



Volatility Derivatives

Peter Carr

Bloomberg/NYU, New York, NY 10022; email: pcarr@nyc.rr.com

Roger Lee

Department of Mathematics, University of Chicago, Chicago, Illinois 60637;
email: RL@math.uchicago.edu

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Abstract

Volatility derivatives are a class of derivative securities where the payoff explicitly depends on some measure of the volatility of an underlying asset. Prominent examples of these derivatives include variance swaps and VIX futures and options. We provide an overview of the current market for these derivatives. We also survey the early literature on the subject. Finally, we provide relatively simple proofs of some fundamental results related to variance swaps and volatility swaps.

1. INTRODUCTION

At its core, the study of finance is fundamentally about the trade-off between risk and expected return. Various measures have been proposed to operationalize the risk component of this trade-off, but since the middle of the last century, the standard deviation of an asset's return has undoubtedly been the most commonly used measure of risk. The term volatility is a well-known shorthand for this standard deviation, with the slang term “vol” favored by harried practitioners.

While asset volatilities are an important input into portfolio theory, they are of even greater significance for derivatives pricing. It is common to hear of hedge funds engaged in volatility trading or to hear of strategists conceptualizing volatility as an asset class. The actual assets in this class comprise any derivative security whose theoretical value is affected by some measure of the volatility of the underlying asset. By market convention, theoretical value refers to a standard option valuation model, such as Black Scholes (BS). Hence, classical examples of such assets would include options but not futures or forward contracts.

Since the mid-1990s, a subset of derivative securities has arisen that place even greater emphasis on volatility. These contingent claims elevate volatility from its traditional role in abetting pricing of nonlinear payoffs to an even more significant role in determining the payoff of the derivative security. These types of derivative securities are generically referred to as volatility derivatives. Prominent examples of these derivatives include variance swaps and futures or options written on a volatility index known as VIX. They would not include standard options because the payoffs of these securities do not depend on volatility, even though the values do.

In this review, we provide an up-to-date description of the market for volatility derivatives. We show that the volatility measure used to define the payoff can be either the implied volatility derived from option prices or the realized volatility defined from prices of some underlying asset. We also show that these derivatives trade both over-the-counter (OTC) and on organized exchanges.

An overview of this article is as follows. In the next section, we provide a history of the volatility derivatives market. We describe the nature of the contracts presently trading, giving an idea of their relative popularity. The following section surveys the early work on the pricing theory, detailing contributions from both academics and practitioners. We close with three technical sections, which present relatively simple proofs of fundamental results on variance swaps and volatility swaps. The first result concerns the order of the error when attempting to replicate a discretely monitored variance swap by delta hedging a log contract. We show that the leading term in the replication error is third order in the daily return. The second result relates the theoretical value of the floating leg of a continuously monitored variance swap to a weighted average of the total implied variances across a standard measure of moneyness. This theoretical value is based on the pricing model that the Chicago Board Options Exchange (CBOE) uses implicitly in its present construction of its volatility index VIX. The final result shows that the volatility swap rate is well approximated by the at-the-money-forward implied volatility in stochastic volatility models with an independence condition. The final section summarizes the paper and suggests some open problems. An extensive bibliography is provided.

2. HISTORY OF VOLATILITY DERIVATIVES MARKET

The first volatility derivatives appeared in the OTC market in the early part of the past decade. The first contracts to enjoy any liquidity were variance swaps. Like any swap, a

variance swap is an OTC contract with zero upfront premium. In contrast to most swaps, a variance swap has a payment only at expiration. At this expiry date, the long side of the variance swap pays a positive dollar amount, which was agreed upon at inception. In return for this fixed payment, the long side receives a positive dollar amount at expiry, called the realized variance of the underlying index. The realized variance is an annualized average of the squared daily returns. Some details of the realized variance calculation will be given in Section 4.

According to Michael Weber, now with J.P. Morgan, the first volatility derivative appears to have been a variance swap dealt in 1993 by him at the Union Bank of Switzerland (UBS). As both an at-the-money-forward (ATMF) straddle and a variance swap are thought by practitioners to have sizeable vega but little delta, UBS initially quoted the variance swap rate at the ATMF implied volatility, less one volatility point for safety. They later valued the variance swap using the method of Neuberger (1990a). Weber recalls that UBS bought one million pounds per volatility point on the FTSE 100. The variance swap rate was quoted at a volatility of 15%, with a cap quoted at a volatility of 23% (so UBS also dealt the first option on realized variance as well). The motivation for the trade was that the UBS book was short many millions of vega at the five-year time horizon and thus the trade lessened this exposure.

Between 1993 and 1998, there are reports of sporadic trading in both variance and volatility swaps. As its name suggests, a volatility swap payoff is linear in realized volatility. Practitioners preferred thinking in terms of volatility, familiar from the notion of implied volatility, rather than variance, and this created a demand for volatility swaps. For example, an article in *Derivatives Strategy* (1998) describes volatility swaps issued by Salomon Smith Barney. However, by 1998, published literature had already suggested that variance swaps enjoyed robust hedges, whereas volatility swaps had no such hedge.

By 1998, variance swaps on stock indices started to take off for this latter reason. To satisfy the desire to communicate in volatility terms, the standard convention quotes the variance swap rate as an annualized volatility. Hence, term sheets routinely describe the floating leg of a variance swap as the square of a realized volatility, with the latter volatility computed separately as the square root of a sum. The typical sampling frequency is daily on close, although weekly variance swaps also trade.

The emergence of variance swaps in 1998 is probably due to the historically high implied volatilities experienced in that year. Hedge funds found it attractive *ex ante* to sell realized variance at rates that exceeded by wide margins the econometric forecasts of future realized variance based on time series analysis of returns on the underlying index.

Meanwhile, the banks on the other side buying realized variance at the historically high rate knew from practitioner research that most of the market risk of the variance swap could be hedged by selling a strip of coterminous options and delta-hedging them. So long as the premium earned on the option sales more than compensated for the historically high fixed payment, banks were more than happy to satisfy their hedge fund customers. In other words, banks were engaging in classical cross-market arbitrage, buying realized variance at a low rate through variance swaps and simultaneously selling it at a high rate through delta-hedged options.

