Deriving Derivatives of Derivative Securities

(Greeks for Geeks)

Overheads for Presentation
by Peter Carr, Principal
Banc of America Securities

Part I

Introduction

Why Study Ancient Greeks?

- Despite its age, the Black-Merton-Scholes (BMS) model is the *lingua franca* of option pricing.
- This paper examines greeks in the BMS model, which continues to enjoy multiple applications:
  - hedging
  - market risk measurement
  - profit and loss attribution
  - model risk assessment
  - optimal contract design
  - implied parameter estimation.
- While symbolic math programs can derive arbitrary greeks, they cannot replace an intuitive understanding of the role, genesis, and relationships among all the various greeks.

What's in the Paper?

- This paper develops methods for understanding and calculating greeks of path-independent claims in the BMS model.
- Theorem 1: Delta, gamma, speed, and higher order price derivatives can always be interpreted as the value of a certain quantoed contingent claim.
- Theorem 2: There is an explicit formula for an arbitrary price greek of European options.
- Theorem 3: *Any* partial derivative w.r.t q, r, t, σ, or S can be expressed in terms of the claim's price derivatives.
- Theorem 4: There is a finite radius of convergence of Taylor series expansions of claim values in stock price, time, or volatility.

Greek 101

- Standard textbooks (eg. Hull (1999)) describe the basic greeks of claim values $V(q, r, t, \sigma, S)$ in the BMS model:
  1. Delta = first derivative w.r.t. stock price S, $\frac{\partial V}{\partial S}$
  2. Gamma = second derivative w.r.t. stock price S, $\frac{\partial^2 V}{\partial S^2}$
  3. Theta = first derivative w.r.t. time t, $\frac{\partial V}{\partial t}$
  4. Vega/Kappa = first derivative w.r.t. volatility σ, $\frac{\partial V}{\partial \sigma}$
  5. Rho = first derivative w.r.t. riskfree rate r, $\frac{\partial V}{\partial r}$
  6. Phi/Lambda = first derivative w.r.t. dividend yield q, $\frac{\partial V}{\partial q}$
- Garman (1995) introduces some additional terminology:
  1. Speed = third derivative w.r.t. stock price S, $\frac{\partial^3}{\partial S^3}$
  2. Charm = cross partial w.r.t. stock price S and time t, $\frac{\partial^2}{\partial S \partial t}$
  3. Color = cross partial of delta w.r.t. S and t, $\frac{\partial^3}{\partial S^2 \partial t}$

Literature Review

- binomial model greeks - Pelsser and Vorst (1994)
- vega hedging
  - Garman (1999)
  - Haug (1993)
  - Hull and White (1987)
- multi-factor greeks
  - Willard (1997)
  - Ross (1998)
- Taylor series in stock price - Estrella (1995)
- price greeks for Monte Carlo simulation
  - Broadie and Glasserman (1995)
  - Curran (1993)
  - Glasserman and Zhao (1999)
- price greeks for level-dependent volatility
  - Bergman, Grundy, and Wiener (1996)
  - Grundy and Wiener (1996)
- strike price greeks
  - Breeden and Litzenberger (1978)
  - Schroder (1995)

The Black-Merton-Scholes (BMS) Model

- The BMS model assumes:
  - frictionless security markets
  - constant riskless rate r
  - constant continuous dividend yield q
  - underlying stock price S is geometric Brownian motion:

$$\frac{dS_t}{S_t} = \alpha_t \, dt + \sigma \, dB_t, \quad t \in [0, T].$$

- Consider a path-independent claim whose final payoff $f(S)$ is a known function of S.
- Let $U(\tau, x)$ be a $C^{1,2}$ function relating the claim's arbitrage-free value, U, to the claim's time to maturity, $\tau \equiv T - t$, and to the log of the stock price, $x = \ln S$. Then $U(\tau, x)$ solves a p.d.e. with constant coefficients:

$$\frac{\partial U(\tau, x)}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 U(\tau, x)}{\partial x^2} + \mu \frac{\partial U(\tau, x)}{\partial x} - rU(\tau, x),$$

