Closed Form Option Valuation with Smiles

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Part I

Introduction
Introduction

• When the stock is regarded as a call on the firm’s assets (Black-Scholes (1973), Merton (1974), Geske (1979)), then the local volatility of the stock depends on the firm’s assets.

• Since there is a one to one map between the assets and the stock, the stock’s local volatility can be said to depend on the stock price.

• This dependence of local volatility on the stock price has been studied by many researchers (eg. Cox (1975), Schroeder (1989), Goldenberg (1991), Li (1998), Bouchev (1998))
Overview

- The purpose of this talk is to characterize the entire class of volatility functions which permit the stock price to be transformed into standard Brownian motion by scale changes alone.
- We’ll show that this class is characterized by a fully nonlinear partial differential equation (p.d.e.), which we are nonetheless able to solve analytically.
- As a result, many new closed form solutions for option prices are obtained.
- We’ll also illustrate our results by obtaining several new formulas involving only normal distribution or density functions.
Part II

Nonlinear P.D.E. and Solutions
Assumptions

1. Frictionless markets, no arbitrage, and the underlying stock price process is a one dimensional diffusion starting from a positive value.

2. Proportional risk-neutral drift of \( r - q \), where \( r \geq 0 \) is the constant risk-free rate and \( q \geq 0 \) is the constant dividend yield.

3. Absolute volatility rate is an arbitrary function \( a(S,t) \) of the stock price \( S > 0 \) and time \( t \in [0,T] \), where \( T \) is some distant horizon exceeding the longest maturity of the option to be priced.

4. Since this process will in general have positive probability that the stock price hits zero, we absorb the stock price at the origin in this event. Let \( \tau_o \) denote the first hitting time of the origin and let \( \tau \equiv T \land \tau_o \) be a stopping time. Letting \( \{S_t : t \in [0,\tau]\} \) denote the risk-neutral stock price process:

\[
    dS_t = (r - q)S_t dt + a(S_t, t)dW_t, \quad t \in [0, \tau],
\]

where \( \{W_t, t \in [0, \tau]\} \) is a standard Brownian motion (SBM) under the risk-neutral measure \( Q \).
The Nonlinear P.D.E.

- Recall:
  \[ dS_t = (r - q)S_t dt + a(S_t, t) dW_t, \quad t \in [0, \tau], \]

- Let \( w(S, t) \) denote a \( C^{2,1} \) function defined on the domain \( S > 0 \) and \( t \in [0, T] \) which for each \( t \), maps the stock price \( S \) to the SBM \( W \), i.e.:
  \[ W_t = w(S_t, t). \]

- Itô's lemma can be used to describe the risk-neutral drift and diffusion of \( W \) in terms of the risk-neutral drift and diffusion of \( S \). Setting the risk-neutral diffusion of \( W \) to unity gives the scale change:
  \[ w(S, t) = \int_{S_0}^{S} \frac{1}{a(Z, t)} dZ + \int_0^t \left[ \frac{1}{2} \frac{\partial a(S, s)}{\partial S} \bigg|_{S=S_0} - \frac{(r - \delta)S_0}{a(S_0, s)} \right] ds. \]

- Zeroing out the risk-neutral drift implies that the following fundamental p.d.e. governs the absolute volatility:
  \[ \frac{a^2(S, t)}{2} \frac{\partial^2 a(S, t)}{\partial S^2} + (r - q)S \frac{\partial a(S, t)}{\partial S} + \frac{\partial a(S, t)}{\partial t} = (r - q)a(S, t), \quad S > 0, t \in [0, \tau]. \]

- Conversely, if \( a(S, t) \) solves the p.d.e., then the process \( W \) defined above is SBM.
A Solution Class for the P.D.E.