At this point, one may wonder why the hedge funds engaged in market timing did not also try to sell realized variance through delta-hedged options. The surprisingly simple answer is that at the time, most hedge funds did not have the infrastructure necessary to delta-hedge even a single option, much less the strip of options needed to theoretically

replicate realized variance. Rather than invest in this infrastructure, hedge funds found it far easier to pay a fee to banks to provide payoffs, such as realized variance, whose time series properties were ostensibly well understood.

While some well-known hedge funds lost money betting in 1998 that volatility would realize below variance swap rates, there was still much support until 2008 for the basic idea that selling realized variance has positive alpha. Institutions with long time horizons, such as insurers and pensions funds, stepped in alongside hedge funds, providing liquidity to the market in this way. Even today, the actuarial and financial sectors have disagreements on valuation of long-dated derivatives. The actuarial view favors the use of law of large numbers, whereas the financial sector favors the application of no-arbitrage principles. The two model values can differ substantially for long-dated derivatives, and this difference has provided the motivation for many trades.

After the successful rollout of variance swaps on indices, the next natural step was to introduce variance swaps on individual stocks. However, a convention in the way that returns are computed on variance swaps required caps in the contractual payoff. The standard convention computes the return as a log price relative. Section 4 offers a possible explanation for the use of log price relatives rather than percentage returns; but for now, we simply note that the contractual payoffs that appear in thousands of term sheets become literally infinite if the underlying ever closes at zero. Although indices are thought to have no chance of vanishing over a variance swap's life, the experience with individual stocks has differed. As a result, another convention that became standard for single-name variance swaps caps the realized volatility at 2.5 times the variance swap rate. After the market meltdown of 2008, this convention also became standard for index variance swaps as well.

In 2005, derivatives houses introduced several innovations in contracts on the realized variance of equity indices. For example, options on realized variance began trading outright, rather than just as a cap on the variance swap. Also, conditional and corridor variance swaps were introduced for indices. The floating leg of these contracts pay the variance realized only during times when the underlying is in a specified corridor. For example, a hedge fund manager who thought the index options skew was too negative could sell downside variance and buy upside variance. Alternatively, a so-called gamma swap weighted each squared return by the gross return on the index since inception. As a result, squared returns realized on days when the market opened below its level at inception received less weight than squared returns realized on days when the market opened above its level at inception.

In 2007, timer puts and calls were introduced. These options deliver the usual hockey stick payoff, but the maturity date is a random stopping time. The stopping time is the first day that the cumulative sum of squared returns surpasses a positive barrier set at inception. The barrier is calculated from an investment horizon, say 3 months, and an annualized volatility, say 15%, which is referred to as a volatility budget. These timer options are also known as mileage options.

Options exchanges reacted to The above buildup of activity in OTC volatility derivatives. In 1993, the Chicago Board Options Exchange (CBOE) introduced its first Volatility Index, known as the VIX, based on the BS option pricing model. Specifically, the VIX construction averaged together several implied volatilities from short-term near-the-money OEX (S&P 100) puts and calls.

Although the CBOE did not introduce VIX derivatives in the first decade after the VIX launch, other exchanges listed some volatility contracts. For example, the OMLX

(the London-based subsidiary of the Swedish exchange OM) launched volatility futures at the beginning of 1997. In 1998, the Deutsche Terminborse (DTB) launched its own futures based on its already established implied volatility index. Suffering a fate shared by many financial innovators, the volume in these contracts failed to materialize and these contracts are now relegated to the dustbins of history.

In 2003, the CBOE introduced the Chicago Futures Exchange (CFE), whose sole purpose was to provide exchange-traded volatility derivatives. The CBOE accomplished this by revising the definition of the VIX in three important ways. First, the underlying changed from the OEX 100 to the S&P 500, a broader and more widely followed index. Second, the annualization changed from calendar days to business days. Third, and of greatest importance from a theoretical perspective, the new construction did not rely on the BS model with its troubling assumption of deterministic volatility. Instead, it relied on a more robust theory for pricing (continuously monitored) variance swap, which almost all dealers have used since their conception to determine profit and loss (P&L) on their discretely monitored variance swap positions. In stark contrast to the BS theory, which could be calibrated to the market price of a single option, this theory required market prices of all out-of-the-money (OTM) coterminous options, whether these options existed on the market or not. In practice, dealers and the CBOE used all listed strikes subject to some liquidity cutoff. A CBOE white paper detailed the weights attached to market prices (CBOE 2003).

The financial press, which has covered the VIX since its inception, often refer to it as the “fear gauge.” If the fear is of variance higher than expected, then the name clearly fits. However, if the fear is of returns realizing lower than expected, then the aptness of the nickname is not immediately apparent, because both the present VIX and the former VIX are constructed from OTM puts and OTM calls, with at-the-money (ATM) defined as the strike that equates call value to put value. According to the weighting scheme used in the CBOE white paper, the price of an OTM put receives more weight than the price of an equally OTM call, even if moneyness is measured by the log of the strike to the futures. This would seem to explain the fear gauge moniker except that in the standard Black model, an OTM put is cheaper than an equally OTM call. However, a detailed calculation shows that if the Black model is holding and a continuum of strikes are available, then more dollars are invested in OTM puts than in OTM calls when replicating a continuously monitored variance swap (with no cap). This calculation also shows that this bias toward OTM puts is purely due to the convention that the ATM strike is the forward price. If the ATM strike is redefined to be the barrier that equates the Black model theoretical values of an up variance swap to a down variance swap, then OTM puts cost as much in the Black model as OTM calls. By this metric, positive and negative returns on the underlying are weighted equally in the Black model. The empirical reality that an OTM put typically has a higher implied volatility than an equally OTM call motivates that the fear in the term “fear gauge” is of returns realizing below the benchmark set by the alternative ATM strike.

Accompanying the introduction of the new VIX, the launch of VIX futures enjoyed mild success. At about the same time, the CBOE introduced realized variance futures, but these did not succeed in taking away any liquidity away from variance swap. In 2005, Eurex also launched futures on three new volatility indices, the VSTOXX, the VDAX-New, and the VSMI. All of these contracts employ the same design. The underlying cash volatility index comes from market prices of index options of two different maturity dates. To obtain a constant 30 day horizon, the weights on the option prices adjust about once

every dozen seconds while the markets are open. The futures expire on the one day each month when only one option contract has positive weight. By 2009, only the VSTOXX futures survived. In May of 2009, this futures contract was revised to carry a lower multiplier.

