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Valuing Bonds with Detachable Warrants

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Abstract

This paper presents analytical valuation formulas for bonds with detachable warrants on Japanese equities. The bonds bear default risk on both coupons and principal. The warrants also bear default risk, and are American-style, with a positive early exercise premium due to the existence of stock dividends.

1. INTRODUCTION

Since the first[2] Euro-bond with detachable Japanese equity warrants was issued in 1982, the market for these instruments have flourished. While the market grew steadily from 1982 to 1985, the combination of a surging Japanese equity market, low Japanese interest rates, and financial market deregulation[3] in 1986 prompted an explosion in popularity. From 1987-89, Japanese companies issued $115 billion of Euro-bonds with warrants attached, bringing the total outstanding to $140 billion.

The Euro-bonds usually mature in four to five years, and the detachable warrants give the right during the Euro-bond's life to buy new shares for slightly more than the share price just after issuance. In return for this right, the coupon rate on the Euro-bond is much lower (eg. 400-450 basis points in March 1989) than for straight Euro-dollar corporate bonds. Although the detachable warrant's strike price and underlying stock are denominated in yen, the Euro-bond is typically denominated in US dollars. The strike price per share and the number of shares controlled by each warrant are chosen so that the yen received from warrant exercise at expiration covers the dollar payout to bondholders (assuming a constant exchange rate).

A Euro-bond issued in July of 1988 by Nippon Steel is a good example of a typical "cum-warrant" bond. The issue amount is $600 million with each bond having a $5,000 face value and one warrant attached. The warrant is detachable and gives the right to purchase 980 shares of Nippon Steel at 687 yen per share at any time over the four year life of the bond. The strike price per share was chosen to be 2.5% above the share closing price of 670 yen per share on the setting date. The number of shares per warrant was chosen so that Nippon Steel's yen inflow from exercise of the warrant at expiration (980 shares x 687 yen per share) covers the dollar outflow required to retire the bond ($5,000), using the spot exchange rate on the setting date (134.70 yen per dollar). In return for attaching a warrant to each bond, the bond's coupon rate is only 3.25% per year.

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[2]A 1981 amendment to the Commercial Code allowed the issuance of Euro-bonds with warrants. The first Euro-bond with warrants on Japanese equity was issued in December 1981, but the warrants were non-detachable. In January 1982, Mitsubishi Kasei (4010) became the first Japanese company to issue a Euro-bond with detachable warrants.

[3]In 1986, The Ministry of Finance abolished restrictions on Japanese investment in the London market.

The warrant described above is attached to a U.S. dollar denominated bond. Warrants on Japanese equity have also been attached to bonds denominated in other foreign currencies. However, the U.S. dollar denominated market is the largest with over 400 issues, of which 200 trade actively (at the end of September 1988). The Swiss Franc market is the second largest with over 170 issues of which about 70 trade actively. Denominations in this market are generally smaller in size than the U.S. dollar market. The other markets are significantly smaller than the U.S. dollar and Swiss Franc denominated markets. There are about 30 issues denominated in Deutschmarks, and a few others denominated in ECU's, Dutch Gilders, Sterling, and Yen.

The four big Japanese security houses (Nomura, Daiwa, Nikko, and Yamaichi) organize almost all of the issues in the primary market (97% in 1989). Issuers typically swap their proceeds into yen immediately after issuance. The secondary market for Euro-warrants is strictly over-the-counter, dominated by foreign brokerage houses such as Baring Securities and Morgan Stanley. Since 1987, Japanese investors comprised about 70 per cent of the end investors. While London has been the traditional marketplace for these issues, the market is slowly drifting to Tokyo, largely due to efforts by the Ministry of Finance.

The popularity of Euro-bonds with detachable warrants from 1987-89 was prompted by favorable market conditions for both issuers and investors. Issuers paid coupons of less than 4%. After swapping their exposure into yen (whose interest rate was then as much as 5% below dollar rates), their cost of capital was either zero or negative. Furthermore, exercise of the warrants financed retirement of the debt, so that the only cost of the debt issue to shareholders in this event was dilution. Investors held a safe debt instrument coupled with warrants whose value soared as the Japanese equity market surged.

