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THE FORWARD PDE FOR EUROPEAN OPTIONS ON STOCKS WITH FIXED FRACTIONAL JUMPS

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We derive a partial integro differential equation (PIDE) which relates the price of a calendar spread to the prices of butterfly spreads and the functions describing the evolution of the process. These evolution functions are the forward local variance rate and a new concept called the *forward local default arrival rate*. We then specialize to the case where the only jump which can occur reduces the underlying stock price by a fixed fraction of its pre-jump value. This is a standard assumption when valuing an option written on a stock which can default. We discuss novel strategies for calibrating to a term and strike structure of European options prices. In particular using a few calendar dates, we derive closed form expressions for both the local variance and the local default arrival rate.

Keywords: Credit risk; default risk; forward equations; jump diffusion.

1. Introduction

Suppose that the risk-neutral evolution of a stock price is Markov in itself and time. Under the further assumptions of continuity and deterministic riskfree rates and dividend yields, Dupire [4] developed a forward partial differential equation (PDE)

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for European option prices. Dupire [5], Savine [12] and Klebaner [6, 7] all provide important extensions in the continuous case. We relax the continuity requirement and develop a forward partial integro differential equation (PIDE) for European calls on a stock following a jump diffusion process. This forward PIDE generalizes ones published previously by Andersen and Andreasen [1] where it is assumed that the jump intensity and jump distribution are level independent; as well as by Lipton [8] which focuses on Exponentially distributed jump sizes. We then specialize the analysis to where the only possible jump size is down by a fixed fraction of the pre-jump price. This particular forward PIDE is relevant to the case of default by the issuer of the asset underlying the option. Merton [10] also considers a jump to zero for the asset underlying a call. In contrast, we allow for a recovery rate and consider both calls and puts. We never consider default by the option writer. We also consider strategies for calibrating to a term and strike structure of European options. Similar work in the convertible bond context has been developed independently by Andersen [2] and Lipton [9]. However unlike others, we do not solve the PIDE numerically but use a few calendar dates to derive closed form expressions for both the local variance and the local default arrival rate.

The rest of the article is organized as follows: Section 2 defines the assumptions and notations for the paper. Section 3 derives the PIDE for the general case. Section 4 treats the fundamental case of Fixed Fractional Jumps. We first derive the PIDE for this specific case. We then deduce an Ordinary Differential Difference Equation (ODDE) which can be simplified into a solvable Ordinary Differential (or Difference) Equation. We finally solve the case where neither the local variance nor the local default arrival rate is known. Section 5 deals with the case of Puts, and Sec. 6 concludes.

2. Assumptions and Notation

We assume frictionless markets and no arbitrage. We also assume a deterministic interest rate $r(t)$ and a deterministic dividend yield $q(t)$. Assume that under a risk-neutral measure Q, the spot price dynamics of a traded asset are given by the following stochastic differential equation:

$$\begin{aligned} dS_t &= [r(t) - q(t)]S_{t-}\,dt + a(S_{t_-}, t)dW_t \\ &\quad + \int_{-S_{t_-}}^{\infty} j[\mu(dj, dt) - \nu(S_{t_-}, j, t)\,dt], \quad t \in [0, \Upsilon], \end{aligned} \tag{2.1}$$

where $S_0 > 0$ is known and where Υ is some arbitrarily distant horizon. This is the most general continuous time model consistent with Markovian stock price dynamics, limited liability, and deterministic rates and yields. Since the stock price process can jump, S_{t_-} denotes the stock price just prior to any jump at t. The function $a(S, t), S \geq 0, t \in [0, \Upsilon]$ is termed the instantaneous (normal) local volatility function. The process W is a Q standard Brownian motion. The integer valued random measure $\mu(dj, dt)$ counts the number of jumps of size j in the stock price at time t. The function $\nu(S, j, t), S \geq 0, j \in \Re - \{0\}, t \in [0, \Upsilon]$ is termed the local Lévy density and is used to compensate the driving jump process $J_t \equiv \int_0^t \int_{-S_{s-}}^{\infty} j\mu(dj, ds)$. As a result, the last term in (2.1) is the increment at t of a Q jump martingale. We restrict the evolution functions $a(S,t)$ and $\nu(S,j,t)$, so that the stock price is always nonnegative and absorbing at the origin.

The evolution functions $a(S,t)$ and $\nu(S,j,t)$ may each be defined as limits:

$$\begin{aligned}
a^2(S,t) &\equiv \lim_{\triangle T\downarrow 0} E^Q\left[\frac{(S_{t+\triangle T}-S_t)^2}{\triangle T}\middle| S_t = S\right], \quad S\geq 0,\ t\in[0,\Upsilon],\\
\nu(S,j,t) &\equiv \lim_{\triangle T\downarrow 0} E^Q\left[\frac{\delta(S_{t+\triangle T}-S_t-j)}{\triangle T}\middle| S_t = S\right], \quad S\geq 0,\ j\in\Re-\{0\},\ t\in[0,\Upsilon],
\end{aligned} \tag{2.2}$$

where $\delta(\cdot)$ is a delta function. Assuming that European calls[a] of all strikes $K > 0$ and maturities $T\in[0,\Upsilon]$ trade, this article will show that the evolution functions can also be expressed in terms of their time t prices $C_t(K,T)$:

$$\begin{aligned}
a^2(S,t) &= \lim_{\triangle T\downarrow 0} \frac{2}{\triangle T\frac{\partial^2}{\partial K^2}C_t(S,t+\triangle T)}C_t(S,t+\triangle T), \quad S\geq 0,\ t\in[0,\Upsilon],\\
\nu(S,j,t) &= \lim_{\triangle T\downarrow 0} \frac{\frac{\partial^2}{\partial K^2}C_t(S+j,t+\triangle T)}{\triangle T}, \quad S\geq 0,\ j\in\Re-\{0\},\ t\in[0,\Upsilon].
\end{aligned} \tag{2.3}$$