Motivated by the success of VIX futures, the CFE eventually introduced options on the cash VIX. These European-style options cash-settle at their expiry. Like the futures, the VIX options mature on the one day each month when only a single maturity is used to compute the cash VIX. After the SPX index options, these VIX options are the CBOE's most liquid option contract. Their popularity stems from the well-known negative correlation between VIX and SPX. Calls on VIX are often compared with puts on SPX. Standard models are presently being developed to integrate the two markets. Calls on VIX have also been compared with calls on credit indices. Models based on a common time change or factor models can be used to integrate these two markets. In May of 2009, Eurex announced that it may list an option on VSTOXX, depending on market demand.

The cataclysm that hit almost all financial markets in 2008 had particularly pronounced effects on volatility derivatives. Historically profitable strategies based on the naked selling of realized variance suffered huge losses in the final quarter of 2008. Dealers learned the hard way that the standard theory for pricing and hedging variance swaps is not nearly as model-free as previously regarded, even in respectable academic publications (see Jiang & Tian 2005b). In particular, sharp moves in the underlying highlighted exposures to cubed and higher-order daily returns. The inability to take positions in deep OTM options when hedging a variance swap later affected the efficacy of the hedging strategy. As the underlying index or stock moved away from its initial level, dealers found themselves exposed to much more vega than a complete hedging strategy would permit. This issue was particularly acute for single names, as the options are not as liquid and the most extreme moves are bigger. As a result, the market for single-name variance swap has evaporated in 2009. As already indicated, the caps previously reserved for single-name variance swap are now in place for index variance swap, fueling the need for robust theories that capture their effect on pricing.

3. EARLY LITERATURE ON VOLATILITY DERIVATIVES

Preceding the development of the VIX, a small but prescient literature advocated the development of volatility indices and the listing of financial products whose payoff is tied to these indices. For example, Gastineau (1977), and Galai (1979) proposed option indices similar in concept to stock indices. Brenner & Galai (1989) proposed realized volatility indices and futures and options contracts on these indices. Fleming et al. (1993) described the construction of the original VIX, while Whaley (1993) proposed derivative contracts written on this index.

Following the success of stochastic volatility extensions of the BS model for pricing vanilla options, the early attempts at pricing volatility derivatives proposed a parametric process for the volatility of the underlying asset. In particular, Brenner & Galai (1993, 1996) develop a valuation model for options on volatility using a binomial process, whereas Grünbichler & Longstaff (1996) value futures and options on volatility by assuming a continuous time GARCH process for the instantaneous variance rate. A drawback of this approach is the dependence of the model value on the particular process used to model the short-term volatility or instantaneous variance. While this problem

plagues most derivatives pricing models, the problem is particularly acute for volatility models because the quantity being modeled is not directly observable. Although an estimate for the initially unobservable state variable can be inferred from market prices of derivative securities, noise in the data generates noise in the estimate, raising doubts that a modeler can correctly select any parametric stochastic process from the menu of consistent alternatives.

Fortunately, an alternative nonparametric approach was proposed just before variance swaps were introduced. The first paper along these lines was a working paper by Anthony Neuberger (1990a) that was published in 1994 (Neuberger 1994). Instead of assuming a particular stochastic process, Neuberger assumed only that the price of the underlying evolves continuously over time and that the limit of the sum of squared returns should exist. The latter restriction holds if the price of the traded underlying is a strictly positive semimartingale. Neuberger argued for the introduction of a contract that would pay the natural log of an underlying asset price at its expiry. He showed that by continuously delta-hedging such a contract using the BS model at a fixed variance rate, the hedging error accumulates to the difference between the realized variance and the fixed variance used in the delta-hedge. Hence the payoff from a continuously monitored variance swap could in theory be replicated under conditions far more robust than usually suggested for path-dependent contingent claims.

Independently of Neuberger, Dupire (1993) developed the same argument and was able to publish it first. Dupire also pointed out that the contract paying the log of the price can be created with a static position in options, as a simple consequence of a more general theory first developed in Breeden & Litzenberger (1978). However, Dupire did not explicitly state the replicating portfolio, perhaps because his main interest lay elsewhere. He showed that a calendar spread of two such log contracts pays the variance realized between the two maturities, and developed the notion of forward variance. Mimicking the Heath, Jarrow, and Morton (HJM) stochastic evolution of a forward interest rate curve (Heath et al. 1992), Dupire modeled the random evolution of the term structure of forward variances over time. The resulting model became the first preference-free stochastic volatility model in the exact same way that Ho & Lee and HJM became the first preference-free interest rate models. Dupire pointed out that his model could be used to price and hedge derivatives written on the path of both the underlying asset price and its instantaneous variance rate. Hence, the model could be used for both standard vanilla options and for volatility derivatives more general than variance swaps.

Building on a prior working paper by Dupire (1996), Carr & Madan (1999) completed the task of developing a robust replicating strategy for continuously monitored variance swap. By showing explicitly the replicating portfolio for the log contract, they assembled all of the ingredients necessary to synthesize such a claim. In theory, the payoff on a continuously monitored variance swap was perfectly replicated by combining static positions in a continuum of options on price with a continuously rebalanced position in the underlying. As first shown by Neuberger (1990a), the number of shares held in the dynamic trading strategy depended only on the price level of the underlying asset. In theory, the dynamic component was independent of both time to maturity and any measure of volatility, whether it be historical or implied. This independence stems from the observation that the Black model volatility value of the log contract is independent of the stock price. As a result, the Black model delta can be derived from the intrinsic value of the log contract, which depends only on the price level of the underlying asset.

The next main breakthrough in the robust pricing of volatility derivatives occurred in path-breaking work by Dupire (1996) and by Derman, Kani, & Kamal (1997) (DDK). The idea of localizing the variance swap payoff in time had already been used in Dupire (1993). These authors observed that it was furthermore possible to localize the variance swap payoff in space, while not restricting dynamics beyond the aforementioned restriction in Neuberger (1990a). By restricting the set of times and price levels for which returns are used in the payoff calculation, one can synthesize a contract that pays off the local variance, i.e., the instantaneous realized variance that will be experienced should the underlying be at a specified price level at a specified future date. By integrating over space and time, Carr & Madan (1998) point out that one could robustly replicate corridor variance swaps, such as the up and down variance swaps that became popular in 2005.

