Commodity Covariance Contracting

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Abstract

We introduce covariance swaps and show how to replicate one between two futures prices by static positions in spread and standard options coupled with dynamic trading in futures and bonds.

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A variance swap is a contract which pays the realized variance of the returns of a specified underlying asset over a specified period of time.

Several authors have shown that under certain conditions, the payoffs to a continuously monitored variance swap can be synthesized by combining continuous trading in the underlying with static positions in standard options maturing with the swap.

While the development of a replication strategy for variance swaps represents a significant theoretical advance, there are at least three problems with the proposed replication strategy:

1. The strategy replicates perfectly only if there are no jumps in the underlying.
2. The strategy assumes that the variance swap is continuously monitored, even though all variance swaps are discretely monitored in practice.
3. The strategy requires continuous trading in the underlying which is problematic in the presence of transactions costs and market closings.
Price Variance Swaps

• We propose a solution which addresses all three drawbacks described on the previous page.

• By changing the definition of variance in the swap from the realized variance of returns to the realized variance of price changes, we show that the new payoff can be perfectly replicated in the presence of jumps, discrete monitoring, and discrete trading opportunities.

• We further show that a contract paying the realized covariance of price changes can also be synthesized in this setting.
• We illustrate our results in the context of commodity options as the mark-
  kets for these structures have many of the features which we require. In 
  particular, to synthesize covariance swaps, we use spread options, which 
  represent one of the few options written on two assets and listed on an 
  organized exchange.

• The standard papers on the valuation of spread options assume that the 
  covariance between the two commodities in the spread is constant.

• We assume that the covariance between the two commodities is random, and 
  furthermore that the stochastic process governing covariance is unknown.

• Rather than price spread options in terms of a fixed covariance, we turn the 
  problem around and show how the covariance between the price changes in 
  two commodity futures can be traded, given the ability to trade dynamically 
  in the futures and to take static positions in spread options and in options 
  written on each component of the spread.
• Consider a single period setting in which investments are made at time $t_0$ with all payoffs received at time $t_n$.

• We assume there exists a futures market in a commodity for delivery at some date $T \geq t_n$.

• We also assume that markets exist for European-style futures options of all strikes.

• Note that listed futures options are generally American-style. However, by setting $T = t_n$, the underlying futures will converge to the spot at $t_n$ and so the assumption is that there exists European-style spot options in this special case.

• The above market structure allows investors to create any smooth function $f(F_n)$ of the terminal futures price $F_n$ by taking a static position at time 0 in options.
• When the theory of static hedging is used to generate desired volatility exposures, only the second derivative of the payoff affects such exposures.

• Consequently, we will always choose \( f \) so that its value and slope vanish at the initial futures price \( F_0 \) (i.e. \( f(F_0) = f'(F_0) = 0 \)).

• In this case, any twice differentiable payoff can be spanned by the following position in out-of-the-money options:

\[
\begin{align*}
f(F_n) &= \int_0^{F_0} f''(K)(K - F_T)^+dK + \int_{F_0}^{\infty} f''(K)(F_T - K)^+dK.
\end{align*}
\]

• In words, to create a twice differentiable payoff \( f(\cdot) \) with value and slope vanishing at \( F_0 \), buy \( f''(K)dK \) puts at all strikes less than \( F_0 \) and buy \( f''(K)dK \) calls at all strikes greater than \( F_0 \).
Valuing the Static Hedge

• Recall the decomposition of $f(\cdot)$ into payoffs from puts and calls:

$$f(F_n) = \int_{F_0}^{F_0^-} f''(K)(K - F_T)^+ dK + \int_{F_0^+}^{\infty} f''(K)(F_T - K)^+ dK.$$ 

• In the absence of arbitrage, a similar decomposition must prevail among the initial values. Specifically, if we let $V_0^f$, $P_0(K)$, and $C_0(K)$ denote the initial prices of the payoff $f(\cdot)$, the put, and the call respectively, then the no arbitrage condition requires that:

$$V_0^f = \int_{F_0}^{F_0^-} f''(K)P_0(K) dK + \int_{F_0^+}^{\infty} f''(K)C_0(K) dK.$$ 

• Thus, the value of an arbitrary payoff can be obtained from the option prices.
Application to Spread Options

- Given the spectrum of European options on the spread of two futures prices \( S = F_1 - F_2 \), one can also create any smooth function of this spread \( g(S) \).