In 1990, these market conditions reversed, dramatically slowing down new issuance. The drop in the Tokyo stockmarket (38% as of September 1990) and the rise in Japanese interest rates to dollar levels has dramatically increased issuers' capital costs. Furthermore, the Japanese government is discouraging new issuance due to its recent concern about the warrrants' dilution effect on stock prices. For many firms, there is a high probability[4] that existing warrants will expire worthless, requiring issuers to refinance their debt at high rates. This prospect has made potential issuers wary of these instruments, and has soured investor demand for them.

In their seminal papers, Black-Scholes[2] and Merton[14] recognized that their option pricing approach could be applied in developing a unified theory of corporate liabilities. Merton[15] priced corporate discount bonds in the presence of default risk. Ingersoll[11] and Brennan and Schwartz[3] extended Merton's analysis to convertible bonds. In particular, under certain assumptions, Ingersoll proved that a non-callable coupon-bearing convertible bond has the same value as an ordinary bond with the same coupons, principal and maturity, plus an attached stock purchase warrant. This warrant has an identical maturity, is exchangeable for as many shares as is the convertible, and has a gross exercise payment equal to the face value of the bond.

Ingersoll states that the bond-warrant combination can differ in value from the con-

[4] About $125 billion of the $140 billion worth of outstanding bonds with detachable warrants were out-of-the-money as of September 1990

vertible bond only when early exercise of the warrant or conversion of the bond may be optimal. He shows that a convertible bond will never be converted prior to maturity under the three assumptions of perfect markets, constant conversion terms, and no dividends. Maintaining these three assumptions, it follows that a bond with European warrants attached may be priced as a convertible bond using the methods given in Ingersoll[11] and Brennan and Schwartz[3],[4].

The purpose of this paper is to price Euro-dollar bonds with Japanese equity warrants attached. Since the attached warrants are American-style and the underlying stock receives dividends, the value of the cum-warrant bond reflects an early exercise premium. The paper provides analytic expressions for the Euro-dollar bond and for this early exercise premium. However, the premium is expressed in terms of a quantity which must be determined numerically[5].

To lend some reality to the analysis, the bond is allowed to bear coupons. To simplify matters, the coupon payout to bondholders and the dividend payout to shareholders are both assumed to be continuous. While bond and equity payouts are discrete in reality, the effect of assuming continuous payout over long horizons is negligible. The coupon payout to bondholders is assumed to be constant over time while the dividend payout to shareholders is assumed to increase with the share price. Again, while dividend payouts tend to be sticky over short horizons in reality, the effect of this dividend assumption over long horizons is probably minimal.

To model the default risk of debt, the firm is allowed to go bankrupt either at the debt's maturity date or beforehand. The model posits that bankruptcy occurs at maturity if the market value of the firm's assets is below the debt's face value. Bankruptcy occurs prior to maturity if the firm is unable to meet its coupon payout. Since the coupon payout is continuous, the firm can always sell off assets as long as its assets have positive value. Thus, bankruptcy occurs prior to maturity when the firm's assets become worthless.

In technical terms, bankruptcy is possible when the stochastic process for the firm's asset value can reach zero in finite time. Financial considerations also require that the origin be an absorbing boundary. For the familiar geometric Brownian motion, the origin is absorbing, but not attainable. Consequently, a Feller process is employed instead. This process has linear drift and instantaneous variance proportional to the firm's value. The parameters describing the drift and diffusion coefficients are chosen so that the origin is both attainable and absorbing.

The paper presents formulas for the value of a bond with warrants attached. The model can be used to determine the coupon rate on new issues and to identify mispriced issues in the secondary market. The model can also be used by issuers and arbitrageurs to hedge the risks associated with long and short positions in these securities.

The remainder of the paper is organized as follows. Section 2 sets up notation and delineates the model. For simplicity, the model ignores exchange rate issues and assumes that the bond is denominated in yen. Section 3 studies the pricing of bonds with warrants attached under the usual assumption that the warrants are European-style. Section 4 then relaxes this assumption and determines the value of the early exercise premium.

[5]The analytic American option pricing formulas of Roll[17] and Geske and Johnson[9] are also expressed in terms of a numerically determined quantity.

The final section summarizes and provides directions for future research. An appendix provides a proof of a mathematical result used in the paper.