The work of Dupire and DDK implied that more complicated volatility derivatives could also be valued, albeit in a less robust fashion. To do so, these authors developed the notion of forward local volatility, which is the fixed rate the buyer of the local variance swap pays at maturity in the event that the specified price level is reached. Given a complete term and strike structure of market option prices, their work shows that the entire initial forward local variance surface can be backed out from the initial market prices of these options. This surface is the two-dimensional analogue of the initial forward rate curve, central to the HJM analysis. Following HJM's lead, Dupire and DDK imposed a diffusion process on the forward local variance surface and derived the risk-neutral dynamics of this surface. Although not as robust as the pricing model for corridor variance swaps, such a model could be used for more general claims, such as standard vanilla options and general volatility derivatives.

The theoretical breakthroughs in volatility trading that occurred in the 1990s led to the introduction of other volatility derivatives besides variance swaps and corridor variance swaps. In 1990, Neuberger (1990) wrote a second working paper, which introduced the notion of a mileage option. Working in a binomial framework for simplicity, Neuberger randomized the length of each discrete time step and considered an option that matured as soon as the running sum of squared returns passed a positive barrier. Bick (1995) significantly extended the analysis to the case where the price is a continuous positive semimartingale. In this setting, he showed that the mileage/timer option payoff admits perfect replication just by dynamically trading in the underlying risky asset. Furthermore, the timer option is valued by the classical BS formula for a fixed maturity option, by replacing the remaining total variance $\sigma^2(T - t)$ with the remaining variance budget.

These fundamental papers laid the groundwork for much subsequent research on volatility derivatives. An extensive bibliography of this research appears at the end of this review.

4. REPLICATION ERROR FOR DISCRETELY MONITORED VARIANCE SWAP

A variance swap is a contract that pays some measure of the realized variance of the returns of a specified underlying asset over a specified period of time. The measure of realized variance requires monitoring the underlying price path discretely, usually at the end of each business day. Under certain conditions, the payoffs to a variance swap can be approximately replicated by combining dynamic trading in the underlying with static positions in standard options maturing with the swap. In this section, we analyze the leading source of error when attempting to replicate the payoff of a (discretely monitored)

variance swap on a futures price. Without loss of generality, we assume that the payoff is computed from daily returns using closing futures prices. We assume that one can take a static position in a continuum of European options and that one can trade futures daily without frictions. The standard model due to Neuberger (1990a) and Dupire (1993) equates the fair fixed payment on a T maturity continuously monitored variance swap to the forward value of a theoretical contract paying $-2\ln(S_T/F_0(T))$ at T , where $F_0(T)$ is the initial forward price. In this section, we argue that, to leading order, this standard methodology underprices actual variance swaps when the risk-neutral expectation of cubed returns is negative. We also present a novel explanation as to why the variance swap payoff involves squaring log price relatives, rather than discretely compounded returns.

4.1. Replication Error for Standard Variance Swaps

Consider a finite set of discrete times $\{t_0, t_1, \dots, t_n\}$ at which one can trade futures contracts. In what follows, we take these times to be daily closings. Let F_i denote the closing price on day i , for $i = 0, 1, \dots, n$. We assume that investors can trade at this price. By day n , a standard measure of the realized annualized variance of returns will be

$$V_{\Delta \ln F} := \frac{N}{n} \sum_{i=1}^n \ln^2(F_i/F_{i-1}),$$

where N is the number of trading days in a year. We next demonstrate a strategy whose terminal payoff approximates the above measure of variance.

Denote the simple return in period i by

$$R_i := \frac{F_i - F_{i-1}}{F_{i-1}}$$

A Taylor expansion of $2 \ln(1 + x)$ implies that

$$\begin{aligned} 2 \ln F_i &= 2 \ln F_{i-1} + \frac{2}{F_{i-1}}(F_i - F_{i-1}) - \frac{1}{F_{i-1}^2}(F_i - F_{i-1})^2 \\ &\quad + \frac{2}{3F_{i-1}^3}(F_i - F_{i-1})^3 + O(R_i^4), \quad i = 1, \dots, n, \end{aligned} \quad (1)$$

where $O(R_i^p)$ denotes $f(R_i)$ for some function f such that $f(x) = O(x^p)$ as $x \rightarrow 0$. Hence,

$$\ln(F_i/F_{i-1}) = R_i - \frac{R_i^2}{2} + O(R_i^3), \quad i = 1, \dots, n. \quad (2)$$

Squaring both sides,

$$\ln^2(F_i/F_{i-1}) = R_i^2 - R_i^3 + O(R_i^4), \quad i = 1, \dots, n. \quad (3)$$

Solving for R_i^2 and substituting in Equation 1 implies

$$2 \ln F_i = 2 \ln F_{i-1} + 2R_i - \ln^2(F_i/F_{i-1}) - \frac{R_i^3}{3} + O(R_i^4), \quad i = 1, \dots, n, \quad (4)$$

Hence,

$$\ln^2(F_i/F_{i-1}) = 2R_i - 2(\ln F_i - \ln F_{i-1}) - \frac{1}{3}R_i^3 + O(R_i^4), \quad i = 1, \dots, n. \quad (5)$$

Summing over i gives a decomposition of the sum of squared log price relatives:

$$\begin{aligned}\sum_{i=1}^n \ln^2(F_i/F_{i-1}) &= \sum_{i=1}^n 2R_i - 2 \sum_{i=1}^n (\ln F_i - \ln F_{i-1}) - \frac{1}{3} \sum_{i=1}^n R_i^3 + \sum_{i=1}^n O(R_i^4) \\ &= \sum_{i=1}^n \frac{2}{F_{i-1}} (F_i - F_{i-1}) - 2(\ln F_n - \ln F_0) - \frac{1}{3} \sum_{i=1}^n R_i^3 + \sum_{i=1}^n O(R_i^4).\end{aligned}\quad (6)$$

Now, a Taylor expansion with remainder of $\ln F_n$ about the point F_0 implies

$$\ln F_n = \ln F_0 + \frac{1}{F_0} (F_n - F_0) + \int_0^{F_0} \frac{-1}{K^2} (K - F_n)^+ dK + \int_{F_0}^{\infty} \frac{-1}{K^2} (F_n - K)^+ dK;$$

hence,

$$-2(\ln F_n - \ln F_0) = -\frac{2}{F_0} \sum_{i=1}^n (F_i - F_{i-1}) + \int_0^{F_0} \frac{2}{K^2} (K - F_n)^+ dK + \int_{F_0}^{\infty} \frac{2}{K^2} (F_n - K)^+ dK. \quad (7)$$