$x \in \Re, \tau \in (0, T)$, where $\mu \equiv r - q - \frac{\sigma^2}{2}$, subject to the initial condition:

$$U(0, x) = \phi(x).$$

Part II

Price Derivatives as Quantoed Claims

Log Price Derivatives

- Recall that $U(\tau, x)$ solves a p.d.e. with constant coefficients:
$$\frac{\partial U(\tau, x)}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 U(\tau, x)}{\partial x^2} + \mu \frac{\partial U(\tau, x)}{\partial x} - rU(\tau, x),$$
$x \in \Re, \tau \in (0, T)$, where $\mu \equiv r - q - \frac{\sigma^2}{2}$, subject to the initial condition:
$$U(0, x) = \phi(x).$$
- Let D_x^ℓ denote the ℓ-th derivative w.r.t. x. Since the p.d.e. has constant coefficients, differentiating it w.r.t. x ℓ times implies that $D_x^\ell U(\tau, x)$ satisfies the same p.d.e. as $U(\tau, x)$.
- Consequently, with $X_t \equiv \ln S_t$, $D_x^\ell U(T - t, X_t)$ is the value at t of a contingent claim with the single payoff $\phi^{(\ell)}(X_T)$ occurring at T.
- For example, when $\ell = 1$, $U_x(T - t, X_t)$ is the value at t of a claim paying $\phi'(X_T)$ at T. Letting $f(S) \equiv \phi(\ln S)$, this final payoff is $S_T f'(S_T)$ at T.
- In the case of a call, $f(S) = \max[0, S - K]$, and so the payoff associated with U_x is that of a gap call $S_T 1(S_T > K)$.
- Letting $V(\tau, S) \equiv U(\tau, x)$, the chain rule implies that $S_t V_s(T - t, S_t)$ is a claim price process.

Log Price Derivatives (con'd)

- Recall that with $X_t \equiv \ln S_t$, $D_x^\ell U(T-t, X_t)$ is the value at t of a contingent claim with the single payoff $\phi^{(\ell)}(X_T)$ occurring at T.
- For $\ell = 2$, $U_{xx}(T-t, X_t)$ is the value at t of a claim paying $\phi''(X_T) = S_T^2 f''(S_T) + S_T f'(S_T)$ at T.
- Since $U_{xx}(T-t, X_t) = S_t^2 V_{ss}(T-t, S_t) + U_x(T-t, X_t)$ and since $U_x(T-t, X_t)$ is also a claim price process, $S_t^2 V_{ss}(T-t, S_t)$ is yet another claim price process, with payoff $S_T^2 f''(S_T)$ at T.
- More generally:
  Lemma 1: *For each $\ell = 0, 1, 2 \ldots$, the process $\{S_t^\ell D_s^\ell V(S_t, T-t), t \in [0, T]\}$ is the arbitrage-free value at t of a claim with payoff $S_T^\ell f^{(\ell)}(S_T)$ at T.*

Delta as a Quantoed Claim

- Recall Lemma 1: *For each* $\ell = 0, 1, 2 \ldots$, *the process* $\{S_t^\ell D_s^\ell V(S_t, T-t), t \in [0,T]\}$ *is the arbitrage-free value at* t *of a claim with payoff* $S_T^\ell f^{(\ell)}(S_T)$ *at* T.
- Since $S_t V_s(T-t, S_t)$ is the price of a claim paying $S_T f'(S_T)$ dollars at T, it follows that delta can be interpreted as the price in shares of a claim paying $f'(S_T)$ shares at T.
- Thus for the call example, delta is the price in shares of a claim paying one share if $S_T > K$ at T.
- Since the function $V_s(\tau, S)$ relates the price *in shares* to the price of the stock *in dollars*, we are in the same situation as when the payoff of an option is defined in terms of a different currency than the price of the underlying.

Delta as a Quantoed Claim (Con'd)

- Recall that delta is interpretable as the price *in shares* of a claim whose underlying is measured *in dollars.*
- The correction for quantoing payoffs from dollars into shares results in the following modification of the usual "risk-neutral" stock price process:

$$\frac{dS_t}{S_t} = (r - q + \sigma^2)dt + \sigma \, dB_t^{(1)}, \quad t \in [0, T],$$

  where $\{B_t^{(1)}; t \in [0, T]\}$ is a $Q^{(1)}$ standard Brownian motion.
- The appropriate discount rate for discounting share-denominated payoffs is the dividend yield q.
- Thus, delta can be represented as:

$$V_s(\tau, S) = e^{-q\tau} E^{(1)}[f^{(1)}(S_T)|S_t = S], \quad S > 0, \tau \in (0, T),$$

  where the operator $E^{(1)}$ indicates that the expectations are calculated using the measure $Q^{(1)}$.

Gamma as a Quantoed Claim

- Again recall Lemma 1: *For each* $\ell = 0, 1, 2 \ldots,$ *the process* $\{S_t^\ell D_s^\ell V(S_t, T-t), t \in [0, T]\}$ *is the arbitrage-free value at* t *of a claim with payoff* $S_T^\ell f^{(\ell)}(S_T)$ *at* T.