Thus, (2.3) states that as the time to maturity $\triangle T$ goes to zero, the local variance rate $a^2(S,t)$ arises as the limiting value at time t and price S of a position in $\frac{2}{\triangle T\frac{\partial^2}{\partial K^2}C_t(S,t+\triangle T)}$ at-the-money calls. It also states that as time to maturity goes to zero, the local Lévy density $\nu(S,j,t)$ is just the limiting value at time t and price S of a position in $\frac{1}{\triangle T}$ butterfly spreads struck at $S+j$ for $j\neq 0$. Note that as the time to maturity approaches zero, the maturity derivative of a claim is just the negative of its theta. Thus, the local Lévy density $\nu(S,j,t)$ can also be interpreted as the negative of the theta at time t and price S of the nearest term butterfly spread struck at $S+j$ for $j\neq 0$.

3. Derivation of Forward PIDE

Using the Meyer-Itô formula (see [11, p. 167]) on the convex function $(S_t-K)^+ e^{\int_t^T r(u)du}$ yields:

$$\begin{aligned}
(S_T-K)^+ &= (S_0-K)^+ e^{\int_0^T r(u)du} + \int_0^T e^{\int_t^T r(u)du} 1(S_{t-}>K)dS_t\\
&\quad + \int_0^T e^{\int_t^T r(u)du}\left[\frac{a^2(S_{t-},t)}{2}\delta(S_{t-}-K) - r(t)(S_{t-}-K)^+\right]dt\\
&\quad + \sum_{0<t\leq T} e^{\int_t^T r(u)du}\left[(S_t-K)^+ - (S_{t-}-K)^+ - 1(S_{t-}>K)\triangle S_t\right],
\end{aligned} \tag{3.1}$$

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[a] All of the results in this document also apply to European puts *mutatis mutandis*. One could also easily extend to other European path-independent claims with a singly kinked payoff.

where $\triangle S_t$ is the random size jump in S at time t (which is zero if there is no jump). Using the counting measure $\mu(dj, dt)$, (3.1) can also be written formally as:

$$\begin{aligned}(S_T - K)^+ &= (S_0 - K)^+ e^{\int_0^T r(u)du} + \int_0^T e^{\int_t^T r(u)du} 1(S_{t-} > K) dS_t \\ &\quad + \int_0^T e^{\int_t^T r(u)du} \left[\frac{a^2(S_{t-}, t)}{2}\delta(S_{t-} - K) - r(t)(S_{t-} - K)^+\right] dt \\ &\quad + \int_0^T \int_{-S_{t-}}^{\infty} e^{\int_t^T r(u)du} \left[(S_{t-} + j - K)^+ - (S_{t-} - K)^+\right. \\ &\quad \left. - 1(S_{t-} > K) j\right] \mu(dj, dt). \end{aligned} \tag{3.2}$$

There is an equivalent representation for the expression in square brackets in the last line of (3.2):

$$\begin{aligned}&(S_{t-} + j - K)^+ - (S_{t-} - K)^+ - 1(S_{t-} > K) j \\ &\quad = 1(S_{t-} \le K)(S_{t-} + j - K)^+ + 1(S_{t-} > K)(K - S_{t-} - j)^+. \end{aligned} \tag{3.3}$$

The equality in (3.3) can be easily proved by considering the $2 \times 2 = 4$ cases which arise from considering whether or not the call is in-the-money at time $t-$ and at time t. Substituting (3.3) in (3.2) leaves:

$$\begin{aligned}(S_T - K)^+ &= (S_0 - K)^+ e^{\int_0^T r(u)du} + \int_0^T e^{\int_t^T r(u)du}\ 1(S_{t-} > K) dS_t \\ &\quad + \int_0^T e^{\int_t^T r(u)du} \left[\frac{a^2(K, t)}{2}\delta(S_{t-} - K) - r(t)(S_{t-} - K)^+\right] dt \\ &\quad + \int_0^T \int_{-S_{t-}}^{\infty} e^{\int_t^T r(u)du} \left[1(S_{t-} \le K)(S_{t-} + j - K)^+\right. \\ &\quad \left. + 1(S_{t-} > K)(K - S_{t-} - j)^+\right] \mu(dj, dt). \end{aligned} \tag{3.4}$$

Subtract and add the term $\int_0^T e^{\int_t^T r(u)du}[r(t) - q(t)] S_{t-} 1(S_{t-} > K) dt$ to (3.4):

$$\begin{aligned}(S_T - K)^+ &= (S_0 - K)^+ e^{\int_0^T r(u)du} \\ &\quad + \int_0^T e^{\int_t^T r(u)du} 1(S_{t-} > K)[dS_t - [r(t) - q(t)] S_{t-}\, dt] \\ &\quad + \int_t^T e^{\int_t^T r(u)du} \left[\frac{a^2(K, t)}{2}\delta(S_{t-} - K)\right. \\ &\quad \left. + [r(t) - q(t)] S_{t-} 1(S_{t-} > K) - r(t)(S_{t-} - K)^+\right] dt \\ &\quad + \int_0^T \int_{-S_{t-}}^{\infty} e^{\int_t^T r(u)du} \left[1(S_{t-} \le K)(S_{t-} + j - K)^+\right. \\ &\quad \left. + 1(S_{t-} > K)(K - S_{t-} - j)^+\right] \mu(dj, dt). \end{aligned} \tag{3.5}$$