2. THE MODEL

The following assumption is maintained in this section and the next only:

A1) European Warrants

The warrants outstanding are European-style.

In contrast, the remaining assumptions hold throughout the paper:

A2) Frictionless Markets

There are no taxes, transactions costs, or other market frictions.

A3) Capital Structure

The firm's capital structure contains bonds and stocks only. The bonds consist of straight debt with warrants attached.

Further assumptions apply to these components of the capital structure:

A3i) Straight Debt

The firm has a single debt issue outstanding, which matures at date T. Bondholders receive coupons continuously at rate c so long as the value of the firm's assets, V_t, is positive. Letting F denote the bond's total face value, the bondholders also receive $\min(V_T, F)$ *at the maturity date T.*

In reality, several issuers have multiple debt issues with detachable warrants (eg. Nippon Steel and Yamaichi). The warrants differ in strike price and maturity. Assumption **A3i)** is made for simplicity. The payoff of the debt at maturity reflects the standard assumption that debtholders liquidate the firm at maturity if its assets do not have sufficient value to cover the principle. If the firm has gone bankrupt prior to maturity, the debtholders receive nothing at expiration.

A3ii) Warrants

There is a single warrant issue, which has a total strike price of K yen and matures with the bond at date T.

Exercise of the warrants augments the value of the firm's assets by K yen and requires the firm to issue new shares to the former warrantholders. Enough new shares are issued so that the former warrantholders own $100 \cdot \alpha$ percent of all the firm's shares. The warrants will only be exercised if the exercise benefit exceeds the exercise cost K. After the value of the firm's assets V_T is augmented by the strike price, the total equity value is $\max[0, V_T + K - F]$. Since the former warrantholders own $100 \cdot \alpha$ percent of this new firm value, their exercise benefit is $\max[0, \alpha(V_T - F + K)]$. Requiring this exercise benefit to exceed the exercise cost K implies that:

$$\alpha V_T > \alpha F + (1 - \alpha)K \equiv X \tag{1}$$

If the warrants are exercised rationally, the warrantholders receive the excess of the exercise benefit over the cost i.e. $\alpha V_T - X$. Conversely, if the warrants expire rationally unexercised, they are worthless. Thus, the terminal value of the warrants is $\max[0, \alpha V_T - X]$.

A3iii) Equity

The stockholders receive dividends continuously[6] *at rate* $a \cdot V_t$. *If the warrants are not exercised at expiration, the shareholders also get* $\max[0, V_T - F]$ *at* T. *If the warrants are (rationally) exercised, they get the smaller amount* $(1-\alpha)(V_T + K - F)$ *instead.*

The bond value B_t has three sources of value, namely, principal P_t, coupons C_t, and European warrants W_t^e:

$$B_t \equiv P_t + C_t + W_t^e \tag{2}$$

In order to price these claims, the following three assumptions are appended:

A4) Continuous Trading

Investors can trade in the firm's assets and liabilities continuously.

A5) Constant Interest Rate

The riskless rate of interest is a positive constant r.

A6) Firm Value Dynamics

The value of the firm's assets V_t *obeys the following diffusion process:*

$$dV_t = \mu(V_t, t)dt + \sigma(V_t, t)dZ_t, t \in [0, T]. \tag{3}$$

The growth rate $\mu(V_t, t)$ *and the instantaneous volatility* $\sigma(V_t, t)$ *are both assumed to be bounded. The Wiener process* $\{W_t; t \in [0, T]\}$ *is defined on the probability space* $(\Omega, \mathcal{F}, Q)$.

Under these assumptions, the preclusion of arbitrage opportunities implies the existence of a probability measure $\hat{Q}$, equivalent to Q, validating risk-neutral pricing (see Harrison and Pliska[10]). Let $\hat{E}_v$ denote expectations under this measure given that the initial firm value is V_0. Then the claims comprising the bond value have the following representation.

$$P_0 = e^{-rT} \hat{E}_v \min(V_T, F) \tag{4}$$

$$C_0 = \hat{E}_v \int_0^T c e^{-rt} 1_{\{V_t > 0\}} dt \tag{5}$$

$$W_0^e = e^{-rT} \hat{E}_v \max(0, \alpha V_T - X), \text{ where } X \equiv \alpha F + (1-\alpha)K. \tag{6}$$

To value these claims, one final assumption is added:

A7) Feller Process

The value of the firm's assets obeys a Feller process:

$$dV_t = [(\mu - a)V_t - c]dt + \sigma\sqrt{V_t}dZ_t, \tag{7}$$

where $\mu, a, c,$ *and* σ *are constants*[7], *with* $a, c, \sigma > 0$.