- Since $S_t^2 V_{ss}(T-t, S_t)$ is the price of a claim paying $S_T^2 f''(S_T)$ at T, it follows that gamma can be interpreted as the price in "squares" of a claim paying $f''(S_T)$ squares at T. By a square, we mean a dividend-paying claim whose value is S_t^2 for all $t \in [0, T]$. The dividend can be shown to be $(2q-r-\sigma^2)S^2$ for a constant dividend yield of $2q - r - \sigma^2$.

- Thus for the call example, gamma is the price in squares of a claim paying $f''(S) = \delta(S_T - K)$ squares at T, where $\delta(\cdot)$ is the Dirac delta function.

- The appropriate discount rate for discounting square denominated payoffs is the dividend yield $2q - r - \sigma^2$.

Gamma as a Quantoed Claim (con'd)

- Recall that gamma is the price in squares of a claim whose underling is the stock price in dollars.
- Since the appropriate discount rate for discounting square denominated payoffs is $2q-r-\sigma^2$, the gamma of a claim can be represented as:

  $$V_{ss}(\tau, S) = e^{-(2q-r-\sigma^2)\tau} E^{(2)}[f''(S_T)|S_t = S], S > 0, \tau \in (0, T),$$

  where the operator $E^{(2)}$ indicates that the expectation of the final gamma, $f''(S_T)$, is calculated from the following geometric Brownian motion:

  $$\frac{dS_t}{S_t} = (r - q + 2\sigma^2)\, dt + \sigma\, dB_t^{(2)}, \quad t \in [0, T],$$

  where $\{B_t^{(2)}; t \in [0, T]\}$ is a $Q^{(2)}$ standard Brownian motion.

- The paper proves the following general theorem:

  Theorem 1: *The value, delta, gamma, and higher order derivatives of path-independent claims in the BMS model are given by:*

  $$\frac{\partial^j V(\tau, S)}{\partial S^j} = e^{-[jq-(j-1)r-(j-1)j\sigma^2/2]\tau} E^{(j)}[f^{(j)}(S_T)|S_t = S],$$

  $j = 0, 1, \ldots,$ *on the region* $S > 0, \tau \in (0, T)$, *where the operator* $E^{(j)}$ *indicates that the expectation of the final* j*-th derivative,* $f^{(j)}(S_T)$, *is calculated from the following geometric Brownian motion:*

  $$\frac{dS_t}{S_t} = (r - q + j\sigma^2)\ dt + \sigma\ dB_t^{(j)}, \quad t \in [0, T],$$

  and where $\{B_t^{(j)}; t \in [0, T]\}$ *is a* $Q^{(j)}$ *standard Brownian motion.*

Interpreting Theorem 1

- Recall from Theorem 1 that the j-th price derivative:

$$\frac{\partial^j V(\tau,S)}{\partial S^j} = e^{-[jq-(j-1)r-(j-1)j\sigma^2/2]\tau} E^{(j)}[f^{(j)}(S_T)|S_t = S].$$

- The discount rate $jq-(j-1)r-(j-1)j\sigma^2/2$ is the dividend yield on a "power claim" whose value is S_t^j for all $t \in [0,T]$.
- The measure $Q^{(j)}$ describes prices of Arrow Debreu securities in terms of these power claims. The payoffs of these Arrow Debreu securities are indexed over paths and also pay out in power claims.
- Since the stock price S generating the paths is still denominated in dollars, a quanto correction is needed, which involves adding $j\sigma^2$ to the proportional drift.

European Option Log Price Derivatives

- To illustrate higher order price derivatives in the case of an option, let c be a call/put indicator:

$$c = \begin{cases} 1 & \text{if call} \\ -1 & \text{if put.} \end{cases}$$

- The BMS formula for European option value in terms of $x \equiv \ln S$ is:

$$eo(x) \equiv c[e^{x-q\tau}N(cd_1) - Ke^{-r\tau}N(cd_2)],$$

where:

$$d_2 \equiv \frac{x - \ln(K) + \mu\tau}{\sigma\sqrt{\tau}}, \quad d_1 = d_2 + \sigma\sqrt{\tau},$$

$\mu \equiv r - q - \sigma^2/2$ and $N(d) \equiv \int_{-\infty}^{d} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$.