- If we assume that the value and slope vanish at the initial spread \( S_0 \in \mathbb{R} \), i.e. \( g(S_0) = g'(S_0) = 0 \), then the analogous expression for the value of this payoff is:

\[
V_0^g = \int_{-\infty}^{S_0-} f''(K)P_0^s(K)dK + \int_{S_0+}^{\infty} f''(K)C_0^s(K)dK,
\]

where \( P_0^s(K) \) and \( C_0^s(K) \) are the initial prices of European put and call spread options struck at \( K \).

- Note that no assumptions were made regarding the stochastic processes governing the futures prices.
• Consider a finite set of discrete times \( \{t_0, t_1, \ldots, t_n\} \) at which one can trade futures contracts.

• Let \( F_i \) denote the price traded at on day \( i \), for \( i = 0, 1, \ldots, n \). By day \( n \), a standard estimator of the realized annualized variance of price changes will be:

\[
\text{Var}(\Delta F) \equiv \frac{N}{n} \sum_{i=1}^{n} (F_i - F_{i-1})^2,
\]

where \( N \) is the number of trading days in a year.

• We next demonstrate a strategy whose terminal payoff matches the above estimator of variance.
Recall that the objective is to find a trading strategy with final payoff of

$$\text{Var}(\triangle F) \equiv \frac{N}{n} \sum_{i=1}^{n} (F_i - F_{i-1})^2,$$

where $N$ is the number of trading days in a year.

By Taylor’s series, we note that:

$$F_i^2 = F_{i-1}^2 + 2F_{i-1}(F_i - F_{i-1}) + (F_i - F_{i-1})^2, \quad i = 1, \ldots, n.$$

Re-arranging and summing implies:

$$\sum_{i=1}^{n} (F_i^2 - F_{i-1}^2) - \sum_{i=1}^{n} 2F_{i-1}(F_i - F_{i-1}) = \sum_{i=1}^{n} (F_i - F_{i-1})^2, \quad i = 1, \ldots, n.$$

The first sum on the left telescopes to $F_n^2 - F_0^2$. Thus, multiplying both sides by $\frac{N}{n}$ implies:

$$\text{Var}(\triangle F) = \frac{N}{n}(F_n^2 - F_0^2) - \sum_{i=1}^{n} \frac{N}{n}F_{i-1}(F_i - F_{i-1}).$$
Derivation (Con’d)

- Recall:
  \[ \text{Var}(\triangle F) = \frac{N}{n}(F_n^2 - F_0^2) - \sum_{i=1}^{n} 2\frac{N}{n}F_{i-1}(F_i - F_{i-1}). \]

- The first term on the RHS can be regarded as a function \( \phi(\cdot) \) of \( F_n \), where:
  \[ \phi(F) \equiv \frac{N}{n}(F - F_0)^2. \]

- The first derivative is given by:
  \[ \phi'(F) = \frac{N}{n}2(F - F_0). \]

- Thus, the value and slope both vanish at \( F = F_0 \). Hence, the payoff \( \phi(F_n) \) can be replicated using options. The number of options held at each strike is proportional to the second derivative of \( \phi \), which is simply:
  \[ \phi''(F) = \frac{N}{n}2. \]

- Substitution implies:
  \[ \text{Var}(\triangle F) = \frac{N}{n}2 \left[ \int_{0}^{F_0^-} (K - F_n)^+ dK + \int_{F_0^+}^{\infty} (F_n - K)^+ dK \right] - \sum_{i=1}^{n} 2\frac{N}{n}F_{i-1}(F_i - F_{i-1}). \]
Derivation (Con’d)

• Recall:

\[
\text{Var}(\Delta F) = \frac{N}{n} 2 \left[ \int_0^{F_0^-} (K - F_n)^+ dK + \int_{F_0^+}^\infty (F_n - K)^+ dK \right] \\
- \sum_{i=1}^n 2\frac{N}{n} F_{i-1}(F_i - F_{i-1}).
\]

• The initial cost of creating the first term on the RHS is:

\[
V_0 = 2\frac{N}{n} \int_0^{F_0^-} P_0(K, T) dK + 2\frac{N}{n} \int_{F_0^+}^\infty C_0(K, T) dK.
\]

• If we assume that interest rates are constant at \( r \), then the second term on the RHS can be regarded as the cumulative marking-to-market proceeds arising from holding \(-e^{-r(t_n-t_i)}2\frac{N}{n} F_{i-1}\) futures contracts from time \( t_{i-1} \) to time \( t_i \).