Suppose we add and subtract $[r(t)-q(t)]K1(S_{t-}>K)$ to $[r(t)-q(t)]S_{t-}1(S_{t-}>K)-r(t)(S_{t-}-K)^+$:

$$\begin{aligned}
&[r(t)-q(t)]S_{t-}1(S_{t-}>K)-r(t)(S_{t-}-K)^+\\
&\quad=[r(t)-q(t)]K1(S_{t-}>K)+[r(t)-q(t)](S_{t-}-K)^+-r(t)(S_{t-}-K)^+\\
&\quad=[r(t)-q(t)]K1(S_{t-}>K)-q(t)(S_{t-}-K)^+. \qquad (3.6)
\end{aligned}$$

Substituting (3.6) in (3.5) implies:

$$\begin{aligned}
(S_T-K)^+ &= (S_0-K)^+e^{\int_0^T r(u)du}\\
&\quad+\int_0^T e^{\int_t^T r(u)du}1(S_{t-}>K)[dS_t-[r(t)-q(t)]S_{t-}\,dt]\\
&\quad+\int_t^T e^{\int_t^T r(u)du}\left[\frac{a^2(K,t)}{2}\delta(S_{t-}-K)+[r(t)-q(t)]K1(S_{t-}>K)\right.\\
&\quad\left.-q(t)(S_{t-}-K)^+\right]dt+\int_0^T\int_{-S_{t-}}^{\infty}e^{\int_t^T r(u)du}[1(S_{t-}\le K)\\
&\quad\times(S_{t-}+j-K)^++1(S_{t-}>K)(K-S_{t-}-j)^+]\mu(dj,dt). \qquad (3.7)
\end{aligned}$$

Taking discounted conditional expectations of (3.7) under Q yields:

$$\begin{aligned}
C_0(K,T) &= (S_0-K)^+\\
&\quad+\int_0^T e^{-\int_0^t r(u)du}\left[\frac{a^2(K,t)}{2}E_0^Q\delta(S_{t-}-K)\right.\\
&\quad\left.+[r(t)-q(t)]KE_0^Q1(S_{t-}>K)-q(t)E_0^Q(S_{t-}-K)^+\right]dt\\
&\quad+E_0^Q\left\{\int_0^T e^{-\int_0^t r(u)du}\int_{-S_{t-}}^{\infty}\left[1(S_{t-}\le K)(S_{t-}+j-K)^+\right.\right.\\
&\quad\left.\left.+1(S_{t-}>K)(K-S_{t-}-j)^+\right]\nu(S_{t-},j,t)dj\,dt\right\}, \qquad (3.8)
\end{aligned}$$

where the law of iterated expectations has been used to replace $\mu(dj,dt)$ with $\nu(S_{t_-},j,t)dj$.

Suppose we define a call maturing at t by:

$$C_0(K,T)=e^{-\int_0^t r(u)du}E_0^Q(S_{t-}-K)^+. \qquad (3.9)$$

Differentiating w.r.t. K:

$$\frac{\partial}{\partial K}C_0(K,T)=-e^{-\int_0^t r(u)du}E_0^Q1(S_{t-}>K). \qquad (3.10)$$

Differentiating w.r.t. K again:

$$\frac{\partial^2}{\partial K^2}C_0(K,T)=e^{-\int_0^t r(u)du}E_0^Q\delta(S_{t-}-K). \qquad (3.11)$$

Substituting (3.9) to (3.11) in (3.8) implies:

$$
\begin{aligned}
C_0(K,T) &= (S_0 - K)^+ \\
&\quad + \int_0^T \left[\frac{a^2(K,t)}{2} \frac{\partial^2}{\partial K^2} C_0(K,t) - [r(t) - q(t)] K \frac{\partial}{\partial K} C_0(K,t) - q(t) C_0(K,t) \right] dt \\
&\quad + E_0^Q \left\{ \int_0^T e^{-\int_0^t r(u)du} \int_{-S_{t-}}^{\infty} \left[1(S_{t-} \leq K)(S_{t-} + j - K)^+ \right. \right. \\
&\quad \left. \left. + 1(S_{t-} > K)(K - S_{t-} - j)^+ \right] \nu(S_{t-}, j, t) dj\, dt \right\}, \qquad (3.12)
\end{aligned}
$$

Differentiate both sides with respect to T:

$$
\begin{aligned}
&\frac{\partial}{\partial T} C_0(K,T) \\
&\quad = \frac{a^2(K,T)}{2} \frac{\partial^2}{\partial K^2} C_0(K,T) - [r(T) - q(T)] K \frac{\partial}{\partial K} C_0(K,T) - q(T) C_0(K,T) \\
&\qquad + E_0^Q e^{-\int_0^T r(u)du} \left\{ \int_{-S_{T_-}}^{\infty} \left[1(S_{T-} \leq K)(S_{T-} + j - K)^+ \right. \right. \\
&\qquad \left. \left. + 1(S_{T-} > K)(K - S_{T-} - j)^+ \right] \nu(S_{T-}, j, T) dj \right\}. \qquad (3.13)
\end{aligned}
$$

The first term involves $a^2(K,T)$, which is called the forward local variance rate, observed at time 0 for maturity T. Analogously, the last term is christened here as the *forward local variation rate*, observed at time 0 for maturity T.