- **Theorem 2:** The ℓ_x-th derivative w.r.t x of a European option is:

$$D_x^{\ell_x} eo(x) = ce^{x-q\tau}N(cd_1) + Ke^{-r\tau}\frac{N'(d_2)}{\sigma\sqrt{\tau}} \sum_{i_x=0}^{\ell_x-2} \frac{H_{i_x}(d_2)}{(-\sigma\sqrt{\tau})^{i_x}},$$

$l_x = 1, 2, \ldots,$ where $H_i(d)$, $i = 0, 1, 2$, are the Hermite polynomials satisfying the recursion:

$$H_{i+1}(d) = dH_i(d) - iH_{i-1}(d), \text{ with } H_0(d) = 1, H_1(d) = d.$$

European Option Stock Price Derivatives

- Recall Theorem 2: The ℓ_x-th derivative w.r.t x of a European option is:

$$D_x^{\ell_x} eo(x) = ce^{x-q\tau} N(cd_1) + Ke^{-r\tau} \frac{N'(d_2)}{\sigma\sqrt{\tau}} \sum_{i_x=0}^{\ell_x-2} \frac{H_{i_x}(d_2)}{(-\sigma\sqrt{\tau})^{i_x}},$$

$l_x = 1, 2, \ldots,$ where $H_i(d), i = 0, 1, 2,$ are the Hermite polynomials.

- The formula for the i-th Hermite polynomial is:

$$H_i(d) = \sum_{g=0}^{[\frac{i}{2}]} \frac{d^{i-2g}}{(i-2g)!} \frac{i!}{g!(-2)^g}.$$

- To derive stock price derivatives, write stock price derivative in terms of log price derivatives:

$$S^{\ell_s} D_s^{\ell_s} = \sum_{i_s=1}^{\ell_s} \mathcal{S}_1(\ell_s, i_s) D_x^{i_s}, \quad \ell_s = 1, 2, \ldots,$$

where $\mathcal{S}_1(\ell_s, i_s)$ denotes Stirling numbers of the first kind, satisfying the recursion:

$$\mathcal{S}_1(\ell, i) = \begin{cases} \mathcal{S}_1(\ell-1, i-1) - (\ell-1)\mathcal{S}_1(\ell-1, i) & l = 1, 2 \ldots, i = 1, 2 \ldots \ell \\ 0 & \text{otherwise, except that:} \end{cases}$$

$\mathcal{S}_1(0, 0) = 1.$

- For $l = 1, 2, \ldots, i = 1, 2, \ldots, \ell$, the solution of the recursion is:

$$\mathcal{S}_1(\ell, i) = \sum_{j=0}^{\ell-i} \sum_{h=j}^{\ell-i} (-1)^{j+h} \binom{h}{j} \binom{\ell-1+h}{\ell-i+h} \binom{2\ell-i}{\ell-i-h} \frac{(h-j)^{\ell-i+h}}{h!}$$

- The above implies that higher order price derivatives of option values are given by a completely explicit formula.

Part III

Rate Derivatives and Operator Calculus

Claim Value Using Operator Calculus

- The BMS p.d.e. governing the value $V(\tau, S)$ can be written as:
$$\frac{\partial V(\tau, S)}{\partial \tau} = \mathcal{L}[V(\tau, S)], \quad \tau > 0,$$
subject to:
$$V(0, S) = f(S),$$
where $\mathcal{L}$ is the following linear operator:
$$\mathcal{L} \equiv \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} + (r - q) S \frac{\partial}{\partial S} - r\mathcal{I}, \quad S > 0.$$

- Operator calculus treats the top equation as an *ordinary* differential equation in τ, by treating $\mathcal{L}$ as a constant and S as a fixed parameter.

- The formal solution of the above initial value problem is then given as:
$$V(\tau, S) = \exp\{\tau \cdot \mathcal{L}\} f(S), \quad S > 0, \tau \in (0, T),$$
where:
$$\exp\{\tau \cdot \mathcal{L}\} \equiv \sum_{j=0}^{\infty} \frac{(\tau \mathcal{L})^j}{j!}.$$

- Differentiating w.r.t. τ verifies the validity of the solution.

Rate Derivatives and Operator Calculus

- Recall that the formal solution for claim value is:

$$V(\tau, S) = \exp\{\tau \cdot \mathcal{L}\} f(S), \quad S > 0, \tau \in (0, T),$$

where $\mathcal{L}$ is the following linear operator:

$$\mathcal{L} \equiv \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} + (r - q) S \frac{\partial}{\partial S} - r\mathcal{I}, \quad S > 0.$$

- Differentiating w.r.t. the dividend yield q, the riskless rate r, or the volatility σ yields useful representations of partial derivatives w.r.t. these variables:

$$\begin{aligned}
\frac{\partial V(\tau, S)}{\partial q} &= \tau \tfrac{\partial \mathcal{L}}{\partial q} \exp\{\tau \cdot \mathcal{L}\} f(S) \ = -\tau S \frac{\partial}{\partial S} V(\tau, S) \\
\frac{\partial V(\tau, S)}{\partial r} &= \tau \tfrac{\partial \mathcal{L}}{\partial r} \exp\{\tau \cdot \mathcal{L}\} f(S) \ = -\tau \left[V(\tau, S) - S \frac{\partial}{\partial S} V(\tau, S) \right] \\
\frac{\partial V(\tau, S)}{\partial \sigma} &= \tau \tfrac{\partial \mathcal{L}}{\partial \sigma} \exp\{\tau \cdot \mathcal{L}\} f(S) \ = \sigma \tau S^2 \frac{\partial^2}{\partial S^2} V(\tau, S), \quad S > 0, \tau > 0.
\end{aligned}$$

- Note that all 3 rate greeks have been expressed in terms of stock price derivatives.
- The three results are easily justified using risk-neutral valuation.