• Let \( s(w, t), w \geq L(t), t \in [0, T] \) be the spatial inverse of \( w(S, t) \), i.e.:
  
  \[ S_t = s(W_t, t), \quad t \in [0, \tau]. \]

• By Itô’s lemma, the stock price process can be written as:
  
  \[ dS_t = \left[ \frac{\partial s}{\partial t}(W_t, t) + \frac{1}{2} \frac{\partial^2 s}{\partial w^2}(W_t, t) \right] dt + \frac{\partial s}{\partial w}(W_t, t) dW_t, \quad t \in [0, \tau]. \]

• Recall: \( dS_t = (r - q)S_t dt + a(S_t, t) dW_t, \quad t \in [0, \tau]. \)

• Equating coefficients on \( dt \) and using the top equation gives a simple linear p.d.e. for the stock pricing function \( s(w, t) \):
  
  \[ \frac{\partial s}{\partial t}(w, t) + \frac{1}{2} \frac{\partial^2 s}{\partial w^2}(w, t) = (r - q)s(w, t), \quad w \geq L(t), t \in [0, T]. \quad (1) \]

• Equating coefficients of \( dW_t \) and using \( S_t = s(W_t, t) \) gives a link between the absolute volatility function \( a(S, t) \) and the stock pricing function \( s(w, t) \):
  
  \[ a(S, t) = \frac{\partial s}{\partial w}(w(S, t), t), \quad w \geq L(t), t \in [0, T]. \quad (2) \]

• Solving \( (1) \) subject to a terminal condition \( s(w, T) = \phi(w) \) and absorbing lower barrier condition, \( (2) \) relates the local volatility to the derivative and inverse of the solution.
Option Pricing

- We first determine the function \( \gamma(w, t) \) relating the value of a European call maturing at \( M \in [t, T] \) to the price of the Brownian derivative \( W_t \) and time \( t \). By Itô’s lemma, this pricing function solves:

\[
\frac{1}{2} \frac{\partial^2 \gamma}{\partial w^2}(w, t) + \frac{\partial \gamma}{\partial t}(w, t) = r \gamma(w, t), \quad w > L(t), t \in [0, M),
\]

subject to the boundary conditions:

\[
\lim_{w \downarrow L(t)} \gamma(w, t) = 0, \quad \lim_{w \to \infty} \gamma(w, t) = s(w, t)e^{-q(M-t)} - Ke^{-r(M-t)}, \quad t \in [0, M),
\]

where \( K > 0 \) is the call strike, and subject to the terminal condition:

\[
\gamma(W, M) = [s(w, M) - K]^+, \quad w > L(M).
\]

- By the Feynman-Kac theorem, the continuous solution to this BVP is:

\[
\gamma(w, t) = e^{-r(M-t)}E_{W_t}^Q[s(W^a_M, M) - K]^+, \quad w > L(t), t \in [0, M \wedge \tau),
\]

where recall \( \{W_u, u \in [t, M]\} \) is an SBM absorbing at the lower barrier \( \{L(u), u \in (t, T)\} \) and starting at \( w \geq L(t) \) at time \( t \).
Option Pricing (con’d)

- To instead relate the call value to the stock price and time, we assume that $L(t) = L$, i.e. that the boundary along which the Brownian derivative vanishes is independent of time. In this case, the probability density function for absorbing SBM is known and if we let $c(S, t) = \gamma(w(S, t), t)$, then:

$$c(S, t) = e^{-r(M-t)} \sum_{0}^{\infty} \frac{|s(z, M) - K|^+}{L \sqrt{2\pi(M-t)}} \left\{ \exp \left[ -\frac{1}{2} \left( \frac{z - w(S, t)}{\sqrt{M - t}} \right)^2 \right] - \exp \left[ -\frac{1}{2} \left( \frac{z + w(S, t) - 2L}{\sqrt{M - t}} \right)^2 \right] \right\} dz,$$

for $S \geq 0$, $t \in [0, M \wedge \tau)$.

- This solution will be an explicit function of $S$ and $t$ if $s(z, M)$ and $w(S, t)$ can both be written explicitly in terms of their arguments.
Risk-Neutral Density

- Differentiating the call pricing formula twice w.r.t. strike yields the risk-neutral density $q$.