Substituting Equation 7 in Equation 6 implies

$$\begin{aligned}\sum_{i=1}^n \ln^2(F_i/F_{i-1}) &= \sum_{i=1}^n \left(\frac{2}{F_{i-1}} - \frac{2}{F_0} \right) (F_i - F_{i-1}) + \int_0^{F_0} \frac{2}{K^2} (K - F_n)^+ dK \\ &\quad + \int_{F_0}^{\infty} \frac{2}{K^2} (F_n - K)^+ dK - \frac{1}{3} \sum_{i=1}^n R_i^3 + \sum_{i=1}^n O(R_i^4).\end{aligned}\quad (8)$$

Finally, multiplying by $\frac{N}{n}$ to annualize gives the desired decomposition:

$$\begin{aligned}V_{\Delta \ln F} &= \sum_{i=1}^n \frac{2N}{n} \left(\frac{1}{F_{i-1}} - \frac{1}{F_0} \right) (F_i - F_{i-1}) \\ &\quad + \left[\int_0^{F_0} \frac{2N}{nK^2} (K - F_n)^+ dK + \int_{F_0}^{\infty} \frac{2N}{nK^2} (F_n - K)^+ dK \right] - \frac{N}{3n} \sum_{i=1}^n R_i^3 + \sum_{i=1}^n O(R_i^4).\end{aligned}\quad (9)$$

The first two terms on the right-hand side of Equation 9 are the P&L from a dynamic position in futures and a static position in options. For the dynamic component, one holds $e^{-r(t_n-t_i)} \frac{2N}{n} \left(\frac{1}{F_{i-1}} - \frac{1}{F_0} \right)$ futures contracts from day $i-1$ to day i . For the static component, one holds $\frac{2N}{nK^2} dK$ puts at all strikes below the initial forward F_0 . One also holds $\frac{2N}{nK^2} dK$ calls at all strikes above the initial forward F_0 . The third term in Equation 9 is the leading source of error when approximating the payoff $V_{\Delta \ln F}$. If the cubed returns sum to a negative number, then this third term realizes to a positive value due to its leading negative sign. If we ignore the small final term in Equation 9, then we see that the payoff from being long options and delta-hedging them falls short of the target payoff when cubed returns sum to a negative number.

The initial cost of creating the payoff generated by the first two terms in Equation 9 is:

$$\int_0^{F_0} \frac{2N}{nK^2} P_0(K, t_n) dK + \int_{F_0}^{\infty} \frac{2N}{nK^2} C_0(K, t_n) dK, \quad (10)$$

where $P_0(K, t_n)$ and $C_0(K, t_n)$, respectively, denote the initial prices of puts and calls struck at K and maturing at t_n . The square root of this quantity is the standard model value for the variance swap rate.

If the risk-neutral mean of cubed returns is negative, i.e., $E_0^Q \sum_{i=1}^n R_i^3 < 0$, and if we ignore the price impact of the final term in Equation 9, then the fair fixed payment for a variance swap is higher than the cost in Equation 10. In other words, the standard pricing scheme underprices the variance swap rate when the risk-neutral mean of cubed returns is negative and higher-order terms are ignored.

Cubic-in-return error estimates were first established by Demeterfi et al. (1995a), who defined the variance payoff not in terms of log returns but rather simple returns. The next section deals with that alternative payoff specification.

4.2. Replication Error for Simple Return Variance Swaps

By a Taylor expansion of $\ln^2(1+x)$,

$$\frac{N}{n} \sum_{i=1}^n \ln^2(F_i/F_{i-1}) - \frac{N}{n} \sum_{i=1}^n R_i^2 = -\frac{N}{n} \sum_{i=1}^n R_i^3 + \sum_{i=1}^n O(R_i^4). \quad (11)$$

Combining this with Equation 8 produces the following expression for simple return variance:

$$\begin{aligned} V_{\Delta F/F} := \frac{N}{n} \sum_{i=1}^n R_i^2 &= \sum_{i=1}^n \frac{2N}{n} \left(\frac{1}{F_{i-1}} - \frac{1}{F_0} \right) (F_i - F_{i-1}) + \int_0^{F_0} \frac{2N}{nK^2} (K - F_n)^+ dK \\ &+ \int_{F_0}^{\infty} \frac{2N}{nK^2} (F_n - K)^+ dK + \frac{2N}{3n} \sum_{i=1}^n R_i^3 + \sum_{i=1}^n O(R_i^4). \end{aligned} \quad (12)$$

In the early days of variance swaps, it would have been plausible to define the floating side payoff as $V_{\Delta F/F}$, rather than $V_{\Delta \ln F}$. Let us refer to the variance swap with the former payoff as a simple return variance swap. We see from Equation 11 that the payoff from a long position in a simple return variance swap differs from the payoff of a standard variance swap by $-\frac{N}{n} \sum_{i=1}^n R_i^3$, ignoring remainder terms. Hence, when cubed returns realize to a negative value, the payoff generated by squaring log price relatives exceeds the payoffs from squaring the simple returns with daily compounding. As mentioned earlier, when variance swaps first became liquid in the late 1990s, banks tended to buy variance swaps from hedge funds, who found the fixed payment that banks were willing to pay was attractive when compared with realized variance. Because the fixed rate at which hedge funds are willing to sell realized variance was probably insensitive to minor details in the contract design, the bank had a profit motive to define the returns in the term sheet as $\ln(F_i/F_{i-1})$ rather than R_i . This may possibly explain the appearance of the natural logarithm in the term sheets of standard variance swap. Prior to the appearance of variance swaps, transcendental functions did not appear in term sheets, although they certainly appeared in valuation formulae.

For a buyer of standard variance swaps, who partially hedges by selling options and trading futures, the above analysis indicates a negative exposure to cubed returns. In contrast, for a buyer of simple return variance swaps, who partially hedges by selling options and trading futures, the above analysis indicates a positive exposure to cubed returns with twice the magnitude. Therefore a position with zero net exposure to cubed returns can be created by buying a simple return variance swap for every two standard

variance swap purchased, with both kinds of variance swaps partially hedged by selling options and trading futures, as Bondarenko (2008) shows. Although market conventions regarding variance swaps are by now firmly established, this observation does suggest the introduction of this new contract, once credit concerns subside.