• Since futures positions are costless, the theoretically fair price to charge for this variance contract is \( V_0 \) as given above.
• Recall the following representation of price variance:
\[
\text{Var}(\Delta F) = \frac{N}{n} (F_n^2 - F_0^2) - \sum_{i=1}^{n} \frac{2N}{n} F_{i-1}(F_i - F_{i-1}).
\]

• Amusingly, the dynamic strategy in futures can be interpreted as an attempt to delta-hedge a path-independent payoff made at \(t_n\), after making the false assumption of zero volatility.

• Given this ridiculous assumption, the value function is \(V_{i-1}^\phi(F_{i-1}, t_{i-1}) = e^{-r(t_n-t_{i-1})} \frac{N}{n} (F_{i-1}^2 - F_0^2)\) for \(i = 1, \ldots, n\). Recognizing that the marking-to-market proceeds are realized one trading day after the position is put on, the zero vol hedger holds \(e^{r(t_{i-1}-t_i)} \frac{\partial V_{i-1}^\phi(F_{i-1}, t_{i-1})}{\partial F} = e^{-r(t_n-t_i)} \frac{N}{n} 2F_{i-1}^2\) futures contracts from time \(t_{i-1}\) to time \(t_i\).

• This is exactly the dynamic strategy needed to create the last term in the top equation.

• Since realized volatility will in fact be positive, an error arises, and the magnitude of this error is given by the left side of the top equation.
Creating a Covariance Contract

• Recall that:

\[ S_i \equiv F_{1,i} - F_{2,i}, \quad i = 0, 1, \ldots, n \]

denotes the spread on day \( t_i \), where \( F_{1,i} \) and \( F_{2,i} \) denote the contemporaneous futures prices of the two components of the spread.

• We now show how to create a contract paying \( \text{Cov}(\Delta F_1, \Delta F_2) \equiv \frac{N}{n} \sum_{i=1}^{n} (F_{1,i} - F_{1,i-1})(F_{2,i} - F_{2,i-1}) \) at time \( t_n \) by combining static positions in options with dynamic trading in the underlying futures.
Creating a Covariance Contract

• Recall the well-known result that:
  \[ \text{Var}(\Delta S) = \text{Var}(\Delta (F_1 - F_2)) = \text{Var}(\Delta F_1) - 2\text{Cov}(\Delta F_1, \Delta F_2) + \text{Var}(\Delta F_2). \]

• Re-arranging this expression gives:
  \[ \text{Cov}(\Delta F_1, \Delta F_2) = \frac{-1}{2} \text{Var}(\Delta S) + \frac{1}{2} \text{Var}(\Delta F_1) + \frac{1}{2} \text{Var}(\Delta F_2). \]

• Thus, one can synthesize a covariance swap by selling half a variance swap on the spread and buying half a variance swap on each of the spread components.
Static Option & Dynamic Futures Positions

- Recall that one can synthesize a covariance swap by selling half a variance swap on the spread and buying half a variance swap on each of the spread components:

\[
\text{Cov}(\Delta F_1, \Delta F_2) = -\frac{1}{2} \text{Var}(\Delta S) + \frac{1}{2} \text{Var}(\Delta F_1) + \frac{1}{2} \text{Var}(\Delta F_2).
\]