Distributing the integral and discounted expectation in the last term over the sum:

$$
\begin{aligned}
&\frac{\partial}{\partial T} C_0(K,T) \\
&\quad = \frac{a^2(K,T)}{2} \frac{\partial^2}{\partial K^2} C_0(K,T) - [r(T) - q(T)] K \frac{\partial}{\partial K} C_0(K,T) - q(T) C_0(K,T) \\
&\qquad + e^{-\int_0^T r(u)du} E_0^Q \left\{ \int_{K - S_{T_-}}^{\infty} 1(S_{T-} \leq K)(S_{T-} + j - K) \nu(S_{T-}, j, T) dj \right\} \\
&\qquad + e^{-\int_0^T r(u)du} E_0^Q \left\{ \int_{-S_{T_-}}^{K - S_{T_-}} 1(S_{T_-} > K)(K - S_{T_-} - j) \nu(S_{T-}, j, T) dj \right\}.
\end{aligned}
\qquad (3.14)
$$

This is the forward PIDE for call values under deterministic interest rates and dividend yields when the underlying stock price follows a continuous time Markov process. From (3.11), one can replace the discounted expectation in the last two terms with integrals over $\frac{\partial^2}{\partial K^2} C_0(K,T)$, but we suspend this substitution. The last two terms represent the contribution from up jumps and down jumps respectively, which cause the stock price to cross the strike price at T.

To generate a forward boundary value problem, note that for some fixed $(0, S_0)$, no arbitrage also implies that call values satisfy the following boundary conditions:

$$\begin{aligned} C_0(K,0) &= (S_0 - K)^+, \quad K \geq 0 \\ \lim_{K \uparrow \infty} C_0(K,T) &= 0, \quad T \in [0,\Upsilon] \\ \lim_{K \downarrow 0} C_0(K,T) &\sim S_0 e^{-\int_0^T q(u)du} - K e^{-\int_0^T r(u)du}, \quad T \in [0,\Upsilon]. \end{aligned} \tag{3.15}$$

Thus, given a specification of the local variance rate $a(S,t)$, $S > 0, t \in [0,T]$ and the local Lévy density $\nu(S,j,t)$, $S > 0$, $j \in \Re$, $t \in [0,T]$, one can numerically solve the boundary value problem (3.13) to (3.15) for the call values. The result is a grid of call values for various strikes and maturities. Note that differentiating the PIDE (3.13) with respect to S does not change its form, so the same approach can be used to efficiently calculate deltas and gammas.

If one is instead given call values across all strikes and maturities at time 0, then one can solve (3.12) for one of $a(S,t)$ or $\nu(S,j,t)$ but not both. When the objective is to determine $\nu(S,j,t)$ given $a(S,t)$, note that the integral in (3.12) is a convolution, so the fast Fourier transform can be used to determine $\nu(S,j,t)$ under some additional assumptions. The details are in Carr [3].

4. Calls on Stocks with a Fixed Fractional Jump

4.1. *The partial integro differential equation*

We now suppose that the only jump which can occur is downward by a fixed fraction of the pre-jump value. This assumption is commonly used to capture default on the stock underlying an option. Note that the option writer is assumed to be solvent. This setting is most appropriate for listed (European) options.

To capture down jumps which are a fixed fraction of the pre-default value, the local Lévy density $\nu(S,j,t)$, is assumed to have the form:

$$\nu(S,j,t) = \lambda(S,t)\delta(j + S\phi(t)), \tag{4.1}$$

where $\phi(t) \in (0,1]$ is called the *forward fractional loss rate*, observed at time 0 for maturity t. The nonnegative function $\lambda(S,t)$ is christened here as the *forward local default arrival rate*, observed at time 0 for maturity t. Thus, at each t, there is a discrete arrival rate on a jump of size $-\phi(t)S_{t-}$ which takes the stock price from S_{t-} to $\alpha(t)S_{t-}$ where $\alpha(t) \equiv 1 - \phi(t)$ is called the *stock recovery rate* at time t. No other jump size can occur. Substituting (4.1) in (3.14) implies:

$$\begin{aligned} \frac{\partial}{\partial T} C_0(K,T) = {} & \frac{a^2(K,T)}{2} \frac{\partial^2}{\partial K^2} C_0(K,T) - [r(T) - q(T)]K \frac{\partial}{\partial K} C_0(K,T) \\ & - q(T)C_0(K,T) + e^{-\int_0^T r(u)du} \\ & \times E_0^Q \left\{ \int_{-S_{T_-}}^{K - S_{T_-}} 1(S_{T_-} > K) 1\left(S_{T-} < \frac{K}{\alpha(T)}\right) \right. \\ & \qquad \left. (K - S_{T_-} - j)\lambda(S_{T-},T)\delta(j + \phi(T)S_{T-})dj \right\}. \end{aligned}$$

The indicator function $1\left(S_{T-} < \frac{K}{\alpha(T)}\right)$ has been inserted in the last term because if $S_{T-} > \frac{K}{\alpha(T)}$, then the down jump does not force the stock price down below the strike and hence the integral vanishes. Using the sifting property of delta functions:

$$\begin{aligned}
&\frac{\partial}{\partial T}C_0(K,T)\\
&\quad= \frac{a^2(K,T)}{2}\frac{\partial^2}{\partial K^2}C_0(K,T) - [r(T)-q(T)]K\frac{\partial}{\partial K}C_0(K,T) - q(T)C_0(K,T)\\
&\qquad + e^{-\int_0^T r(u)du}E_0^Q\left[1(S_{T\_} > K)1\left(S_{T-} < \frac{K}{\alpha(T)}\right)[K-\alpha(T)S_{T-}]\lambda(S_{T-},T)\right]\\
&\quad= \frac{a^2(K,T)}{2}\frac{\partial^2}{\partial K^2}C_0(K,T) - [r(T)-q(T)]K\frac{\partial}{\partial K}C_0(K,T) - q(T)C_0(K,T)\\
&\qquad + \int_K^{\frac{K}{\alpha(T)}} \lambda(L,T)[K-\alpha(T)L]\frac{\partial^2}{\partial K^2}C_0(L,T)dL. \qquad (4.2)
\end{aligned}$$

Note that the butterfly spreads maturing at T do not include the jump at T, if any, in their payoff. In other words, their payoff at T is $\delta(S_{T-} - K)$ as opposed to $\delta(S_T - K)$.

