{1}{R_t} dS_t - \frac{1}{R_t} \sigma_{rs}(S_t, t) S_t dt,
\end{aligned}
$$

  where $\sigma_{rs}(S_t, t)$ is the covariance of dR/R and dS/S.

- Substituting $dS_t = R_t \frac{dS_t}{R_{t+dt}} + \sigma_{rs}(S_t, t) S_t dt$ in the top equation, we get:

$$
\begin{aligned}
V(S_T, T) &= V(S_0, 0)e^{r_£ T} \\
&+ \int_0^T e^{r_£(T-t)} \frac{\partial V}{\partial S}(S_t, t) R_t \frac{dS_t}{R_{t+dt}} + \\
&\int_0^T e^{r_£(T-t)} \left\{ \frac{\sigma_s^2(S_t, t) S_t^2}{2} \frac{\partial^2 V}{\partial S^2}(S_t, t) + \sigma_{rs}(S_t, t) S_t \frac{\partial V}{\partial S}(S_t, t) \right. \\
&\left. - r_£ V(S_t, t) + \frac{\partial V}{\partial t}(S_t, t) \right\} dt.
\end{aligned}
$$

- Recall:

$$
\begin{aligned}
V(S_T,T) &= V(S_0,0)e^{r_{\pounds}T} \\
&+ \int_0^T e^{r_{\pounds}(T-t)}\frac{\partial V}{\partial S}(S_t,t)R_t\frac{dS_t}{R_{t+dt}} + \\
&\int_0^T e^{r_{\pounds}(T-t)}\left\{\frac{\sigma_s^2(S_t,t)S_t^2}{2}\frac{\partial^2 V}{\partial S^2}(S_t,t) + \sigma_{rs}(S_t,t)S_t\frac{\partial V}{\partial S}(S_t,t)\right. \\
&\left. -r_{\pounds}V(S_t,t) + \frac{\partial V}{\partial t}(S_t,t)\right\}dt.
\end{aligned}
$$

- Make a further adjustment so that the second term represents the P&L from a zero-cost self-financing strategy:

$$
\begin{aligned}
V(S_T,T) &= V(S_0,0)e^{r_{\pounds}T} \\
&+ \int_0^T e^{r_{\pounds}(T-t)}\frac{\partial V}{\partial S}(S_t,t)R_t\left[\frac{dS_t}{R_{t+dt}} - (r_{\$}-\delta)\frac{S_t}{R_t}dt\right] \\
&+ \int_0^T e^{r_{\pounds}(T-t)}\left\{\frac{\sigma_s^2(S_t,t)S_t^2}{2}\frac{\partial^2 V}{\partial S^2}(S_t,t) + [r_{\$}-\delta+\sigma_{rs}(S_t,t)]S_t\frac{\partial V}{\partial S}(S_t,t)\right. \\
&\left. -r_{\pounds}V(S_t,t) + \frac{\partial V}{\partial t}(S_t,t)\right\}dt.
\end{aligned}
$$

- Choose $V(S,t)$ to solve the following generalized fundamental PDE:

$$
\begin{aligned}
&\frac{\sigma_s^2(S,t)S^2}{2}\frac{\partial^2 V}{\partial S^2}(S,t) + [r_{\$}-\delta+\sigma_{rs}(S,t)]S\frac{\partial V}{\partial S}(S,t) - r_{\pounds}V(S,t) \\
&+\frac{\partial V}{\partial t}(S,t) = -i(S_t,t), \text{with } V(S,T) = f(S).
\end{aligned}
$$

- Then we get $f(S_T) + \int_0^T e^{r_{\pounds}(T-t)}i(S_t,t)dt$

$$
= V(S_0,0)e^{r_{\pounds}T} + \int_0^T e^{r_{\pounds}(T-t)}\frac{\partial V}{\partial S}(S_t,t)R_t\left[\frac{dS_t}{R_{t+dt}} - (r_{\$}-\delta)\frac{S_t}{R_t}dt\right].
$$

- In this case, the dynamic strategy is to hold $\frac{\partial V}{\partial S}(S_t,t)R_t$ shares of the underlying stock at each $t \in [0,T]$, financed by borrowing in dollars and with gains converted into pounds.

Example: Butterfly Spread

- We assume that the volatilities and the covariance are constant.
- Consider no intermediate payoffs and the Dirac final payoff, where the underlying stock is denominated in dollars and the payoff is quantoed into pounds:

$$f(S) \;=\; \delta(S-K).$$

- Compare the PDE with no quantoing:

$$\frac{\sigma_s^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}(S,t) + (r_\$ - \delta) S \frac{\partial V}{\partial S}(S,t) - r_\$ V(S,t) + \frac{\partial V}{\partial t}(S,t) \;=\; 0,$$

to the PDE with quantoing:

$$\frac{\sigma_s^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}(S,t) + (r_\$ - \delta + \sigma_{rs}) S \frac{\partial V}{\partial S}(S,t) - r_£ V(S,t) + \frac{\partial V}{\partial t}(S,t) \;=\; 0.$$

- If we used "risk-neutral valuation", we would have an additional term σ_{rs} in the drift and we would have a new discount rate $r_£$.
- Recall the solution for the non-quantoed butterfly spread:

$$BS(S,t) \;=\; \frac{e^{-r_\$(T-t)}}{\sqrt{2\pi\sigma_s^2(T-t)}K}\exp\left\{-\frac{1}{2}\left[\frac{\ln(K/S)-\mu(T-t)}{\sigma_s\sqrt{T-t}}\right]^2\right\},$$

where $\mu \equiv (r_\$ - \delta - \frac{\sigma_s^2}{2})$.