Part IV

Arbitrary Greeks

Representation in terms of Log Price Derivatives

- Just as phi, rho, and vega are all simple functions of a claim's value, delta, and gamma, the BMS p.d.e. implies that a claim's theta can also be expressed in terms of the first three spatial derivatives.
- The paper generalizes these results by expressing an *arbitrary greek* in terms of its *spatial derivatives*.
- By an arbitrary greek, I mean a partial derivative of the form $D_q^{\ell_q} D_r^{\ell_r} D_t^{\ell_t} D_\sigma^{\ell_\sigma} D_s^{\ell_s} V(q, r, t, \sigma, S)$, where D_q, D_r, D_t, D_σ, and D_s denote the first order derivative operators w.r.t. q, r, t, σ, and S respectively.
- By a spatial derivative, I mean a partial derivative w.r.t. the *log* stock price. This representation is convenient if finite differences are used to approximate the solution to the constant cofficient p.d.e.
- The formula given in Theorem 3 of the paper is complicated (it has 7 nested sums!).

Representation in terms of Stock Price Derivatives

- If a representation of the arbitrary greek in terms of stock price derivatives is desired, then combine Theorem 3 with the following general relationship between a log price derivative and stock price derivatives:

$$D_x^{\ell_x} = \sum_{i_x}^{\ell_x} \mathcal{S}_2(\ell_x, i_x) S^{i_x} D_s^{i_x},$$

where $\mathcal{S}_2(\ell_s, i_s)$ denotes Stirling numbers of the second kind, satisfying the recursion:

$$\mathcal{S}_2(\ell, i) = \begin{cases} \mathcal{S}_2(\ell - 1, i - 1) + i\mathcal{S}_2(\ell - 1, i) & \text{for } l = 1, 2, \ldots, i = 1, 2 \ldots \ell \\ 0 & \text{otherwise, except that:} \end{cases}$$

$\mathcal{S}_2(0, 0) = 1.$

- The solution of this recursion is:

$$\mathcal{S}_2(\ell, i) = \frac{1}{\ell!} \sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} j^{\ell}, \quad l = 1, 2, \ldots, i = 1, 2 \ldots, \ell.$$

Part V

Taylor Series

Taylor Series Expansion of Claim Value

- By standard calculus, the Taylor series expansion of a claim value $V(q, r, t, \sigma, S)$ about the point $(q_0, r_0, t_0, \sigma_0, S_0)$ is:

$$V(q, r, t, \sigma, S) = V(q_0, r_0, t_0, \sigma_0, S_0) + \sum_{m=1}^{\infty} \sum_{\ell_q=0}^{m} \sum_{\ell_r=0}^{m-\ell_q} \sum_{\ell_t=0}^{m-\ell_q-\ell_r} \sum_{\ell_\sigma=0}^{m-\ell_q-\ell_r-\ell_t}$$
$$\frac{m! D_q^{\ell_q} D_r^{\ell_r} D_t^{\ell_t} D_\sigma^{\ell_\sigma} D_s^{\ell_s}}{\ell_q! \ell_r! \ell_t! \ell_\sigma! \ell_s!} \frac{(\triangle q)^{\ell_q} (\triangle r)^{\ell_r} (\triangle t)^{\ell_t} (\triangle \sigma)^{\ell_\sigma} (\triangle S)^{\ell_s}}{m!} V(q_0, r_0, t_0, \sigma_0, S_0),$$

where $\ell_s \equiv m - \ell_q - \ell_r - \ell_t - \ell_\sigma$ and $\triangle q \equiv q - q_0$, $\triangle r \equiv r - r_0$, $\triangle t \equiv t - t_0$, $\triangle \sigma \equiv \sigma - \sigma_0$ and $\triangle S \equiv S - S_0$.

- Substituting my formula for an arbitrary greek (with 7 nested sums) in the above results in an even more complex formula (with 12 nested sums). This formula expresses a Taylor series expansion of a path-independent claim value in the 5 variables in terms of derivatives w.r.t. $x = \ln S$.

- Further substituting the formula in Theorem 2 for option price derivatives w.r.t. x results in an explicit expansion formula (with 13 nested sums).