If the default rate function $\lambda(S,t)$, recovery rate $\alpha(t)$, and all call prices $C_0(K,T)$ are known, but the forward local variance rate $a^2(S,t)$ is not, then one can learn it from (4.2):

$$\begin{aligned}
&a^2(K,T)\\
&= 2\times\frac{\frac{\partial C_0(K,T)}{\partial T} + [r(T)-q(T)]K\frac{\partial C_0(K,T)}{\partial K} + q(T)C_0(K,T) - \int_K^{\frac{K}{\alpha(T)}}\lambda(L,T)[K-\alpha(T)L]\frac{\partial^2 C_0(L,T)}{\partial K^2}dL}{\frac{\partial^2 C_0(K,T)}{\partial K^2}}. \qquad (4.3)
\end{aligned}$$

In contrast, if the volatility function $a(S,t)$, recovery rate $\alpha(t)$, and all call prices $C_0(K,T)$ are known, but the function $\lambda(S,t)$ is not, then one can treat the forward PIDE (4.2) as an integral equation determining the function $\lambda(S,t)$.[b]

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[b] Two special cases are worth being mentioned here:

First, the special case of *Total Ruin* where the default event drives the stock price to zero. We can solve the integral equation analytically and

$$\lambda(K,T) = q(T) - r(T) - \frac{\frac{\partial^2\gamma_0(K,T)}{\partial T\partial K} - \frac{\partial}{\partial K}\left[\frac{a^2(K,T)}{2K}\frac{\partial^2 C_0(K,T)}{\partial K^2}\right] + q(T)\frac{\partial}{\partial K}\gamma_0(K,T)}{\frac{\partial^2}{\partial K^2}C_0(K,T)},$$

with $\gamma_0(K,T) \equiv \frac{C_0(K,T)}{K}$.

Second, the case of a *Deterministic Default Rate* where the local Lévy density $\nu(S,j,t)$ has the form $\nu(S,j,t) = \lambda(t)\delta(j+\phi(t)S)$. In this case, the forward PIDE simplifies into a forward partial differential difference equation (PDDE)

$$\begin{aligned}
\frac{\partial}{\partial T}C_0(K,T) = {}& \frac{a^2(K,T)}{2}\frac{\partial^2}{\partial K^2}C_0(K,T) - [r(T)+\lambda(T)\phi(T)-q(T)]K\frac{\partial}{\partial K}C_0(K,T)\\
&- [q(T)+\lambda(T)\alpha(T)]C_0(K,T) + \lambda(T)\alpha(T)C_0\left(\frac{K}{\alpha(T)},T\right).
\end{aligned}$$

4.2. *An ordinary differential difference equation*

One can turn the problem into solving an ordinary differential difference equation holding for each T. To do so, differentiate (4.2) w.r.t. K:

$$\begin{aligned}\frac{\partial^2}{\partial T \partial K} C_0(K,T) = {} & \frac{a^2(K,T)}{2} \frac{\partial^3 C_0(K,T)}{\partial K^3} \\ & - \left\{ [r(T) - q(T)]K - a(K,T) \frac{\partial a(K,T)}{\partial S} \right\} \frac{\partial^2 C_0(K,T)}{\partial K^2} \\ & - r(T) \frac{\partial C_0(K,T)}{\partial K} - \lambda(K,T) K \phi(T) \frac{\partial^2}{\partial K^2} C_0(K,T) \\ & + \int_K^{\frac{K}{\alpha(T)}} \lambda(L,T) \frac{\partial^2}{\partial K^2} C_0(L,T) dL. \end{aligned} \tag{4.4}$$

Differentiate w.r.t. K again:

$$\begin{aligned}\frac{\partial^3 C_0(K,T)}{\partial T \partial K^2} = {} & \frac{a^2(K,T)}{2} \frac{\partial^4}{\partial K^4} C_0(K,T) \\ & - \left\{ [r(T) - q(T)]K - 2a(K,T) \frac{\partial a(K,T)}{\partial S} \right\} \frac{\partial^3}{\partial K^3} C_0(K,T) \\ & - \left\{ 2r(T) - q(T) - \left[\frac{\partial a(K,T)}{\partial S} \right]^2 - a(K,T) \frac{\partial^2 a(K,T)}{\partial S^2} \right\} \\ & \times \frac{\partial^2}{\partial K^2} C_0(K,T) - \frac{\partial \lambda(K,T)}{\partial K} K \phi(T) \frac{\partial^2 C_0(K,T)}{\partial K^2} \\ & - \lambda(K,T) \left\{ \phi(T) \frac{\partial}{\partial K} \left[K \frac{\partial^2}{\partial K^2} C_0(K,T) \right] + \frac{\partial^2 C_0(K,T)}{\partial K^2} \right\} \\ & + \frac{\lambda\left(\frac{K}{\alpha(T)}, T\right)}{\alpha(T)} \frac{\partial^2 C_0\left(\frac{K}{\alpha(T)}, T\right)}{\partial K^2}. \end{aligned} \tag{4.5}$$