Since $c > 0$, Feller (1951) has shown that this process is absorbed at the origin. Note that under the "risk-neutral" probability measure $\hat{Q}$, one can define a Wiener process $\hat{Z}_t$ such that the following equation holds:

[6] It is possible to model the stockholders' dividend as $c_t^s + a_t^s \cdot V_t$, where c_t^s and a_t^s are non-negative functions of time. In this case, the bondholders' coupon payments would be modeled as $c_t^b + a_t^b \cdot V_t$, where $c_t^b \equiv c - c_t^s$ and $a_t^b \equiv a - a_t^s$.

[7] For what follows, the expected rate of return in the firm's assets μ can be an unknown bounded function of V and t.

$$dV_t = [(r - a)V_t - c]dt + \sigma\sqrt{V_t}d\tilde{Z}_t. \tag{8}$$

This equation is the starting point for the analysis in the next section.

3. EUROPEAN WARRANTS

This section is concerned with pricing bonds with detachable European warrants. The bond value B_0 has three components, the first of which is the principal P_0. From (4):

$$\begin{aligned} P_0 &= e^{-rT}\tilde{E}_0 \min(V_T, F) \\ &= \int_0^F e^{-rT} V_T f(V_T, T; V_0, 0) dV_T + Fe^{-rT}\int_F^\infty f(V_T, T; V_0, 0) dV_T, \end{aligned} \tag{9}$$

where $f(V_T, T; V_0, 0)$ is the transition density of the risk-neutral Feller process (8), which is given in the Appendix. In words, the bond's principal is the present value of the partial mean of the terminal firm value, V_T, summed with the product of the default-free price, Fe^{-rT}, and the complementary distribution function of V_T.

Cox and Ross[6] determine the partial mean as:

$$\begin{aligned} M(F,T) &\equiv \int_0^F e^{-rT} V_T f(V_T, T; V_0, 0) dV_T \\ &= V_0 e^{-aT} \sum_{n=0}^{\infty} \frac{(n+1)e^{-\lambda_T/2}(\lambda_T/2)^{n+2c/\sigma^2}\Gamma(k_T F, n+2)}{\Gamma(n+2+2c/\sigma^2)}, \end{aligned} \tag{10}$$

where $\lambda_T \equiv 2k_T V_0 e^{(r-a)T}, k_T \equiv \frac{2(r-a)}{\sigma^2(e^{(r-a)T}-1)}$, and where $\Gamma(k_T F; n+2)$ is the gamma distribution function evaluated at $k_T F$ with parameter $n+2$

$$\Gamma(z;\alpha) \equiv \frac{1}{\Gamma(\alpha)}\int_0^z e^{-x}x^{\alpha-1}dx, \tag{11}$$

with $\Gamma(\alpha)$ as the gamma function $\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx$.

It is shown in the Appendix that the complementary distribution function of the random variable V_T can be transformed into the non-central chi-squared distribution function:

$$\int_F^\infty f(V_T, T; V_0, 0)dV_T = \chi^2(\lambda_T; 2(1+2c/\sigma^2), 2k_T F), \tag{12}$$

where $\chi^2(\lambda_T; \nu, 2k_T F)$ is the non-central chi-square distribution function (see Johnson and Kotz[13], p. 133) evaluated at λ_T with $2(1+2c/\sigma^2)$ degrees of freedom and non-centrality parameter $2k_T F$. If $Z_1, \ldots, Z_n$ are independent standard normal random variables and $\delta_1, ..., \delta_n$ are constants, then the sum $\sum_{j=1}^{\nu}(Z_i + \delta_i)$ has a non-central chi-square distribution with ν degrees of freedom and non-centrality parameter $\sum_{j=1}^{\nu}\delta_i^2$. The right side of (12) is the probability that this sum is less than the number $\lambda_T \equiv 2k_T V_0 e^{(r-a)T}$, when the sum has $2(1+2c/\sigma^2)$ degrees of freedom and its non-centrality parameter is $2k_T F$. This is also the probability that the firm is solvent at expiration, under the process (8).