- Alternatively, the change of variables $S = s(w, t)$ for the absorbing SBM transition density expresses this probability as:

\[
q(Z, M; S, t) = \frac{\partial w(Z, M)}{\partial S} \frac{1}{\sqrt{2\pi(M-t)}} \left\{ \exp \left\{ -\frac{1}{2} \left[ \frac{w(Z, M) - w(S, t)}{\sqrt{M - t}} \right]^2 \right\} \right.
\]

\[
- \exp \left\{ -\frac{1}{2} \left[ \frac{w(Z, M) + w(S, t)}{\sqrt{M - t}} \right]^2 \right\} \}.
\]

- Since $\frac{\partial w}{\partial S}(Z, M) = \frac{1}{\partial w} = \frac{1}{a(Z,M)}$,

\[
q(Z, M; S, t) = \frac{1}{a(Z,M) \sqrt{2\pi(M-t)}} \left\{ \exp \left\{ -\frac{1}{2} \left[ \frac{w(Z, M) - w(S, t)}{\sqrt{M - t}} \right]^2 \right\} \right.
\]

\[
- \exp \left\{ -\frac{1}{2} \left[ \frac{w(Z, M) + w(S, t)}{\sqrt{M - t}} \right]^2 \right\} \}.
\]

for $S > 0, t \in [0, M \wedge \tau)$. 
Part III

Examples
Example 1: Stock Price is Hyperbolic Sine

- The Black Scholes model fits in our framework by taking the stock payoff function as exponential.
- Suppose more generally that the final payoff $\phi(w)$ is given by the difference of two exponential functions.

$$
\phi(w) = \begin{cases} 
\beta \sinh[\alpha(w - L)] & \text{if } w > L; \\
0 & \text{if } w < L,
\end{cases}
$$

where recall $\sinh(x) \equiv \frac{e^x - e^{-x}}{2}$.

- Recall the linear p.d.e. for the stock pricing function:

$$
\frac{\partial s}{\partial t}(w, t) + \frac{1}{2} \frac{\partial^2 s}{\partial w^2}(w, t) = (r - q)s(w, t), \quad w \geq L(t), \ t \in [0, T].
$$

- Imposing the lower boundary condition $s(L, t) = 0$ yields the following solution:

$$
s(w, t) = \beta e^{-\mu(T-t)} \sinh[\alpha(w - L)], \quad t \in [0, T], w > L,
$$

where:

$$
\mu \equiv r - q - \alpha^2 / 2.
$$

- Setting $s(0, 0) = S_0$ relates the scale parameter $\beta$ to the initial stock price:

$$
\beta = S_0 e^{\mu T} \text{csch}(\alpha L).$$
Stock Price is Hyperbolic Sine (con’d)

• Recall the link between the absolute volatility function $a(S, t)$ and the stock pricing function $s(w, t)$:

$$ a(S, t) = \frac{\partial s}{\partial w}(w(S, t), t), \quad w \geq L(t), t \in [0, T]. $$

• Carrying out the indicated operations on $s(w, t)$ and dividing by $S$ yields the “lognormal” local volatility surface:

$$ \sigma(S, t) \equiv \frac{a(S, t)}{S} = \alpha \sqrt{1 + \left(\frac{\beta}{Se^\mu(T-t)}\right)^2}, \quad S > 0, t \in [0, \tau), $$

where:

$$ \beta = S_0e^\mu T \text{csch}(-\alpha L). $$
Stock Price is Hyperbolic Sine (con’d)

- Using the standard change of variables formula, the risk-neutral stock pricing density is:

\[
q(Z, M; S, t) = \frac{1}{\sqrt{2\pi(M - t)}} \frac{1}{\alpha \sqrt{Z^2 + \beta^2 e^{-2\mu(T-M)}}} \\
\left\{ \exp \left\{ -\frac{1}{2} \left[ \frac{\sinh^{-1}\left( \frac{Z e^{\mu(T-M)}}{\beta} \right) - \sinh^{-1}\left( \frac{S e^{\mu(T-t)}}{\beta} \right)}{\alpha \sqrt{M - t}} \right]^2 \right\} \\
- \exp \left\{ -\frac{1}{2} \left[ \frac{\sinh^{-1}\left( \frac{Z e^{\mu(T-M)}}{\beta} \right) + \sinh^{-1}\left( \frac{S e^{\mu(T-t)}}{\beta} \right)}{\alpha \sqrt{M - t}} \right]^2 \right\} \right\},
\]

for \( S > 0, t \in [0, M \land \tau) \) and where:

\[ \beta = S_0 e^{\mu T} \text{csch}(-\alpha L). \]
Stock Price is Hyperbolic Sine (con’d)