Taylor Series Expansion of European Option Values in 5 Variables

- Figure 1 shows a well-behaved expansion of a European call at a point $(q_1, r_1, t_1, \sigma_1, S_1) = (.05, .1, .25, .25, 110)$ about $(q_0, r_0, t_0, \sigma_0, S_0) = (.02, .06, 0, .2, 100)$ for $(K, T) = (100, 1)$.

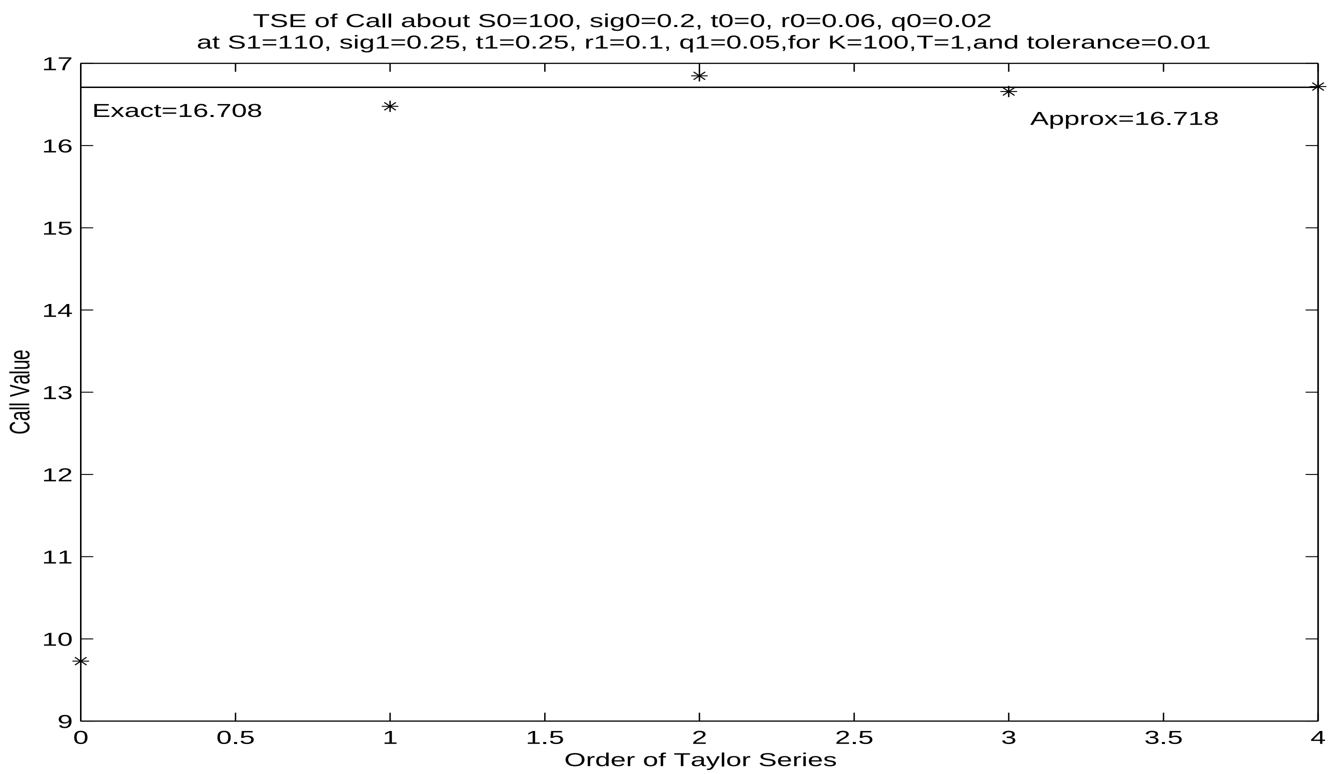

Figure 1: Taylor Series Expansion of European Call Value in all Variables

- As the order is increased from 1 to 4, the truncated Taylor series gets closer to the correct value until the specified tolerance of .01 is achieved.

- The example illustrates that high order truncated expansions are sometimes needed to achieve the desired result.

Taylor Series Expansion of European Option Values in Volatility

- Figure 2 focuses on a univariate Taylor series expansion in volatility, holding (q, r, t, S, K, T) constant at $(.02, .06, 0, 100, 100, 1)$.