5. VARIANCE SWAP RATE AS A WEIGHTED AVERAGE OF IMPLIED VARIANCES

The theoretical value of the floating leg of a continuously monitored variance swap can be expressed as an integral across strikes of OTM option prices. Market practitioners prefer to work with Black-implied volatilities rather than option prices because there is an arbitrage-free economy where the former are constant with respect to strike and maturity. In contrast, the only arbitrage-free economy where OTM option prices are invariant to strike and maturity is one with no uncertainty.

Because the theoretical value of the floating leg of a continuously monitored variance swap is an explicit function of option prices and each option price is an explicit function of its own implied variance, it follows that this theoretical value is an explicit function of implied variances. When the strike price is used to capture the moneyness of the OTM option, this explicit formula is exceedingly messy. In this section, we present an alternative measure of the moneyness of an option. The dependence of theoretical value on the smile becomes exceedingly simple when implied variances are expressed in terms of this alternative moneyness measure. The main result of this section was developed by A. Matytsin (private communication) and first published by Gatheral (2006). Other interesting uses of the alternative moneyness measure appear in Matytsin (2000).

Let K denote the strike of an option and let F denote the current forward price. With K and F both assumed positive, let $x \equiv \ln(K/F)$ and let $\bar{\sigma}(x)$ be the Black-implied volatility as a function of x . We can define a new moneyness measure of a call as

$$y \equiv \tilde{d}_2(x) \equiv \frac{-x}{\bar{\sigma}(x)\sqrt{T}} - \frac{\bar{\sigma}(x)\sqrt{T}}{2}. \quad (13)$$

We suppose that the implied volatility function $\bar{\sigma}(x)$ is such that the moneyness measure $\tilde{d}_2(x)$ defined in Equation 13 is monotonically strictly decreasing in x . As a result, the function $\tilde{d}_2^{-1}(y)$ exists and Equation 13 can be inverted to write x as a function of y :

$$x = \tilde{d}_2^{-1}(y). \quad (14)$$

Under this condition, we may express implied volatility in terms of y rather than x . Similarly, the total implied variance $\bar{\sigma}^2 T$ can be expressed in terms of the moneyness measure y , rather than x . Below, we see that the use of y as the moneyness measure allows the theoretical value of the floating leg of a continuously monitored variance swap to be expressed as a weighted average of total implied variances, and that the weight function is just the standard normal density function n .

To make these statements precise, let $P(K)$ and $C(K)$ respectively denote the forward prices in the market of a European put and a European call as a function of strike K . We assume these prices are arbitrage-free. We show that

$$\int_F^\infty \frac{2}{K^2} C(K) dK + \int_0^F \frac{2}{K^2} P(K) dK = T \int_{-\infty}^{+\infty} n(y) \sigma^2(y) dy,$$

where $\sigma(y) := \hat{\sigma}(d_2^{-1}(y))$ is the implied volatility as a function of y . Our notation suppresses the dependence of both the option prices and the implied variance on the maturity date T since it is treated as a constant.

To show¹ the above result, let

$$I \equiv \int_F^\infty \frac{1}{K^2} C(K) dK + \int_0^F \frac{1}{K^2} P(K) dK.$$

From integration by parts,

$$I = \int_F^\infty \frac{1}{K} C'(K) dK + \int_0^F \frac{1}{K} P'(K) dK - \frac{C(K)}{K} \Big|_{K=F}^{K=\infty} - \frac{P(K)}{K} \Big|_{K=0}^{K=F} \quad (15)$$

From put call parity, $C(F) = P(F)$. From the strict positivity of the futures price process, $\lim_{K \downarrow 0} \frac{P(K)}{K} = 0$. As a result, the boundary terms in Equation 15 vanish:

$$I = \int_F^\infty \frac{1}{K} C'(K) dK + \int_0^F \frac{1}{K} P'(K) dK. \quad (16)$$

Let $\hat{\sigma}(K)$ be the implied volatility as a function of strike K . Recall the defining property of implied volatility in terms of the Black (1976) formulae for the forward price of the options:

$$\begin{aligned} C(K) &= FN(\hat{d}_2(K) + \hat{\sigma}(K)\sqrt{T}) - KN(d_2(K)) \\ P(K) &= -FN(-\hat{d}_2(K) - \hat{\sigma}(K)\sqrt{T}) + KN(-\hat{d}_2(K)), \end{aligned} \quad (17)$$

where N is the standard normal CDF and

$$\hat{d}_2(K) \equiv \frac{\ln(F/K) - \frac{\hat{\sigma}^2(K)T}{2}}{\hat{\sigma}(K)\sqrt{T}}. \quad (18)$$

We can interpret $\hat{d}_2(K)$ as a moneyness measure because it is defined as the number of standard deviations $\hat{\sigma}(K)\sqrt{T}$ that the mean of the terminal log forward $E \ln F_T = \ln F - \frac{\hat{\sigma}^2(K)T}{2}$ exceeds the log of the strike, $\log K$, in an imagined Black economy with $\hat{\sigma}(K)$ as the constant volatility.

Differentiating the Black Equations 17 in strike K ,

$$\begin{aligned} C'(K) &= -N(\hat{d}_2(K)) + \sqrt{T} n(\hat{d}_2(K)) K \hat{\sigma}'(K) \\ P'(K) &= N(-\hat{d}_2(K)) + \sqrt{T} n(\hat{d}_2(K)) K \hat{\sigma}'(K). \end{aligned} \quad (19)$$

Substituting Equation 19 in Equation 15 implies

$$\begin{aligned} I &= \int_F^\infty \frac{1}{K} \left(-N(\hat{d}_2(K)) + \sqrt{T} n(\hat{d}_2(K)) K \hat{\sigma}'(K) \right) dK \\ &\quad + \int_0^F \frac{1}{K} \left(N(-\hat{d}_2(K)) + \sqrt{T} n(\hat{d}_2(K)) K \hat{\sigma}'(K) \right) dK, \end{aligned} \quad (20)$$

¹We thank Alexey Polishchuk of Bloomberg for this elegant derivation.