- Integrating the call’s payoff against this density yields the following pricing formula:

\[
C(S, t) = \frac{e^{-q(M-t)}}{2} \left( S + \sqrt{S^2 + \beta^2 e^{-2\mu(T-t)}} \right) \cdot \\
\left[ N(d_+ + \alpha \sqrt{M - t}) + N(d_- - \alpha \sqrt{M - t}) \right] \\
- \frac{\beta^2 e^{-q(M-t)}}{2 e^{2\mu(T-t)}} \frac{1}{S + \sqrt{S^2 + \beta^2 e^{-2\mu(T-t)}}} \cdot \\
\left[ N(d_+ - \alpha \sqrt{M - t}) + N(d_- + \alpha \sqrt{M - t}) \right] \\
- Ke^{-r(M-t)}[N(d_+) - N(d_-)],
\]

where:

\[
d_\pm = \frac{\pm \sinh^{-1} \left( \frac{S e^{\mu(T-t)}}{\beta} \right) - \sinh^{-1} \left( \frac{K e^{\mu(T-t)}}{\beta} \right)}{\alpha \sqrt{M - t}},
\]

and where:

\[
\beta = S_0 e^{\mu T} \text{csch}(-\alpha L).
\]
Example 2: Stock Price is Depressed Cubic

- Suppose that the final payoff \( \phi(w) \) is described by the following depressed cubic:

\[
\phi(w) = \begin{cases} 
\beta [(w - L)^3 + 3(\gamma - T)(w - L)] e^{(r-q)T} & \text{if } w > L; \\
0 & \text{if } w < L,
\end{cases}
\]

for \( \beta > 0, \gamma > T \).

- Imposing the lower boundary condition and solving the linear p.d.e. for the stock pricing function, we get:

\[
s(w, t) = \beta [(w - L)^3 + 3(\gamma - t)(w - L)] e^{(r-q)t}, \quad w > L, t \in [0, T].
\]
Stock Price is Depressed Cubic (con’d)

- Recall the link between the absolute volatility function $a(S,t)$ and the stock pricing function $s(w,t)$:

$$a(S,t) = \frac{\partial s}{\partial w}(w(S,t), t), \quad w \geq L(t), t \in [0,T].$$

- Carrying out the indicated operations on $s(w,t)$ yields:

$$a(S,t) = 3\beta[\Delta^2(S,t) + \gamma - t]e^{(r-q)t}, \quad S > 0, t \in [0,T),$$

where:

$$\beta = -\frac{S_0}{L^3 + 3\gamma L},$$

$$\Delta(S,t) \equiv \rho_+^{1/3}(S,t) - \rho_-^{1/3}(S,t), \quad S \geq 0, t \in [0,T],$$

and:

$$\rho_{\pm}(S,t) \equiv \pm \frac{S}{2\beta e^{(r-q)t}} + \sqrt{\left(\frac{S}{2\beta e^{(r-q)t}}\right)^2 + (\gamma - t)^3}, \quad S \geq 0, t \in [0,T].$$

- Dividing by $S$ yields the local volatility surface:

$$\sigma(S,t) \equiv \frac{a(S,t)}{S} = \frac{3}{S}\beta[\Delta^2(S,t) + \gamma - t]e^{(r-q)t}, \quad S > 0, t \in [0,T).$$
Stock Price is Depressed Cubic (con’d)