For each T, (4.5) is a linear inhomogeneous ordinary differential difference equation (ODDE) governing $\lambda(K,T)$ as a function of just K. This ODDE can be put in the form:

$$\frac{\partial \lambda(K,T)}{K} + p_0(K,T)\lambda(K,T) + q_0(K,T)\lambda\left(\frac{K}{\alpha(T)}, T\right) = f_0(K,T), \tag{4.6}$$

where:

$$\begin{aligned} p_t(K,T) & \equiv \frac{\phi(T) \frac{\partial}{\partial K} \left[K \frac{\partial^2}{\partial K^2} C_t(K,T) \right] + \frac{\partial^2 C_t(K,T)}{\partial K^2}}{K \phi(T) \frac{\partial^2 C_t(K,T)}{\partial K^2}} \\ & = \frac{\partial}{\partial K} \ln \left[K \frac{\partial^2 C_t(K,T)}{\partial K^2} \right] + \frac{1}{\phi(T) K}, \qquad t \in [0,T] \\ q_t(K,T) & \equiv - \frac{\frac{\partial^2 C_t\left(\frac{K}{\alpha(T)}, T\right)}{\partial K^2}}{\alpha(T) K \phi(T) \frac{\partial^2 C_t(K,T)}{\partial K^2}} \end{aligned}$$

$$f_t(K,T) \equiv \frac{\frac{a^2(K,T)}{2}\frac{\partial^4 C_t(K,T)}{\partial K^4} - \left\{[r(T)-q(T)]K - 2a(K,T)\frac{\partial a(K,T)}{\partial S}\right\}\frac{\partial^3 C_t(K,T)}{\partial K^3}}{K\phi(T)\frac{\partial^2 C_t(K,T)}{\partial K^2}} - \frac{\left\{2r(T)-q(T)-\left[\frac{\partial a(K,T)}{\partial S}\right]^2 - a(K,T)\frac{\partial^2 a(K,T)}{\partial S^2}\right\}\frac{\partial^2 C_t(K,T)}{\partial K^2} + \frac{\partial^3 C_t(K,T)}{\partial K^2 \partial T}}{K\phi(T)\frac{\partial^2 C_t(K,T)}{\partial K^2}}. \tag{4.7}$$

This ODDE can be solved numerically. Note that the problem is embarassingly parallel, so that one can devote different processors for different values of T. However we use an alternative approach leading us to closed form expressions for $\lambda(S,t)$ and $a(S,t)$.

Indeed, the calculation for $\lambda(S,t)$ can be done analytically if we observe call prices at two times, $t=t_0$ and $t=t_1$. From the call prices at time t_1, we have:

$$\frac{\partial\lambda(K,T)}{\partial K} + p_1(K,T)\lambda(K,T) + q_1(K,T)\lambda\left(\frac{K}{\alpha(T)},T\right) = f_1(K,T), \tag{4.8}$$

where $p_1(K,T)$, $q_1(K,T)$, and $f_1(K,T)$ are defined from evaluating (4.7) at $t=t_1$.

Hence, one can combine (4.4) and (4.8) to create either an ordinary difference equation or an ordinary differential equation. As we will see, both these equations are solvable.

4.3. *An ordinary difference equation*

Subtracting ODDE (4.8) from the ODDE (4.6) implies:

$$\begin{aligned}&[p_1(K,T)-p_0(K,T)]\lambda(K,T) + [q_1(K,T)-q_0(K,T)]\lambda\left(\frac{K}{\alpha(T)},T\right)\\ &\quad = f_1(K,T) - f_0(K,T).\end{aligned} \tag{4.9}$$

which could obviously be rewritten as

$$\lambda(K,T) = \tilde{q}(K,T)\lambda\left(\frac{K}{\alpha(T)},T\right) + \tilde{f}(K,T), \tag{4.10}$$

with

$$\begin{aligned}\tilde{q}(K,T) &= \tilde{q}_{0,1}(K,T) = \frac{q_0(K,T)-q_1(K,T)}{p_1(K,T)-p_0(K,T)},\\ \tilde{f}(K,T) &= \tilde{f}_{0,1}(K,T) = \frac{f_1(K,T)-f_0(K,T)}{p_1(K,T)-p_0(K,T)}.\end{aligned} \tag{4.11}$$

Using induction on $\alpha(T)$, we shall have for any integer N

$$\begin{aligned}\lambda(K,T) = {} & \left(\prod_{j=0}^{N}\tilde{q}\left(\frac{K}{\alpha(T)^j},T\right)\right)\lambda\left(\frac{K}{\alpha(T)^{N+1}},T\right) + \tilde{f}(K,T)\\ &+ \sum_{j=1}^{N}\left(\prod_{k=0}^{j-1}\tilde{q}\left(\frac{K}{\alpha(T)^k},T\right)\right)\tilde{f}\left(\frac{K}{\alpha(T)^j},T\right)\end{aligned} \tag{4.12}$$

We assume that $\left(\prod_{j=0}^{+\infty} \tilde{q}\left(\frac{K}{\alpha(T)^j}, T\right)\right) \lambda(+\infty, T) = 0$, or in other words that the default arrival rate $\lambda(S,t)$ goes to zero sufficiently fast as the stock price S goes to infinity. We then use the fact that $1/\alpha(T) > 1$, make $N \to +\infty$ and

$$\lambda(K,T) = \tilde{f}(K,T) + \sum_{j=1}^{+\infty} \left(\prod_{k=0}^{j-1} \tilde{q}\left(\frac{K}{\alpha(T)^k}, T\right)\right) \tilde{f}\left(\frac{K}{\alpha(T)^j}, T\right) \tag{4.13}$$

which is the solution to the Ordinary Difference Equation.