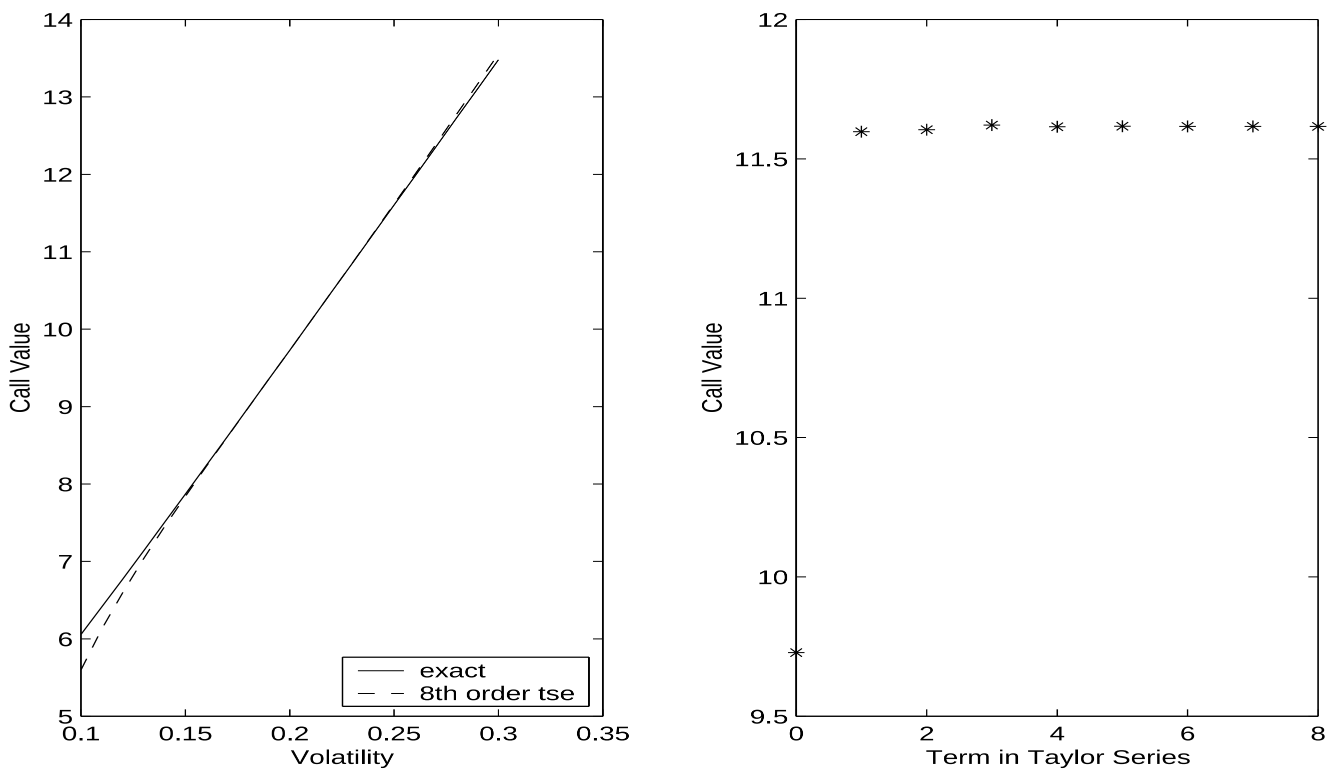

Figure 2: Taylor Series Convergence of European Call Value in Volatility

- The left panel shows that even an 8th order truncated Taylor series expansion is insufficient if the volatility changes by a sufficient magnitude.
- However, the right panel shows that for the small change in volatility from .2 to .25, convergence occurs rapidly.

Convergence Region for Taylor Series Expansion of Claim Values

- Since truncated Taylor series are sometimes used in place of a recalculation, it is worth investigating the region of convergence for the 5 independent variables.
- The following theorem holds for any path-independent payoff function:

  Theorem 4: *Let ρ_y denote the radius of convergence of the variable y. Then $\rho_r = \infty, \rho_q = \infty, \rho_s = S, \rho_t = T - t$, and $\rho_\sigma = \frac{\sigma}{\sqrt{2}}$.*
- Thus:
  1. the radius of convergence is unbounded for r and for q.
  2. for the stock price, Estrella (1995) proves that the radius of convergence is S, so that Taylor series should not be used if the stock price is increased by a factor of 2 or more.
  3. for the time variable t, a similar analysis shows that the radius of convergence is $T - t$, so that Taylor series should not be used to move backwards in calendar time by more than the time to maturity.
  4. For volatility, the radius of convergence is $\frac{\sigma}{\sqrt{2}}$, so that Taylor series should not be used if volatility is raised or lowered by more than about 70%.

Convergence Region for Taylor Series Expansion of Claim Values

- Recall that Taylor series should not be used if volatility is raised or lowered by more than about 70%.
- Implied volatilities have risen by more than this amount following a crash. Figure 3 shows an expansion in volatility when volatility doubles from .2 to .4, holding (q, r, t, S, K, T) constant at $(.02, .06, 0, 100, 100, 1)$.