- The risk-neutral stock pricing density is:

\[
q(Z, M; S, t) = \frac{1}{a(Z, M) \sqrt{2\pi(M-t)}} \left\{ \exp \left\{ -\frac{1}{2} \left[ \frac{\Delta(Z, M) - \Delta(S, t)}{\sqrt{M-t}} \right]^2 \right\} 
- \exp \left\{ -\frac{1}{2} \left[ \frac{\Delta(Z, M) + \Delta(S, t)}{\sqrt{M-t}} \right]^2 \right\} \right\},
\]

where recall:

\[
a(Z, M) = 3\beta[\Delta^2(Z, M) + \gamma - M]e^{(r-q)M}, \quad Z > 0, M \in [0, T),
\]

\[
\beta = -\frac{S_0}{L^3 + 3\gamma L},
\]

\[
\Delta(S, t) \equiv \rho_+^{1/3}(S, t) - \rho_-^{1/3}(S, t), \quad S \geq 0, t \in [0, T],
\]

and:

\[
\rho_\pm(S, t) \equiv \pm \frac{S}{2\beta e^{(r-q)t}} + \sqrt{\left(\frac{S}{2\beta e^{(r-q)t}}\right)^2 + (\gamma - t)^3}, \quad S \geq 0, t \in [0, T].
\]
Stock Price is Depressed Cubic (con’d)

- Integrating the call’s payoff against this density yields the following pricing formula:

\[
C(S, t) = \beta e^{rt-qM} \sqrt{M-t} \\
\left\{ \left[ 3\gamma - 2t - M + [\triangle(S, t) - \triangle(K, M)]^2 + 3\triangle(S, t)\triangle(K, M) \right] N'(d_+) \right. \\
\left. - \left[ 3\gamma - 2t - M + [\triangle(S, t) + \triangle(K, M)]^2 - 3\triangle(S, t)\triangle(K, M) \right] N'(d_-) \right\} \\
+ Se^{-q(M-t)}[N(d_+) + N(d_-)] - Ke^{-r(M-t)}[N(d_+) - N(d_-)],
\]

where:

\[
d_\pm = \frac{\pm \triangle(S, t) - \triangle(K, M)}{\sqrt{M-t}}.
\]

where:

\[
\beta = -\frac{S_0}{L^3 + 3\gamma L},
\]

\[
\triangle(S, t) \equiv \rho_+^{1/3}(S, t) - \rho_-^{1/3}(S, t), \quad S \geq 0, t \in [0, T],
\]

and:

\[
\rho_\pm(S, t) \equiv \pm \frac{S}{2\beta e^{(r-q)t}} + \sqrt{\left( \frac{S}{2\beta e^{(r-q)t}} \right)^2 + (\gamma - t)^3}, \quad S \geq 0, t \in [0, T].
\]
Example 3: Stock Price is Depressed Cubic of Hyperbolic Sine

- Suppose that the final payoff $\phi(w)$ is described by the following depressed cubic:

$$\phi(w) = \begin{cases} 
\beta \left[ \sinh^3[\alpha(w - L)] + 3\gamma \sinh[\alpha(w - L)] \right] & \text{if } w > L; \\
0 & \text{if } w < L,
\end{cases}$$

for $\beta > 0$, $\gamma > 0$.

- Imposing the absorbing lower boundary condition and solving the linear p.d.e. for the stock pricing function, we get:

$$s(w, t) = \beta e^{-\mu(T-t)} \left\{ \sinh^3[\alpha(w - L)] + 3p(t) \sinh[\alpha(w - L)] \right\},$$

for $t \in [0, T], w > L$, where:

$$\mu \equiv r - q - \frac{9}{2} \alpha^2,$$

and:

$$p(t) \equiv \frac{1 - (1 - 4\gamma)e^{-4\alpha^2(T-t)}}{4}.$$
Stock Price is Depressed Cubic of Hyperbolic Sine (con’d)

- Recall the link between the absolute volatility function \( a(S, t) \) and the stock pricing function \( s(w, t) \):

\[
a(S, t) = \frac{\partial s}{\partial w}(w(S, t), t), \quad w \geq L(t), t \in [0, T].
\]