The non-central chi-square distribution has widespread application, including coverage problems in ballistics and estimating the power of the chi-square test. Algorithms for exact valuation and analytic approximation formulas for the distribution function are given in Schroder[19] and Sankaran[18] respectively.

Putting these results together gives the valuation formula for the principal component of the bond's value:

$$P_0 = V_0 e^{-aT} M(F,T) + F e^{-rT} \chi^2(\lambda_T; 2(1+2c/\sigma^2), 2k_T F), \tag{13}$$

where $k_T \equiv \frac{2(r-a)}{\sigma^2(e^{(r-a)T}-1)}$ and $\lambda_T \equiv 2k_T V_0 e^{(r-a)T}$.

The second component of bond value is the coupon stream c received so long as firm value is positive, i.e., from (5):

$$\begin{aligned} C_0 &= \tilde{E}_v \int_0^T c e^{-rt} 1_{\{V_t>0\}} dt \\ &= c \int_0^T e^{-rt} \tilde{Q}(\{\omega : V_t(\omega) > 0\}) dt, \end{aligned} \tag{14}$$

where $\tilde{Q}$ is the equivalent martingale measure. Feller[7] gives the probability that the process (8) has not been absorbed by date t so that:

$$C_0 = c \int_0^T e^{-rt} \Gamma(\lambda_t; 1 + 2c/\sigma^2) dt. \tag{15}$$

The final component of bond value is the European warrant. From (6):

$$W_0^e = e^{-rT} \tilde{E}_v \max(0, \alpha V_T - X), \tag{16}$$

where $X \equiv \alpha F + (1-\alpha)K$. Using the transition probability density function:

$$W_0^e = \alpha \int_X^\infty e^{-rT} V_T f(V_T, T; V_0, 0) dV_T - X e^{-rT} \int_X^\infty f(V_T, T; V_0, 0) dV_T. \tag{17}$$

From (10) and (12):

$$W_0^e = \alpha V_0 e^{-aT}[1 - M(X,T)] - X e^{-rT} \chi^2(\lambda_T; 2(1+2c/\sigma^2), 2k_T X). \tag{18}$$

Note that this equation is roughly similar to the Black-Scholes formula.

Summing the three components of bond value together gives the following valuation result:

$$\begin{aligned} B_0(c) &= P_0 + C_0 + W_0^e \\ &= V_0 e^{-aT} M(F,T) + F e^{-rT} \chi^2(\lambda_T; 2(1+2c/\sigma^2), 2k_T F) \\ &\quad + c \int_0^T e^{-rt} \Gamma(\lambda_t; 1 + 2c/\sigma^2) dt \\ &\quad + \alpha V_0 e^{-aT}[1 - M(X,T)] - X e^{-rT} \chi^2(\lambda_T; 2(1+2c/\sigma^2), 2k_T X), \end{aligned} \tag{19}$$

where $k_T \equiv \frac{2(r-a)}{\sigma^2(e^{(r-a)T}-1)}$, $\lambda_T \equiv 2k_T V_0 e^{(r-a)T}$, and $X \equiv \alpha F + (1-\alpha)K$.

Bonds are usually issued at par. To determine the coupon rate which prices the bond at par, set the initial bond value equal to its face value ($B_0(c) = F$) and solve for c

g a numerical search routine. Since the bond price is increasi g in the coupon rate $\frac{(c)}{}$ > 0) and warrant value is strictly positive, the inclusion o warrants lowers the on rate required to price bonds at par. To value the equity, one need merely subtract Merton[15] proves that the Modigliani-Miller[16] Theorem hol s in this framework. sequently, the equity is valued by subtracting the initial bond price from the initial value:

$$S_0 = V_0 - B_0. \tag{20}$$

AMERICAN WARRANTS

he warrants are now assumed to be American rather than European. Since the holders receive proportional dividends, the ability to exerci e early has positive . This section determines the magnitude of this early exe cise premium, using ts in Carr, Jarrow, and Myneni[5] and Jamshidian[12]. For sin plicity, it is assumed if the warrants are exercised, they are exercised as a block nd that the exercise eeds are paid out as a special dividend. Upon exercise, the fo mer warrant-holders $100 \cdot \alpha$ percent of the firm's shares, so they receive $100 \cdot \alpha$ p rcent of this special end, as before.