4.4. *An ordinary differential equation*

Alternatively to create a first order ordinary differential equation, multiply (4.6) by $q_1(K,T)$ and (4.8) by $q_0(k,T)$ and difference the results:

$$\frac{\partial \lambda(K,T)}{\partial K} + p(K,T)\lambda(K,T) = f(K,T), \tag{4.14}$$

where:

$$p(K,T) \equiv \frac{q_1(K,T)p_0(K,T) - q_0(K,T)p_1(K,T)}{q_1(K,T) - q_0(K,T)},$$
$$f(K,T) \equiv \frac{q_1(K,T)f_0(K,T) - q_0(K,T)f_1(K,T)}{q_1(K,T) - q_0(K,T)}. \tag{4.15}$$

By introducing an integrating factor, it is well known that the solution to (4.14) is:

$$\lambda(K,T) = e^{\int_K^\infty p(L,T)dL} \lambda(\infty,T) - \int_K^\infty e^{\int_K^M p(L,T)dL} f(M,T)dM. \tag{4.16}$$

We assume that $e^{\int_K^\infty p(L,T)dL} \lambda(\infty,T) = 0$, or in other words that the default arrival rate $\lambda(S,t)$ goes to zero sufficiently fast as the stock price S goes to infinity. Hence (4.16) simplifies to:

$$\lambda(K,T) = -\int_K^\infty e^{\int_K^M p(L,T)dL} f(M,T)dM, \tag{4.17}$$

where $p(\cdot,\cdot)$ and $f(\cdot,\cdot)$ are given in (4.15).

This gives us a second way to determine $\lambda(S,t)$.

4.5. *Separating the variables in the ODDE*

In the previous paragraphs, we derived two equivalent closed form expressions for $\lambda(S,t)$ under the assumption that $a(S,t)$ was known. We now suppose we have no knowledge of *either* the default arrival rate $\lambda(S,t)$ or the local volatility $a(S,t)$ and would like to deduce them from market data separately.

The Ordinary Differential Difference Equation (4.5) could be rewritten as

$$D_0(\lambda) = C_0^{(4)} X(a) + C_0^{(3)} Y(a) + C_0^{(2)} Z(a), \tag{4.18}$$

with

$$C_0^{(j)} = \frac{\partial^j}{\partial K^j} C_0(K,T),$$

for $j = 2, 3, 4$ and

$$\begin{aligned} D_0(\lambda) = {} & \frac{\partial^3}{\partial T \partial K^2} C_0(K,T) + \frac{\partial \lambda(K,T)}{\partial K} K \phi(T) \frac{\partial^2}{\partial K^2} C_0(K,T) \\ & + \lambda(K,T) \left\{ \phi(T) \frac{\partial}{\partial K} \left[K \frac{\partial^2}{\partial K^2} C_0(K,T) \right] + \frac{\partial^2}{\partial K^2} C_0(K,T) \right\} \\ & - \frac{1}{\alpha(T)} \lambda \left(\frac{K}{\alpha(T)}, T \right) \frac{\partial^2}{\partial K^2} C_0 \left(\frac{K}{\alpha(T)}, T \right) \\ & + [r(T) - q(T)] K \frac{\partial^3}{\partial K^3} C_0(K,T) + [2r(T) - q(T)] \frac{\partial^2}{\partial K^2} C_0(K,T), \end{aligned}$$

and

$$\begin{aligned} X(a) &= \frac{1}{2} a^2(K,T) \\ Y(a) &= 2a(K,T) \frac{\partial a(K,T)}{\partial S} = \frac{\partial}{\partial S} a^2(K,T) \\ Z(a) &= \left[\frac{\partial a(K,T)}{\partial S} \right]^2 + a(K,T) \frac{\partial^2 a(K,T)}{\partial S^2} = \frac{1}{2} \frac{\partial^2}{\partial S^2} a^2(K,T). \end{aligned}$$

Ignoring the factor 1/2, the interpretation for $(X\ , Y, Z)$ is : the value, the slope and the convexity of the local variance $a^2(S,t)$ with respect to S, taken at $S = K$ and $t = T$.

Taking the same equation (4.18) on calendar dates t_1 and t_2 (in addition to t_0) we will have a 3×3 linear system

$$\begin{aligned} D_0(\lambda) &= C_0^{(4)} X(a) + C_0^{(3)} Y(a) + C_0^{(2)} Z(a), \\ D_1(\lambda) &= C_1^{(4)} X(a) + C_1^{(3)} Y(a) + C_1^{(2)} Z(a), \\ D_2(\lambda) &= C_2^{(4)} X(a) + C_2^{(3)} Y(a) + C_2^{(2)} Z(a), \end{aligned}$$

which we can solve via a Gauss-Jordan method.

In simpler terms, we have

$$\begin{aligned} D_0(\lambda) C_1^{(4)} - D_1(\lambda) C_0^{(4)} &= (C_0^{(3)} C_1^{(4)} - C_1^{(3)} C_0^{(4)}) Y(a) + (C_0^{(2)} C_1^{(4)} - C_1^{(2)} C_0^{(4)}) Z(a), \\ D_0(\lambda) C_2^{(4)} - D_2(\lambda) C_0^{(4)} &= (C_0^{(3)} C_2^{(4)} - C_2^{(3)} C_0^{(4)}) Y(a) + (C_0^{(2)} C_2^{(4)} - C_2^{(2)} C_0^{(4)}) Z(a), \end{aligned}$$

which we rewrite as

$$\begin{aligned} h_{0,1}(\lambda) &= f_{0,1} Y(a) + g_{0,1} Z(a), \\ h_{0,2}(\lambda) &= f_{0,2} Y(a) + g_{0,2} Z(a), \end{aligned}$$

with

$$h_{0,j}(\lambda) = D_0(\lambda)C_j^{(4)} - D_j(\lambda)C_0^{(4)},$$
$$f_{0,j} = C_0^{(3)}C_j^{(4)} - C_j^{(3)}C_0^{(4)},$$
$$g_{0,j} = C_0^{(2)}C_j^{(4)} - C_j^{(2)}C_0^{(4)},$$

for $j = 1, 2$.