where

$$n(x) \equiv N'(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}. \quad (21)$$

Changing the variable of integration in Equation 20 from K to

$$x \equiv \log\left(\frac{K}{F}\right) \quad (22)$$

implies

$$\begin{aligned} I = & - \int_0^{\infty} N(\tilde{d}_2(x)) dx + \int_0^{\infty} \sqrt{T} n(\tilde{d}_2(x)) \tilde{\sigma}'(x) dx \\ & + \int_{-\infty}^0 N(-\tilde{d}_2(x)) dx + \int_{-\infty}^0 \sqrt{T} n(\tilde{d}_2(x)) \tilde{\sigma}'(x) dx, \end{aligned} \quad (23)$$

where $\tilde{\sigma}(x) \equiv \hat{\sigma}(Fe^x)$ is the implied volatility and $\tilde{d}_2(x) \equiv \hat{d}_2(Fe^x)$ is the moneyness measure, with both quantities expressed as functions of x rather than K . Combining the second and fourth integrals and integrating by parts on the first and third integrals,

$$\begin{aligned} I = & \int_{-\infty}^{+\infty} \sqrt{T} \tilde{\sigma}'(x) n(\tilde{d}_2(x)) dx + \int_{-\infty}^{+\infty} x n(\tilde{d}_2(x)) \tilde{d}_2'(x) dx \\ & - x N(\tilde{d}_2(x)) \Big|_0^{\infty} + x N(-\tilde{d}_2(x)) \Big|_{-\infty}^0. \end{aligned}$$

A sufficient condition for the boundary terms to vanish, by Lee (2004), is the existence of some $\varepsilon > 0$ such that F_T has finite risk-neutral moments of orders $1 + \varepsilon$ and $-\varepsilon$. In that case,

$$I = \int_{-\infty}^{+\infty} \sqrt{T} \tilde{\sigma}'(x) n(\tilde{d}_2(x)) dx + \int_{-\infty}^{+\infty} x n(\tilde{d}_2(x)) \tilde{d}_2'(x) dx. \quad (24)$$

Integrating just the first integral by parts implies

$$\begin{aligned} I = & \int_{-\infty}^{+\infty} \sqrt{T} \tilde{\sigma}(x) \tilde{d}_2'(x) n(\tilde{d}_2(x)) \tilde{d}_2'(x) dx + \sqrt{T} \tilde{\sigma}(x) n(\tilde{d}_2(x)) \Big|_{x=-\infty}^{x=+\infty} \\ & + \int_{-\infty}^{+\infty} x n(\tilde{d}_2(x)) \tilde{d}_2'(x) dx. \end{aligned} \quad (25)$$

As the boundary terms vanish, this simplifies to

$$I = \int_{-\infty}^{+\infty} \sqrt{T} \tilde{\sigma}(x) \tilde{d}_2'(x) n(\tilde{d}_2(x)) \tilde{d}_2'(x) dx + \int_{-\infty}^{+\infty} x n(\tilde{d}_2(x)) \tilde{d}_2'(x) dx. \quad (26)$$

Pulling out the common factor in the two integrands, we have

$$I = \int_{-\infty}^{+\infty} n(\tilde{d}_2(x)) \tilde{d}_2'(x) \left[\sqrt{T} \tilde{\sigma}(x) \tilde{d}_2'(x) + x \right] dx. \quad (27)$$

Using Equation 13, Equation 27 simplifies to

$$I = \int_{-\infty}^{+\infty} n(\tilde{d}_2(x)) \frac{\tilde{\sigma}^2(x) T}{2} (-\tilde{d}_2'(x)) dx.$$

The change of integration variable from x to $y \equiv \tilde{d}_2(x)$ leads to

$$I = \int_{-\infty}^{+\infty} n(y) \frac{\sigma^2(y)}{2} T dy,$$

where $\sigma(y) := \tilde{\sigma}(d_2^{-1}(y))$ is the implied volatility as a function of y . Doubling both sides gives the desired result.

6. VOLATILITY SWAP RATE AND AT-THE-MONEY IMPLIED VOLATILITY

Volatility swaps are financial contracts traded over the counter that pay the difference between the realized volatility and a constant. Like all swaps, the constant is initially chosen so that the contract has zero cost to enter. Since a volatility swap involves only a single fixed payment made at maturity, a volatility swap is also a forward contract on realized volatility. The actual market volatility swap rate is therefore a risk-neutral expectation of future realized volatility. As such, the volatility swap rate serves as a forecast of subsequent realized volatility, with the caveat that the difference between risk-neutral and statistical probabilities can lead to bias in this forecast.

Unfortunately, market volatility swap rates are not widely distributed, so the information content of this forecast is only available to those obtaining quotes over the counter. In contrast, the market prices of standard options on the asset underlying a volatility swap are widely available. This has led to a common practice of treating the at-the-money implied (ATMI) as a forecast of subsequent realized volatility. However, because the BS model assumes that volatility is deterministic, there is no economic motivation for this procedure. Nonetheless, a substantial body of empirical work assesses the forecast ability of implied volatility. After much debate, the present conclusion of this literature is that ATMI is an efficient although biased forecast of subsequent realized volatility.

The objectives of this section are twofold. First, we provide, for the volatility swap rate, a simple approximation which is easily obtainable from observables. Second, we provide an economic justification for the use of ATMI as a forecast of subsequent realized volatility. We satisfy both objectives simultaneously by giving sufficient conditions under which the volatility swap rate is well approximated by the ATMI. These conditions are that a risk-neutral measure exists (hence no frictions and no arbitrage), that the underlying asset price is positive and continuous over time (hence no bankruptcy and no price jumps), and that increments in instantaneous volatility are independent of returns (hence no leverage effect).

Under these assumptions, the initial volatility swap rate is closely approximated by just the ATMI—a single point on the implied volatility smile. Besides linking the volatility swap rate with an easily observed quantity, this analysis gives an economic motivation for the use of ATMI in forecasting subsequent realized volatility. More specifically, if the ATMI differs from the volatility swap rate by more than the approximation error, then in this setting, an arbitrage opportunity arises [see Carr & Lee (2009b) for the exact trading strategy]. This result also suggests that the forecasting bias in implied volatilities is due to the difference between risk-neutral and statistical probabilities, together with leverage effects.