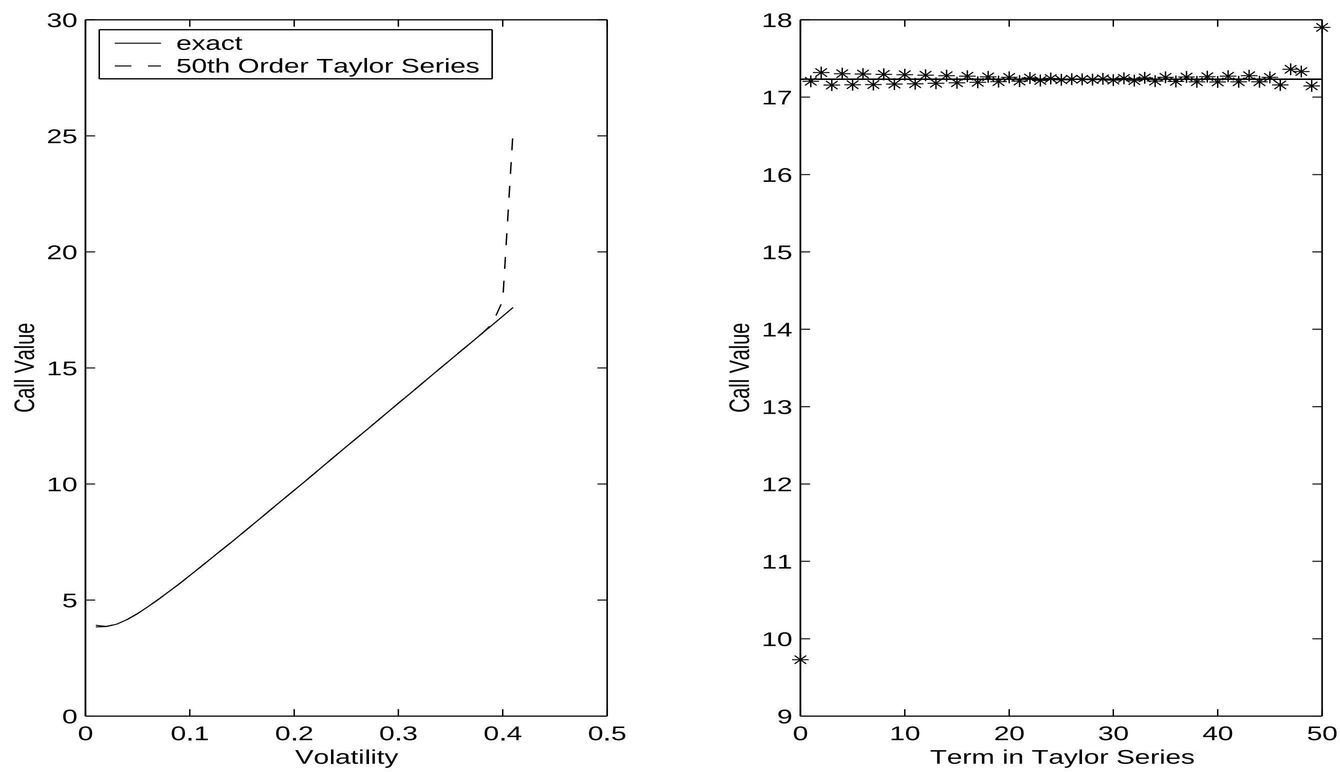

Figure 3: Taylor Series Divergence of European Call Value in Volatility

- Surprisingly, the expansion works until about the 40th term when it begins to diverge. This example shows the danger in increasing the order of a truncated Taylor series when convergence is not guaranteed.

Summary

- For path-independent claims in the BMS model, delta, gamma, speed, and higher order price derivatives can all be interpreted as the values of certain quantoed contingent claims.
- This interpretation allows their values to be calculated as a discounted expectation. Any partial derivative w.r.t. q, r, t, σ, and/or S can be expressed in terms of the security's spatial derivatives.
- Since the latter are easily determined, Taylor series in all 5 variables becomes feasible.
- However, for sufficiently large changes in S, t, or σ, Taylor series diverge.
- The results of this paper realize their greatest practical significance when numerical methods must be employed to value a claim.
- The same technique used to numerically value the claim can be used to numerically determine spatial derivatives. Given numerical results for these spatial derivatives, the other derivatives can be determined.
- Thus, computational resources should be spent accurately determining the claim's spatial derivatives, rather than attempting a coarser approximation of all the greeks.

Extensions

- Our results should easily extend to contingent claims with intermediate payouts and to multiple state variables.
- It might be interesting to explore fractional derivatives or repeated integrals.
- The more difficult extension of the paper's result to more complex stochastic processes or payout structures should be explored.