- Carrying out the indicated operations on \( s(w, t) \) yields:

\[
a(S, t) = \frac{3\alpha \beta \sqrt{1 + \Delta^2(S, t)}}{e^\mu(T-t)} \frac{\rho_+(S, t) + \rho_-(S, t)}{\rho_+^{1/3}(S, t) + \rho_-^{1/3}(S, t)}, \quad S > 0, t \in [0, T),
\]

where:

\[
\beta = \frac{S_0 e^{\mu T}}{\sinh^3[-\alpha L] + 3p(0) \sinh[-\alpha L]},
\]

\[
\Delta(S, t) \equiv \rho_+^{1/3}(S, t) - \rho_-^{1/3}(S, t), \quad S \geq 0, t \in [0, T],
\]

but now:

\[
\rho_\pm(S, t) \equiv \pm \frac{Se^{\mu(T-t)}}{2\beta} + \sqrt{\left(\frac{Se^{\mu(T-t)}}{2\beta}\right)^2 + p^3(t)}.
\]

- Dividing by \( S \) yields the local volatility surface:

\[
\sigma(S, t) \equiv \frac{a(S, t)}{S} = a(S, t) = \frac{3\alpha \beta \sqrt{1 + \Delta^2(S, t)}}{Se^\mu(T-t)} \frac{\rho_+(S, t) + \rho_-(S, t)}{\rho_+^{1/3}(S, t) + \rho_-^{1/3}(S, t)}, \quad S > 0
\]
Stock Price is Depressed Cubic of Hyperbolic Sine (con’d)

- The risk-neutral stock pricing density is:

\[
q(Z, M; S, t) = \frac{1}{a(Z, M)\sqrt{2\pi(M - t)}} \exp \left\{ -\frac{1}{2} \left[ -\frac{\alpha}{\sqrt{M - t}} \sinh^{-1}[\Delta(Z, M)] - \sinh^{-1}[\Delta(S, t)] \right]^2 \right\}
\]

\[
- \exp \left\{ -\frac{1}{2} \left[ -\frac{\alpha}{\sqrt{M - t}} \sinh^{-1}[\Delta(Z, M)] + \sinh^{-1}[\Delta(S, t)] \right]^2 \right\}
\]

where

\[
a(Z, M) = \frac{3\alpha\beta\sqrt{1 + \Delta^2(Z, M)}}{e^{\mu(T - M)}} \frac{\rho_+(Z, M) + \rho_-(Z, M)}{\rho_+^{1/3}(Z, M) + \rho_-^{1/3}(Z, M)}, \quad Z > 0, M \in [0, T],
\]

where:

\[
\beta = \frac{S_0e^{\mu T}}{\sinh^3[-\alpha L] + 3p(0) \sinh[-\alpha L]},
\]

\[
\Delta(S, t) \equiv \rho_+^{1/3}(S, t) - \rho_-^{1/3}(S, t), \quad S \geq 0, t \in [0, T],
\]

and:

\[
\rho_{\pm}(S, t) \equiv \pm \frac{Se^{\mu(T - t)}}{2\beta} \sqrt{\left( \frac{Se^{\mu(T - t)}}{2\beta} \right)^2 + p^3(t)} \quad S \geq 0, t \in [0, T].
\]
Stock Price is Depressed Cubic of Hyperbolic Sine (con’d)

• Integrating the call’s payoff against this density yields the following pricing formula:

\[
C(S, t) = \frac{e^{-\mu(T-t)-q(M-t)}}{8} \left[ \Delta(S, t) + \sqrt{\Delta^2(S, t) + 1} \right]^3 \cdot \left[ N(d_+ + 3\alpha\sqrt{M-t} + N(d_- - 3\alpha\sqrt{M-t}) \right] \\
-3[1 - 4p(M)] \frac{e^{-(r-q^2_2)(M-t)-\mu(T-M)}}{8} \left[ \Delta(S, t) + \sqrt{\Delta^2(S, t) + 1} \right] \cdot \left[ N(d_+ + \alpha\sqrt{M-t} + N(d_- - \alpha\sqrt{M-t}) \right] \\
+3[1 - 4p(M)] \frac{e^{-(r-q^2_2)(M-t)-\mu(T-M)}}{8} \frac{1}{\Delta(S, t) + \sqrt{\Delta^2(S, t) + 1}} \cdot \left[ N(d_+ - \alpha\sqrt{M-t} + N(d_- + \alpha\sqrt{M-t}) \right] \\
e^{-\mu(T-t)-q(M-t)} \frac{1}{8} \frac{1}{\left[ \Delta(S, t) + \sqrt{\Delta^2(S, t) + 1} \right]^3} \cdot \left[ N(d_+ - 3\alpha\sqrt{M-t} + N(d_- + 3\alpha\sqrt{M-t}) \right] \\
-Ke^{-r(M-t)}[N(d_+) - N(d_-)],
\]