rom previous work on American option valuation, there exist an optimal exercise dary B_t^*, independent of V_t, above which it is optimal to exer ise early, i.e., letting lenote the American warrant value:

$$\begin{aligned} &\text{if } V_t \geq B_t^*, \quad \text{then} \quad W_t^a = \max[0, \alpha[V_t - P_t - C_t] - (1-\alpha)K] \\ &\text{if } V_t < B_t^*, \quad \text{then} \quad W_t^a > \max[0, \alpha[V_t - P_t - C_t] - (1-\alpha)K] \end{aligned} \tag{21}$$

the American warrant value is given by (21) if the firm value tarts at or above the al exercise boundary, assume henceforth that the warrant is initially alive on the tion date 0, i.e., $V_0 < B_0^*$.

he intrinsic value of the warrant is:

$$\begin{aligned} I(V,t) &\equiv \max[0, \alpha[V - P(V,t) - C(V,t)] - (1-\alpha)K] \\ &= \max[0, \alpha V - X(V,t)], \end{aligned} \tag{22}$$

$X(V,t) \equiv \alpha[P(V,t) + C(V,t)] + (1-\alpha)K$. Thus, the warra t may be viewed as written on $100 \cdot \alpha$ percent of the firm's assets, with exercise p ice X depending on rm value V and time t. Define B_t^x as the value of the firm sol ing:

$$\alpha B_t^x = X(B_t^x, t) \text{ or } \alpha[B_t^x - P(B_t^x, t) - C(B_t^x, t)] = (1-\alpha)K. \tag{23}$$

$V_t > B_t^x$, exercise of the warrants has positive value and the warrants are in-the- y. Since the warrant holder is never forced to exercise:

$$B_t^* \geq B_t^x, t \in [0, T]. \tag{24}$$

time evolves, the straight debt value, $P(V,t) + C(V,t)$, incr ases, *ceteris paribus*. t B_t^x must increase to maintain equality in (23). As time evol es, the gap between d B_t^x narrows. At expiration, equation (23) is:

$$\alpha B_T^\pi = \alpha \min(B_T^\pi, F) + (1-\alpha)K. \tag{25}$$

If $B_T^\pi < F$, a contradiction arises so:

$$B_T^\pi = F + \frac{1-\alpha}{\alpha}K. \tag{26}$$

Applying the results of Van Moerbeke[20] shows that $B_T^* = B_T^\pi$, if $a \geq r$. However, if $0 < a < r$, then $B_T^* = \frac{r}{a}B_T^\pi > B_T^\pi$, reflecting the decreased value of early exercise.

While alive, the warrant value satisfies the fundamental partial differential equation:

$$\mathcal{L}W^a(V,t) \equiv \frac{\partial W^a}{\partial t} + \frac{\sigma^2 V}{2}\frac{\partial^2 W^a}{\partial V^2} + [(r-a)V - c]\frac{\partial W^a}{\partial V} - rW^a = 0 \tag{27}$$

To gain an understanding of this equation, define the discounted process $Y(V,t) \equiv e^{-rt}W^a(V,t)$. Recall the "risk-neutral" process (8):

$$dV_t = [(r-a)V_t - c]dt + \sigma\sqrt{V_t}d\hat{Z}_t. \tag{28}$$

From Itô's lemma:

$$dY_t = e^{-rt}\mathcal{L}W^a(V,t)dt + \frac{\partial W^a}{\partial V}\sigma\sqrt{V_t}d\hat{Z}_t. \tag{29}$$

Thus, the fundamental partial differential equation (27) asserts that a necessary condition for the preclusion of arbitrage opportunities is that the discounted warrant process Y_t is a (local) martingale.

In the exercise region, (29) still obtains. However, the drift need not vanish since the warrant, if it exists, should be exercised, not held. In the exercise region, $V_t > B_t^* > B_t^\pi$ from (24). Consequently:

$$W_t^a = \alpha[V - P(V,t) - C(V,t)] - (1-\alpha)K \text{ from (21)}. \tag{30}$$

In the exercise region, the warrant price is the sum of the values of two claims. The value of the first claim is $100 \cdot \alpha$ percent of the firm's assets net of the straight debt obligation. The second claim is a short position in a bond paying interest continuously so that its value is constant over time at $(1-\alpha)K$.