Although this section's independence assumption does not hold in typical equity markets, the impact of correlation is mitigated by the pricing methods of Carr & Lee (2009b), which exploit the information in the full volatility smile, not just a single point. Moreover, Carr & Lee (2009b) show that the volatility swap payoff can be perfectly replicated, under

this section's framework, by dynamically trading European options and futures which mature with the swap.

6.1. Analysis

We assume frictionless markets, and for simplicity, zero interest rates².

Let futures prices, option prices, and volatility swap values be martingales with respect to some pricing measure \mathbb{Q} and some filtration F . This assumption rules out arbitrage among these assets. We further assume that the T -delivery futures price process $\{F_t, t \in [0, T]\}$ is always positive and continuous; in particular, we assume that

$$\frac{dF_t}{F_t} = \sigma_t dW_t, \quad t \in [0, T],$$

where $F_0 > 0$ is known; W is a standard F -Brownian motion under \mathbb{Q} ; and the σ process is F -adapted, independent of W , and satisfies $E_0^{\mathbb{Q}} \int_0^T \sigma_t^2 dt < \infty$. The σ process is otherwise unrestricted and unspecified. In particular, the instantaneous volatility σ_t can have stochastic drift, stochastic volatility of its own, and a stochastic jump component.

Consider a volatility swap with continuous path monitoring and a notional of one dollar. By definition, this volatility swap has a payoff at T of

$$\bar{\sigma} - v_0^s, \quad (28)$$

where $\bar{\sigma}$ is the random realized volatility:

$$\bar{\sigma} \equiv \sqrt{\frac{1}{T} \int_0^T \sigma_u^2 du}, \quad (29)$$

and where v_0^s is the initial volatility swap rate. This swap rate is chosen so that the initial value of the volatility swap is zero. Hence, taking the risk-neutral expected value of Equation 28 and setting the result to zero implies that

$$v_0^s = E_0^{\mathbb{Q}} \bar{\sigma}.$$

To determine the volatility swap rate, we assume zero interest rates for simplicity. Hence the initial price of a European call is given by

$$C_0(K, T) = E_0^{\mathbb{Q}}(F_T - K)^+.$$

Using the law of iterated expectations, we have

$$C_0(K, T) = E_0^{\mathbb{Q}}[E_0^{\mathbb{Q}}(F_T - K)^+ | \mathbb{F}_T^{\sigma}], \quad (30)$$

where \mathbb{F}^{σ} is the information filtration generated by (just) the process for instantaneous volatility. As first pointed out by Hull & White (1987), the inner expectation in Equation 30 is just the Black model forward value of a European call on a futures price with a deterministic volatility process:

$$E_0^{\mathbb{Q}}[(F_T - K)^+ | \mathbb{F}_T^{\sigma}] = B(F_0, K, \bar{\sigma}), \quad (31)$$

²All of the results extend easily to deterministic interest rates.

where B is the Black formula:

$$B(F_0, K, \sigma) \equiv F_0 N\left(\frac{\ln(F_0/K) + \frac{\sigma\sqrt{T}}{2}}{\sigma\sqrt{T}}\right) - KN\left(\frac{\ln(F_0/K) - \frac{\sigma\sqrt{T}}{2}}{\sigma\sqrt{T}}\right). \quad (32)$$

It is important to note that this call value depends on the volatility path only through its $L^2[0, T]$ mean $\bar{\sigma}$.

Substituting Equation 31 in Equation 30 implies that

$$C_0(K, T) = E_0^Q[B(F_0, K, \bar{\sigma})], \quad (33)$$

where the risk-neutral expectation is over the random realized terminal volatility $\bar{\sigma}$ defined in Equation 29.

Now suppose the call is ATM so that setting $K = F_0$ in Equation 32 implies

$$A(F_0, \sigma) \equiv B(F_0, F_0, \sigma) = F_0 \left[N\left(\frac{\sigma\sqrt{T}}{2}\right) - N\left(-\frac{\sigma\sqrt{T}}{2}\right) \right]. \quad (34)$$

A Taylor series expansion of each normal distribution function about zero implies

$$A(F_0, \sigma) = \frac{F_0}{\sqrt{2\pi}} \sigma\sqrt{T} + O(\sigma^3 T^{3/2}). \quad (35)$$

Setting $K = F_0$ in Equation 33 and substituting Equation 35 in implies that

$$C_0(F_0, T) \approx E_0^Q \left[\frac{F_0}{\sqrt{2\pi}} \bar{\sigma}\sqrt{T} \right], \quad (36)$$

and hence the volatility swap rate is approximated by

$$v_0^s = E_0^Q \bar{\sigma} \approx \frac{\sqrt{2\pi}}{F_0\sqrt{T}} C_0(F_0, T). \quad (37)$$

The ATMI $a_0(T)$ is defined implicitly by

$$C_0(F_0, T) = A(F_0, a_0(T)) = F_0 \left[N\left(\frac{a_0(T)\sqrt{T}}{2}\right) - N\left(-\frac{a_0(T)\sqrt{T}}{2}\right) \right], \quad (38)$$

from Equation 34. As shown in Brenner & Subrahmanyam (1988), we have

$$a_0(T) \approx \frac{\sqrt{2\pi}}{F_0\sqrt{T}} C_0(F_0, T). \quad (39)$$

Comparing Equations 37 and 39, we conclude that the volatility swap rate is approximated by the ATMI:

$$v_0^s \approx a_0(T), \quad (40)$$

as shown in Feinstein (1989).

7. SUMMARY AND OPEN ISSUES

In this review, we provided a history of the volatility derivatives market. We then surveyed the early work on the pricing theory, detailing contributions from both academics and

practitioners. We closed with three technical sections that presented fundamental results on variance and volatility swaps.

The technical sections each suggest some open problems. The first is to provide alternative robust replicating strategies for discretely monitored variance swaps. The objective is to reduce the error and/or reduce the difficulty in putting on the hedge. A second open problem is to further explore the link between volatility derivatives and the implied volatilities of vanilla options, in more realistic models. A third major research goal is to integrate all of the various interrelated markets. As an aid to the researcher in tackling these problems, we have provided an extensive bibliography on volatility derivatives.

DISCLOSURE OF BIAS

The authors are not aware of any affiliations, memberships, funding, or financial holdings that might be perceived as affecting the objectivity of this review.

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