Option Pricing and the FFT

Peter Carr
NationsBanc Montgomery Securities

Dilip Madan
University of Maryland
Motivation

• Let \( s \equiv \ln S_T \) denote the log of the terminal stock price and \( k \equiv \ln k \) denote the log of the strike.

• The standard approach to option valuation expresses the option value \( C_T(k) \) as an expectation of a discounted payoff:

\[
C_T(k) \equiv e^{-rT} \int_k^\infty \left( e^s - e^k \right) q_T(s) \, ds.
\]

• This approach requires that the risk-neutral density \( q_T(s) \) be known and fast valuation requires that it be simple.

• The approach taken in our paper is to regard the option price as the inverse of its Fourier transform in some variable. For example, when the call is regarded as a function of the log of its strike, its Fourier transform is:

\[
\psi_T(v) \equiv \int_{-\infty}^{\infty} e^{ivk} c_T(k) \, dk.
\]

• This approach requires that the Fourier transform be known and fast valuation requires that it be simple.
Levy Processes and Levy Khintchine Theorem

• In his brilliant talk, Professor Yor gave several new examples of exponential Levy processes.

• By the Levy Khintchine theorem, all Levy processes (i.e. right continuous left limits processes with stationary independent increments) have a characteristic function (FT of PDF of terminal log stock price $s_T$) given by:

$$E e^{i\omega s_T} = \exp \left\{ T \left[ i\hat{\mu}\omega - \frac{\sigma^2\omega^2}{2} + I_1(\omega) + I_2(\omega) \right] \right\},$$

where:

$$I_1(\omega) \equiv \int_{|j|\geq 1} (e^{i\omega j} - 1) \ell(dj)$$

$$I_2(\omega) \equiv \int_{0<|j|<1} (e^{i\omega j} - 1 - i\omega j) \ell(dj)$$

and:

$$\hat{\mu} = \mu - \int_{|j|\geq 1} \frac{j}{1 + j^2} \ell(dj) - \int_{0<|j|<1} \left( \frac{j}{1 + j^2} - j \right) \ell(dj).$$

• The Levy process is specified by the drift $\mu$, the diffusion coefficient $\sigma$, and the Levy measure $\ell(dj)$. 

What We Do

- Our paper shows how to analytically relate the FT of the call price to the characteristic function.
- We then numerically invert this FT using the FFT.
- Thus one can numerically value all options written on exponential Levy processes, even if the transition density is unknown.
- In general, the option valuation problem is reduced to determining the characteristic function of the terminal log price.
- If the transition density is known, our numerical results show that it may still be faster to work in Fourier space, as some Levy processes generate terminal random variables whose characteristic function is simpler than its PDF (eg. VG).
- Our approach outputs a vector of option prices of all strikes so our approach is particularly fast when this is the output needed.
Literature Review

• Much of the recent literature on option valuation has successfully applied Fourier analysis to determine option prices:
  - Bakshi and Chen (1997),
  - Scott (1997),
  - Bates (1996),
  - Heston (1993),
  - Chen and Scott (1992).

• Assuming that the characteristic function of the risk-neutral density is known analytically, these authors numerically solve for the delta and for the risk-neutral probability of finishing in-the-money:

\[ \Pr(S_T > K) = \Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Re} \left[ \frac{e^{-i\omega k\phi_T(\omega)}}{i\omega} \right] d\omega, \]

where \( k = \ln K \) is the log of the strike. Similarly, the delta of the option, denoted \( \Pi_1 \), is numerically obtained as:

\[ \Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Re} \left[ \frac{e^{-i\omega k\phi_T(\omega-i)}}{i\omega\phi_T(-i)} \right] d\omega. \]

Assuming no dividends and a constant riskless rate \( r \), the option value is then determined as:

\[ C = S\Pi_1 - Ke^{-rT}\Pi_2. \]
FFT

• Previous applications of Fourier analysis are unable to harness the considerable computational power of the Fast Fourier Transform (FFT), which represents one of the major advances in scientific computing.

• The purpose of this presentation is to describe two new approaches for numerically determining option values, which are both designed to use the FFT to invert Fourier transforms.

• Assuming that the characteristic function of the terminal log price is known analytically, we present closed form solutions for the FT of (modified) option prices.

• The modification is needed because option prices are not integrable functions of log strike.
Fourier Transform of Dampened Call Value

- Let $C_T(k)$ be the desired value of a $T$ maturity call option with log strike $k$. Let the risk-neutral density of the terminal log price $s_T$ be $q_T(s)$. The characteristic function of this density is defined by:
  \[
  \phi_T(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega s} q_T(s) ds.
  \]

- The initial call value $C_T(k)$ is related to the risk-neutral density $q_T(s)$ by:
  \[
  C_T(k) \equiv e^{-rT} \int_k^{\infty} (e^s - e^k) q_T(s) ds.
  \]
  Note that this call pricing function is not square integrable.

- Consider the Fourier transform of the dampened call price, $e^{\alpha k}C_T(k)$, $\alpha > 0$, defined by:
  \[
  \psi_T(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega k} e^{\alpha k} C_T(k) dk = \ldots = \frac{e^{-rT} \phi_T(\omega - (\alpha + 1)i)}{\alpha^2 + \alpha - \omega^2 + i(2\alpha + 1)\omega}.
  \]

- Call prices can be determined by inverting both the Fourier transform and the dampening factor:
  \[
  C_T(k) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega k} \psi_T(\omega) d\omega = \frac{\exp(-\alpha k)}{\pi} \int_0^{\infty} e^{-i\omega k} \psi_T(\omega) d\omega.
  \]
Fourier Transform of (Modified) Time Value

- A second approach for modifying call values which does not require specifying a dampening coefficient is to subtract off the intrinsic value.