- As these variance swaps are unlikely to be explicitly available, they can be synthesized. Substitution implies:

\[
\text{Cov}(\Delta F_1, \Delta F_2) = -\frac{N}{n} \left[ \int_{0}^{S_0^-} (K - S_n)^+ dK + \int_{S_0^+}^{\infty} (S_n - K)^+ dK \right] + \sum_{i=1}^{n} \frac{N}{n} S_{i-1}(S_i - S_{i-1}) + \frac{N}{n} \left[ \int_{0}^{F_0^{(1)}} (K - F_n^{(1)})^+ dK + \int_{F_0^{(1)}}^{\infty} (F_n^{(1)} - K)^+ dK \right] - \sum_{i=1}^{n} \frac{N}{n} F_{i-1}^{(1)}(F_i^{(1)} - F_{i-1}^{(1)}) + \frac{N}{n} \left[ \int_{0}^{F_0^{(2)}} (K - F_n^{(2)})^+ dK + \int_{F_0^{(2)}}^{\infty} (F_n^{(2)} - K)^+ dK \right] - \sum_{i=1}^{n} \frac{N}{n} F_{i-1}^{(2)}(F_i^{(2)} - F_{i-1}^{(2)}).
\]

- The 2nd term on the RHS can be created by dynamic trading in futures on the spread components:

\[
\sum_{i=1}^{n} \frac{N}{n} S_{i-1}(S_i - S_{i-1}) = \sum_{i=1}^{n} \frac{N}{n} S_{i-1}[(F_i^{(1)} - F_i^{(2)}) - (F_{i-1}^{(1)} - F_{i-1}^{(2)})] = \sum_{i=1}^{n} \frac{N}{n} S_{i-1}(F_i^{(1)} - F_{i-1}^{(1)}) - \sum_{i=1}^{n} \frac{N}{n} S_{i-1}(F_i^{(2)} - F_{i-1}^{(2)}).
\]
Substituting the bottom equation of the previous page into the one above it implies (after simplifying):

\[
\text{Cov}(\Delta F_1, \Delta F_2) = -\frac{N}{n} \left[ \int_0^{S_0^-} (K - S_n)^+ dK + \int_{S_0^+}^{\infty} (S_n - K)^+ dK \right] \\
+ \frac{N}{n} \left[ \int_0^{F_0^{(1)}} (K - F_n^{(1)})^+ dK + \int_{F_0^{(1)}}^{\infty} (F_n^{(1)} - K)^+ dK \right] \\
- \sum_{i=1}^{n} \frac{N}{n} F_{i-1}^{(2)} (F_i^{(1)} - F_{i-1}^{(1)}) \\
+ \frac{N}{n} \left[ \int_0^{F_0^{(2)}} (K - F_n^{(2)})^+ dK + \int_{F_0^{(2)}}^{\infty} (F_n^{(2)} - K)^+ dK \right] \\
- \sum_{i=1}^{n} \frac{N}{n} F_{i-1}^{(1)} (F_i^{(2)} - F_{i-1}^{(2)}).
\]

Since futures positions are costless, the fair price to charge for the covariance swap is the cost of creating the static options position:

\[
V_0 = -\frac{N}{n} \int_0^{S_0^-} P_0^s(K, T) dK - \frac{N}{n} \int_{S_0^+}^{\infty} C_0^s(K, T) dK \\
+ \frac{N}{n} \int_0^{F_0^{(1)}} P_0^{(1)}(K, T) dK + \frac{N}{n} \int_{F_0^{(1)}}^{\infty} C_0^{(1)}(K, T) dK \\
+ \frac{N}{n} \int_0^{F_0^{(2)}} P_0^{(2)}(K, T) dK + \frac{N}{n} \int_{F_0^{(2)}}^{\infty} C_0^{(2)}(K, T) dK.
\]

The investor must also trade futures on a daily basis, holding \(-e^{-r(t_n-t_i)} F_{2,i-1}\) units of the first futures contract and \(-e^{-r(t_n-t_i)} F_{1,i-1}\) units of the second from time \(t_{i-1}\) to time \(t_i\).
Summary & an Extension

• We showed that by combining static positions in options with dynamic trading in futures, investors can synthesize contracts paying the realized variance of a commodity or paying the realized covariance between two commodities.

• Importantly, these contracts were created without assuming anything about the underlying price.

• It would be interesting to extend our results to other payoffs besides variance and covariance. Indeed, Carr, Lewis, and Madan (2000) characterize the entire set of continuously paid cash flows which can be spanned in our structure.

• If interested, this paper (called On the Nature of Options) can be downloaded from:
  www.petercarr.net or
  www.math.nyu.edu\carrp\research\papers