where:

\[
d_{\pm} \equiv \pm \sinh^{-1} \left[ \Delta(S, t) \right] - \sinh^{-1} \left[ \Delta(K, M) \right] / \alpha\sqrt{M-t} \\
\beta = \frac{S_0e^{\mu T}}{\sinh^3[-\alpha L] + 3p(0)\sinh[-\alpha L]},
\]

\[
\Delta(S, t) \equiv \rho^{1/3}_{+}(S, t) - \rho^{1/3}_{-}(S, t), \quad S \geq 0, t \in [0, T],
\]

and:

\[
\rho_{\pm}(S, t) \equiv \pm \frac{Se^{\mu(T-t)}}{2\beta} + \sqrt{\left( \frac{Se^{\mu(T-t)}}{2\beta} \right)^2 + p^3(t)} \quad S \geq 0, t \in [0, T].
\]
Figure 1: The Stock Payoff Function and Its Inverse
Figure 2: The Stock Pricing Function
Volatility vs Spot and Time

Figure 3: The Local Volatility Surface
Figure 4: The Arcsinhnormal Probability Density Function
Figure 5: The Arcsinhnormal Call Value and Time Value vs. Black-Scholes
Figure 6: The Arcsinhnormal Call Valuation Function
Figure 7: The Stock Payoff Function for the Depressed Cubic ($L=-4, \gamma-T=1$)
Figure 8: The Stock Pricing Function ($L = -4, \gamma - T = 1$)
Figure 9: The Local Volatility Surface
Figure 10: The Cube Root Probability Density Function
Figure 11: The Call Valuation Function in the Depressed Cubic Model
Figure 12: The Stock Payoff Function for the Depressed Cubic in Hyperbolic Sine\((L=-4, \alpha = .1, \gamma = 1)\)
Figure 13: The Stock Pricing Function ($L=-4$, $\alpha = .1$, $\gamma = 1$)
Local Vol vs Spot Price and Time($L=-4, \alpha=.1, \gamma=2, S_0=100, T=1$)

Figure 14: The Local Volatility Surface
Figure 15: The Cube Root ArcSinh Probability Density Function
Figure 16: The Call Valuation Function in the Depressed Cubic in Hyperbolic Sine Model
Part IV

Summary and Extensions
Summary and Extensions

- Extensions to this analysis would include developing other tractable payoff functions as well as developing pricing formulas for path-dependent options.
- For stock or option payoff functions which are intractable analytically, the present analysis has implications for numerical analysis.
- A generalization of the present analysis would change the clock as well as the scale. This would let an OU process be the driver.
- A powerful alternative to a time change is a change of probability measure.
  1. It can be shown that for any pair of time and space-dependent drift and volatility functions, a change of scale and measure can be developed which reduces the drift to zero and the absolute volatility to unity.
  2. The generalization of the linear p.d.e. for the stock pricing function will in general be a nonlinear reaction diffusion equation.
Summary and Extensions (con’d)

- Drivers other than absorbing standard Brownian motion could also be considered. For example, one could consider the stock as an even function (e.g., square, fourth power, or hyperbolic cosine) of reflecting Brownian motion, or more generally of a Bessel process.

- The stock price process could depend on the driver’s path statistics such as the extrema or local time.

- Stochastic time changes of standard Brownian motion can make the driver a pure jump Levy process.

- Given the lateness of the hour, my co-author Michael Tari will deal with these extensions tomorrow.