- Define $z_T(k) \equiv C_T(k) - e^{-rT}(F_0 - K)^+$ as the time value of the call, where $F_0$ is the initial forward price.

- Let $\zeta_T(\omega)$ denote the Fourier transform of $z_T(k)$:

$$\zeta_T(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega k} z_T(k) dk = \ldots = e^{-rT} \left[ \frac{1}{1 + i\omega} - \frac{e^{rT}}{i\omega} - \frac{\phi_T(\omega - i)}{\omega^2 - i\omega} \right].$$

- Although there is no issue regarding the behavior of the time value as the log strike $k$ approaches positive or negative infinity, the time value at $k = 0$ can get quite steep for low maturities. This causes the transform to be wide and oscillatory rendering it difficult to numerically integrate.

- Consider the transform of the modified time value $\sinh(\alpha k) z_T(k)$. This function vanishes at $k = 0$ and $\alpha$ controls the steepness of the integrand near zero. The FT of the modified time value is a simple function of the transformed time value:

$$\gamma_T(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega k} \sinh(\alpha k) z_T(k) dk = \ldots = \frac{\zeta_T(\omega - i\alpha) - \zeta_T(\omega + i\alpha)}{2}.$$

- Inverting the transform and solving for the call price gives:

$$C_T(k) = e^{-rT}(F_0 - K)^+ + \frac{1}{\sinh(\alpha k) \pi} \int_{0}^{\infty} e^{-i\omega k} \gamma_T(\omega) d\omega.$$
The VG Model

• The VG process is a Levy process obtained by evaluating arithmetic Brownian motion with drift $\theta$ and volatility $\sigma$ at a random time given by a gamma process having a mean rate per unit time of 1 and a variance rate of $\nu$.

• The resulting process $X_t(\sigma, \theta, \nu)$ is a pure jump process with two additional parameters $\theta$ and $\nu$ relative to the Black Scholes model, providing control over skewness and kurtosis respectively.

• The risk-neutral process for the stock price is:

$$S_t = S_0 e^{rt + X_t(\sigma, \theta, \nu) + \eta t}, \quad t \in [0, T],$$

where by setting $\eta = \ln(1 - \theta \nu - \sigma^2 \nu / 2) / \nu$, the mean rate of return on the stock equals the riskfree rate $r$.

• The characteristic function for the log of $S_T$ is simply:

$$\phi_T(\omega) = \frac{S_0 e^{(r+\eta)T}}{(1 - i\theta \nu \omega + \sigma^2 \omega^2 \nu / 2)^{T/\nu}}.$$

• A closed form option pricing formula can be obtained by analytically inverting this characteristic function to get the density function and then integrating this density function against the option payoff.

• Alternatively, the Fourier transform of the distribution functions can be inverted.

• Finally, our two FFT methods can be employed.
Numerical Results

- Table 1 compares the following 4 methods for obtaining VG option prices:

1. VGP: the analytic formula in Madan, Carr, and Chang
2. VGPS: computing delta and the risk-neutral probability of finishing in-the-money by Fourier inversion of the distribution function
3. VGFFTC: using FFT to invert the dampened call price
4. VGFFTTV: using FFT to invert the modified time value
5. The computation times for the first two methods involve 160 strike levels. The first 4 rows of Table 1 display 4 combinations of parameter settings, while the last 4 rows show computation times in seconds:

<table>
<thead>
<tr>
<th></th>
<th>CASE 1</th>
<th>CASE 2</th>
<th>CASE 3</th>
<th>CASE 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>.12</td>
<td>.25</td>
<td>.12</td>
<td>.25</td>
</tr>
<tr>
<td>$\nu$</td>
<td>.16</td>
<td>2.0</td>
<td>.16</td>
<td>2.0</td>
</tr>
<tr>
<td>$\theta$</td>
<td>-.33</td>
<td>-.10</td>
<td>-.33</td>
<td>-.10</td>
</tr>
<tr>
<td>$t$</td>
<td>1</td>
<td>1</td>
<td>.25</td>
<td>.25</td>
</tr>
<tr>
<td>VGP</td>
<td>22.41</td>
<td>24.81</td>
<td>23.82</td>
<td>24.74</td>
</tr>
<tr>
<td>VGPS</td>
<td>288.50</td>
<td>191.06</td>
<td>181.62</td>
<td>197.97</td>
</tr>
<tr>
<td>VGFFTC</td>
<td>6.09</td>
<td>6.48</td>
<td>6.72</td>
<td>6.52</td>
</tr>
<tr>
<td>VGFFTTV</td>
<td>11.53</td>
<td>11.48</td>
<td>11.57</td>
<td>11.56</td>
</tr>
</tbody>
</table>

- Our first FFT method is fastest, while our second FFT method is second fastest. The analytical formula finishes third, while the slowest (and least accurate in case 4) method inverts for the delta and for the probability of paying off.
I Summary and Conclusion

• We analytically developed two Fourier transforms in terms of the characteristic function of the log of the price at maturity:

  1. FT of the dampened call price, where the dampening arises by multiplying by an exponential.
  2. FT of the modified time value, where the modification involves multiplying by the hyperbolic sine.

• In both cases, call prices are obtained by inverting both the modification and the Fourier transform, with FFT used to speed up the Fourier inversion.

• We illustrated our methods for the VG option pricing model and found that both FFT approaches are considerably faster than other available methods. Thus, we recommend the use of our FFT methods whenever the characteristic function of the underlying uncertainty is available in closed form.

• We anticipate that the advantages of the FFT are not connected to the particular characteristic function we chose to analyse. In fact, we have observed similar speed improvements when we work with the characteristic functions associated with other jump processes given in Geman, Madan, and Yor.