Since $\mathcal{L}$ is a linear operator

$$\begin{aligned}\mathcal{L}W_t^a &= \alpha[\mathcal{L}V - \mathcal{L}P(V,t) - \mathcal{L}C(V,t)] - (1-\alpha)\mathcal{L}K \\ &= \alpha[-aV - c - 0 - (-c) - (1-\alpha)rK \\ &= \alpha aV - (1-\alpha)rK\end{aligned} \tag{31}$$

from Merton[15].

Thus on the whole region, the warrant value $W^a(V,t)$ satisfies:

$$\mathcal{L}W_t^a = 1_{\{V_t > B_t^*\}}[\alpha a V_t - (1-\alpha)rK] \tag{32}$$

ect to the following boundary conditions:

$$W^a(V,t) = \max[0, \alpha \max[0, V_T - F] - (1-\alpha)K] \tag{33}$$

$$W^a(B_t^*,t) = \alpha B_t^* - X(B_t^*,t) \tag{34}$$

$$\frac{\partial W^a(B_t^*,t)}{\partial V} = \alpha - \frac{\partial X(B_t^*,t)}{\partial V} \tag{35}$$

$$\lim_{V \downarrow 0} W(V,t) = 0. \tag{36}$$

[T]he first boundary condition reflects the fact that the America[n] warrant is European [at e]xpiration. The next two boundary conditions reflect the result t[h]at along the optimal [exer]cise boundary, the warrant value and its derivative are contin[u]ous. The final condi[tion] serves to uniquely determine the warrant value $W^a(V,t)$ and the optimal exercise [bou]ndary B_t^*.

[U]sing the Feynman-Kac Theorem (see Friedman[8]):

$$[W]^a(V,t) = e^{-r(T-t)}\tilde{E}\max[\alpha V - X] + \tilde{E}\int_t^T e^{-r(u-t)} 1_{\{V_u > B_u^*\}}[\alpha a V_u - (1-\alpha)rK]du.$$

[The] first term is the value of the European warrant. Therefore:

$$W_0^a = W_0^e + \pi_0, \tag{37}$$

[wher]e π_0 is the early exercise premium given by:

$$[\pi_0] \equiv \alpha a V_0 \int_0^T e^{-at}[1 - M(B_t^*,t)]dt - (1-\alpha)rK \int_0^T e^{-rt}\chi^2(\lambda_t; 2(1+2c/\sigma^2), 2k_t B_t^*)dt,$$

where $k_t \equiv \frac{2(r-a)}{\sigma^2(e^{(r-a)t}-1)}$, $\lambda_t \equiv 2k_t V_0 e^{(r-a)t}$.

[T]he early exercise premium is non-negative. It has the same v[a]lue as a claim which [pays] $\alpha a V_t - (1-\alpha)rK$ continuously whenever the value of the firm V_t exceeds the optimal [exer]cise boundary B_t^*. The value of this boundary can be determin[e]d implicitly from the [value] matching condition (34):

$$W^a(B_t^*,t) = \alpha B_t^* - X(B_t^*,t), \tag{38}$$

[and fr]om the high contact condition (35):

$$\frac{\partial W^a(B_t^*,t)}{\partial V} = \alpha - \frac{\partial X(B_t^*,t)}{\partial V}. \tag{39}$$

[Eithe]r integral equation can be solved numerically.

[5. S]UMMARY AND FUTURE RESEARCH

[T]his paper valued bonds with detachable warrants. The bond[s] bear default risk on [the] coupons and principal. The warrants are American with a [p]ositive early exercise [prem]ium due to the existence of stock dividends. A major aven[u]e for future research [woul]d incorporate exchange rate risk into the bond's price while retaining the risk of default. A second avenue for future research would allow for stochastic interest rates while retaining the American feature of the warrants. A third research direction involves the imposition of an exogenous stochastic process on stock prices rather than firm values. In such a model, the effect of future warrant exercise on current stock prices should be explicitly taken into account.