- Thus, the quantoed butterfly spread value is:

$$QBS(S,t) \;=\; \frac{e^{-r_£(T-t)}}{\sqrt{2\pi\sigma_s^2(T-t)}K}\exp\left\{\frac{-1}{2}\left[\frac{\ln(K/S)-(\mu+\sigma_{rs})(T-t)}{\sigma_s\sqrt{T-t}}\right]^2\right\}.$$

Risk-Neutral Valuation of Quantoed Derivatives

- Recall the risk-neutral valuation formula for a path-independent derivative, when the payoff and underlying are both denominated in dollars:

$$V(S,t) = e^{-r_\$(T-t)} E^{Q^\$}[f(S_T)|S_t = S] \equiv e^{-r_\$(T-t)} E^{Q^\$}_{S,t} f(S_T),$$

where under the risk-neutral measure $Q^\$$, the risk-neutral stock price process is:

$$\frac{dS_u}{S_u} = (r_\$ - \delta)du + \sigma(S_u, u)dW^\$_u, \qquad u \in [t,T], \text{ where } S_t = S.$$

- Also recall that f is unitless, while $Q^\$$ is measured in time T dollars.

- If the payoff f is quantoed into pounds, the dollar value of the derivative changes. If we wish to continue using American forward prices of paths (i.e. $Q^\$$), then the change in dollar value arises from a change in the magnitude of the payoff:

$$QV(S,t)R_0 = e^{-r_\$(T-t)} E^{Q^\$}_{S,t} R_T f(S_T).$$

Here, QV is in pounds, $R_T f(S_T)$ is unitless, while $Q^\$$ continues to be measured in time T dollars.

- Alternatively, one can use British forward prices of the dollar denominated stock price paths. Denoting these forward prices by $Q^£_\$$, quantoed values are obtained by:

$$QV(S,t) = e^{-r_£(T-t)} E^{Q^£_\$}_{S,t} f(S_T),$$

where under our new risk-neutral measure $Q^£_\$$, the risk-neutral stock price process is:

$$\frac{dS_u}{S_u} = (r_\$ - \delta + \sigma_{rs})du + \sigma(S_u, u)dW^£_{\$,u}, \qquad u \in [t,T], \text{ where } S_t = S.$$

Risk-Neutral Valuation of Quantoed Derivatives(con'd)

- Recall the two approaches for valuing a quantoed derivative:

$$
\begin{aligned}
QV(S,t) &= e^{-r_\$(T-t)} E_{S,t}^{Q^\$} \frac{R_T}{R_t} f(S_T), \\
QV(S,t) &= e^{-r_£(T-t)} E_{S,t}^{Q_\$^£} f(S_T).
\end{aligned}
$$

- Thus, the change in the magnitude of the payoff of the ***derivative*** from $f(S_T)$ to $\frac{R_T}{R_0} f(S_T)$ is handled by keeping the payoff fixed at $f(S_T)$ and simply changing the growth rate of the ***underlying***, and changing the currency in which the premium is financed.

- In this problem, the Radon Nikodym derivative is $D_t = \frac{dQ^£}{dQ^\$} = e^{(r_£ - r_\$)(T-t)} \frac{R_T}{R_t}$. The quantoing of the payoff is handled by a change of measure which results in a change of (risk-neutral) drift in accordance with Girsanov's theorem.

- The intuition for the quanto correction and Girsanov's theorem arises from the property of any continuous model that at any time $t \in [0,T)$, the dollar value of a standard derivative at $t+dt$ is known to be *linear* in the time $t+dt$ dollar prices of the bond and stock. Thus, quantoing the payoff is equivalent to quantoing the bond and the stock and the effect of the latter quantoing is to change the risk-neutral drift.

Interpreting the Standard European Call Black Scholes Formula

- First, we re-write the final payoff of the standard European call as:

$$\begin{aligned} C_T &= (S_T - K)^+ \\ &= S_T 1(S_T > K) - K1(S_T > K). \end{aligned}$$

- In the last expression, the second term is K binary calls and its value at time 0 is $Ke^{-r_\$ T}N(d_2)$ dollars, where:

$$d_2 \equiv \frac{\ln(S_0/K) + (r - \delta - \sigma_s^2/2)T}{\sigma_s\sqrt{T}}.$$

- To interpret the first term, we quanto the payoff into shares and adjust the magnitude of the payoff to preserve value.

- If the new payoff currency is to be shares, then the old magnitude of $S_T 1(S_T > K)$ relevant for dollars must be changed to $1(S_T > K)$, since a payoff of $S_T 1(S_T > K)$ dollars is clearly equivalent to a payoff of $1(S_T > K)$ shares.

- Since we are now using shares instead of pounds as the "currency" we are quantoing into, $R_t \equiv S_t$ and $r_£ \equiv \delta$. This makes two changes to the standard binary call valuation:

  1. New drift: since $\sigma_{rs} = \sigma_s^2$ in this case, we add σ_s^2 to $r_\$ - \delta - \frac{\sigma_s^2}{2}$.
  2. New discount rate: use δ instead of $r_\$$ to discount.

- Making these changes, the value of the binary call quantoed into shares becomes $e^{-\delta T}N(d_2 + \sigma\sqrt{T})$ shares. Therefore, its value in dollars is $S_0 e^{-\delta T}N(d_2 + \sigma\sqrt{T})$, which is the same as the first term in the Black-Scholes formula.

Summary and More Relationships

- We have explored the relationships between the following probabilistic and financial concepts:
  1. expected value and no arbitrage value
  2. integration by parts and gains process
  3. martingale problem and Arrow Debreu path securities
  4. transition density and butterfly spread
  5. Girsanov's theorem and quantoing
- Many other relationships exist such as:
  1. reflection principle and static hedging
  2. quadratic variation and volatility contracts
  3. local time and local volatility
- Please see me during the conference if you are intersted in exploring these relationships further.