We can then write

$$h_{0,1}(\lambda)f_{0,2} - h_{0,2}(\lambda)f_{0,1} = (g_{0,1}f_{0,2} - g_{0,2}f_{0,1})Z(a),$$

which allows us to determine $Z(a)$, $Y(a)$ and $X(a)$ as

$$\begin{aligned} Z(a) = Z_{0,1,2}(\lambda) &= \frac{h_{0,1}(\lambda)f_{0,2} - h_{0,2}(\lambda)f_{0,1}}{g_{0,1}f_{0,2} - g_{0,2}f_{0,1}}, \\ Y(a) = Y_{0,1,2}(\lambda) &= \frac{h_{0,1}(\lambda) - g_{0,1}Z_{0,1,2}(\lambda)}{f_{0,1}}, \\ X(a) = X_{0,1,2}(\lambda) &= \frac{D_0(\lambda) - C_0^{(3)}Y_{0,1,2}(\lambda) - C_0^{(2)}Z_{0,1,2}(\lambda)}{C_0^{(4)}}. \end{aligned}$$

Note that all the expressions in $(\lambda(K,T), \lambda(K/\alpha,T), \partial\lambda(K,T)/\partial K)$ remain *linear*.

Finally, using a fourth calendar date t_3 we can use the above solutions in

$$D_3(\lambda) = C_3^{(4)}X_{0,1,2}(\lambda) + C_3^{(3)}Y_{0,1,2}(\lambda) + C_3^{(2)}Z_{0,1,2}(\lambda),$$

which will give us a linear ODDE in $\lambda(S,t)$ similar to (4.6) except for the fact that the coefficients *do not depend on* $a(S,t)$. At this point we can apply the same techniques as in the previous sections and obtain an analytically solvable Ordinary Difference (or Differential) Equation. Note that this would involve the introduction of a fifth calendar date t_4.

Having solved for $\lambda(S,t)$ in this manner, we can then obtain $a(S,t)$ directly from (4.3).

We therefore have solved *analytically* for $\lambda(S,t)$ and $a(S,t)$ separately, using the options market data on five distinct calendar dates.

5. Puts on Stocks With Fixed Fractional Jumps

To derive a forward PIDE for puts, we can use put call parity:

$$C_0(K,T) = S_0e^{-\int_0^T q(t)\,dt} - Ke^{-\int_0^T r(t)\,dt} + P_0(K,T). \tag{5.1}$$

Differentiating w.r.t. K:

$$\frac{\partial}{\partial K}C_0(K,T) = -e^{-\int_0^T r(t)\,dt} + \frac{\partial}{\partial K}P_0(K,T). \tag{5.2}$$

Re-arranging and multiplying by K implies:

$$Ke^{-\int_0^T r(t)\,dt} = K\frac{\partial}{\partial K}P_0(K,T) - K\frac{\partial}{\partial K}C_0(K,T). \tag{5.3}$$

Substituting (5.3) in (5.1) implies:

$$S_0 e^{-\int_0^T q(t)\,dt} = C_0(K,T) - K\frac{\partial}{\partial K}C_0(K,T) - \left[P_0(K,T) - K\frac{\partial}{\partial K}P_0(K,T)\right]. \tag{5.4}$$

Differentiating (5.1) w.r.t. T:

$$\frac{\partial}{\partial T}C_0(K,T) = -q(T)S_0 e^{-\int_0^T q(t)\,dt} + r(T)Ke^{-\int_0^T r(t)\,dt} + \frac{\partial}{\partial T}P_0(K,T), \tag{5.5}$$

Substituting (5.3) and (5.4) in (5.5):

$$\begin{aligned} &\frac{\partial}{\partial T}C_0(K,T) + [r(t) - q(t)]K\frac{\partial}{\partial K}C_0(K,T) + q(t)C_0(K,T), \\ &= \frac{\partial}{\partial T}P_0(K,T) + [r(t) - q(t)]K\frac{\partial}{\partial K}P_0(K,T) + q(t)P_0(K,T). \end{aligned} \tag{5.6}$$

Differentiating (5.2) w.r.t. K:

$$\frac{\partial^2}{\partial K^2}C_0(K,T) = \frac{\partial^2}{\partial K^2}P_0(K,T). \tag{5.7}$$

The forward PIDE (4.2) can be re-written as:

$$\begin{aligned} &\frac{\partial}{\partial T}C_0(K,T) + [r(T) - q(T)]K\frac{\partial}{\partial K}C_0(K,T) + q(T)C_0(K,T) \\ &= \frac{a^2(K,T)}{2}\frac{\partial^2}{\partial K^2}C_0(K,T) + \int_K^{\frac{K}{\alpha(T)}} \lambda(L,T)[K - \alpha(T)L]\frac{\partial^2}{\partial K^2}C_0(L,T)dL. \end{aligned} \tag{5.8}$$

Substituting (5.2), (5.6) and (5.7) in (5.8): implies:

$$\begin{aligned} &\frac{\partial}{\partial T}P_0(K,T) + [r(T) - q(T)]K\frac{\partial}{\partial K}P_0(K,T) + q(T)P_0(K,T) \\ &= \frac{a^2(K,T)}{2}\frac{\partial^2}{\partial K^2}P_0(K,T) + \int_K^{\frac{K}{\alpha(T)}} \lambda(L,T)[K - \alpha(T)L]\frac{\partial^2}{\partial K^2}P_0(L,T)dL. \end{aligned} \tag{5.9}$$

Comparing (5.8) and (5.9), we see that the forward PIDE is the same for calls and puts.

The other derivations could therefore be done in a similar manner.

6. Summary and Extensions

We first derived the call forward PIDE in a Markov setting with deterministic interest rates and dividend yields. We then specialized this result to a fixed fractional jump size. We showed how the volatility function and default arrival rate function can be determined by market option prices, either singly or jointly in closed form. However we do need to use more than one calendar date for this derivation.

Empirical work using this methodology could be carried out upon Options as well as Convertible Bonds. In the interest of brevity, this work is left for future research.

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