APPENDIX

Theorem 1 *Let $f(V_T, T; V_0, 0)$ denote the transition density of the Feller process $dV_t = [(r-a)V_t - c]dt + \sigma\sqrt{V_t}d\bar{Z}_t$. Let $\chi^2(x;\nu,\lambda)$ denote the non-central chi-square distribution function with ν degrees of freedom and noncentrality parameter λ. Then:*

$$\int_F^\infty f(V_T, T; V_0, 0)dV_T = \chi^2(\lambda_T\ ; 2(1 + 2c/\sigma^2), 2k_T F), \tag{40}$$

where $k_T \equiv \frac{2(r-a)}{\sigma^2(e^{(r-a)T}-1)}$ and $\lambda_T \equiv 2k_T V_0 e^{(r-a)T}$.

Proof

Let:

$$I \equiv \int_F^\infty f(V_T, T; V_0, 0)dV_T, \tag{41}$$

where from Cox and Ross ([6], p. 161, equation 35):

$$\begin{aligned} f(V_T, T; V_0, 0) &= \left(\frac{2(r-a)}{\sigma^2(e^{(r-a)T}-1)}\right)\left(\frac{V_0 e^{(r-a)T}}{V_T}\right)^{(1+2c/\sigma^2)/2} \\ &\cdot \exp\left[-\frac{2(r-a)(V_0 e^{(r-a)T} + V_T)}{\sigma^2(e^{(r-a)T}-1)}\right] \\ &\cdot I_{1+2c/\sigma^2}\left[\frac{4(r-a)\sqrt{V_T V_0 e^{(r-a)T}}}{\sigma^2(e^{(r-a)T}-1)}\right], \end{aligned} \tag{42}$$

and where $I_q(z) \equiv \left(\frac{z}{2}\right)^q \sum_{j=0}^\infty \frac{(z^2/4)^j}{j!\Gamma(q+j+1)}$ is the modified Bessel function of the first kind of order q (see Abramowitz and Stegun ([1], p. 375, equation 9.6.10).

Performing the change of variables:

$$x \equiv 2k_T V_T \text{ where } k_T \equiv \frac{2(r-a)}{\sigma^2(e^{(r-a)T}-1)} \tag{43}$$

in (41) yields:

$$\begin{aligned} I &= \int_{2k_T F}^\infty \frac{1}{2}\left(\frac{2k_T V_0 e^{(r-a)T}}{x}\right)^{(1+2c/\sigma^2)/2} \\ &\cdot \exp[-(2k_T V_0 e^{(r-a)T} + x)/2] I_{1+2c/\sigma^2}(\sqrt{2k_T V_0 e^{(r-a)T} x})dx. \end{aligned} \tag{44}$$

Letting:

$$\lambda_T \equiv 2k_T V_0 e^{(r-a)T} \text{ and } \nu \equiv 2(1 + 2c/\sigma^2) \tag{45}$$

simplifies the result to:

$$I = \int_{2k_T F}^{\infty} \frac{1}{2} \left(\frac{\lambda_T}{x}\right)^{\nu/4} I_{\nu/2}(\sqrt{\lambda_T x}) \exp[-(\lambda_T + x)/2] dx. \tag{46}$$

From Johnson and Kotz[13], the integrand is recognized as the probability density function $p(\lambda_T; \nu+2, x)$ of a non-central chi-squared random variable, with $\nu+2$ degrees of freedom and non-centrality parameter x. Schroder, ([19], p. 213), proves that:

$$\int_{2k_T F}^{\infty} p(\lambda_T; \nu+2, x) dx = \chi^2(\lambda_T; \nu, 2k_T F), \tag{47}$$

where $\chi^2(\lambda_T; \nu, 2k_T F)$ is the non-central chi-squared distribution function:

$$\begin{aligned} \chi^2(\lambda_T; \nu, 2k_T F) &\equiv \int_0^{\lambda_T} p(w; \nu, 2k_T F) dw \\ &= \int_0^{\lambda_T} \frac{1}{2} \left(\frac{w}{2k_T F}\right)^{(\nu-2)/4} I_{\frac{\nu-2}{2}}(\sqrt{2k_T F w}) \exp\left[-\frac{2k_T F + w}{2}\right] dw. \end{aligned}$$

Q.E.D.

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