

# American Put Call Symmetry

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## **Abstract**

We derive a simple relationship between the values and exercise boundaries of American puts and calls. The relationship holds for options with the same “moneyness”, although the absolute level of the strike price and underlying may differ. The result holds in both the Black Scholes model and in a more general diffusion setting.

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# I Introduction

This paper describes a simple relationship between both the values and the exercise boundaries of American calls and puts. The relationship holds for an American call and an American put with the same “money-ness”, which is formally defined below. The result holds in both the classical Black-Scholes[3]/Merton[13] model, and in a more general diffusion setting in which the volatility depends on the level of the underlying in a certain manner described below.

The result is useful for transferring knowledge about the fair pricing and optimal exercise of one type of option to the other. Given the difficulty in pricing American options efficiently when volatility smiles exist, the result may also be used to dramatically simplify the coding of one type of option, once the other has been coded. Finally, violations of the result for futures options implies the existence of arbitrage opportunities, so long as the volatility’s dependence on the underlying displays a certain symmetry described below.

Our results are related to three strands of prior research concerning the relationship between puts and calls on the same underlying with the same maturity. The first strand is put-call parity, which appears to have originated in Castelli[6], and to have been rediscovered in Kruizenga[10][11] and in Stoll[17][18]. This robust result relates European options (see Merton[14]) with the same strike in a very general setting. In contrast, a second strand relates *American* calls and puts with the same strike and in particular valuation models. Building on the earlier work of Grabbe[9], McDonald and Schroder[12] first recognized the relationship in the binomial model in a clever working paper. Chesney and Gibson[7] extended this idea to the diffusion setting when interest rates or dividend yields are stochastic. Schroder[16] has significantly extended this approach to stochastic volatility and jumps. Bates[1] originated a third line of research concerning the relationship between European calls and puts with different strikes in lognormal models. Bates[2]<sup>1</sup> and Byun and Kim[4] also extend the relationship to American futures options.

Our contribution over the foregoing papers is modest. In contrast to put-call parity, our results are developed for American options in a diffusion setting. In contrast to the second strand, we consider American options with different strikes. In contrast to the third strand, we deal with American options on both the spot and futures price, and allow for a more general, although still highly specialized, volatility

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<sup>1</sup>This paper also contains an excellent exposition of the implications of asymmetry for implying out crash premia.

structure. In contrast to all three strands, we allow the underlying asset price to differ for the call and the put. This situation can arise when comparing the values which an American call and put will have at a future date (eg. just after a stock split). Although the stock prices and strike prices can differ, our results apply only if the “moneyness” and maturity date are the same. It is also possible to be comparing American call and put values at two different times, when the moneyness, time to maturity and the volatility structure happen to be the same. For example, the comparison may be between a 10% in-the-money one month call on Jan. 1 with a 10% in-the-money one month put one year later. This result is potentially of interest to issuers of over-the-counter options, who may for example be deciding between the issuance of an American call today with an American put in one year, where the strikes and maturity dates are chosen so that the moneyness and time to maturity at issuance are the same. With the introduction of flex options on listed exchanges, buyers of options may also face the same choice.

The structure of this paper is as follows. The next section defines some terminology and specifies our assumptions. Section III delineates our American Put Call Symmetry Condition and its implications, for options on both the cash and futures price of the underlying. The final section summarizes the paper and gives directions for future research. The appendix proves our main results.

## II Definitions and Assumptions

We define the moneyness of an American call as the log price relative of the underlying to the strike. For example, the moneyness of an American call struck at 16 with the underlying priced at 8 is -69%. In contrast, we define the moneyness of an American put as the log price relative of the strike to the underlying. For example, the moneyness of an American put struck at 1 with the underlying priced at 2 is also -69%. Our objective is to relate the value and exercise boundary of the above American call to the value and exercise boundary of the above American put, when the two options have the same underlying asset and time to maturity.

To achieve this objective, we assume the standard model of frictionless markets and no arbitrage. For simplicity, we assume a constant interest rate  $r > 0$  and a constant continuous dividend yield  $\delta > 0$ . Both parameters are assumed to be positive so that there is positive probability of early exercise of each option.

The price process of the common underlying security is assumed to be an Itô process:

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)dW_t, \quad t \geq 0. \quad (1)$$

As usual, the function  $\mu(S_t, t)$  in the drift is essentially arbitrary. However, we note that both  $\mu(S_t, t)$  and the local volatility  $\sigma(S_t, t)$  must depend on the spot price only through the ratio of the spot price to some fixed amount  $\bar{K}$  denominated in the same numeraire (eg. dollars). The dependence on this ratio must arise in any economically sensible model, since the drift and volatility rate are independent of the measurement units of the spot price, while the spot price itself is clearly not. Note that changes in the spot price due to the arrival of information will change the volatility level, since the ratio  $S_t/\bar{K}$  will change. However, a change of measurement units from say dollars to francs, will not affect the ratio or the volatility. The particular choice of  $\bar{K}$  is arbitrary, since any other choice of  $\bar{K}$  can be undone with the appropriate choice of volatility functions. To fix matters, we choose  $\bar{K}$  to be the geometric mean of the call strike  $K_c$  and the put strike  $K_p$ :

$$\bar{K} \equiv \sqrt{K_c K_p}.$$

Since  $\sigma$  depends on  $S_t$  only through  $S_t/\bar{K}$ , this dependence can be made explicit by defining a new function:

$$v\left(\frac{S}{\bar{K}}, T-t\right) \equiv \sigma(S, t), \quad S > 0, t \in [0, T],$$

where  $T$  is the common time to maturity at the comparison time of the call and the put. Note that the second argument of  $v$  is time to maturity  $\tau \equiv T - t$ , rather than calendar time  $t$ .

We now impose additional structure by assuming that this new function obeys the following symmetry condition:

$$v\left(\frac{S}{\bar{K}}, \tau\right) = v\left(\frac{\bar{K}}{S}, \tau\right), \quad \text{for all } S > 0 \text{ and } \tau \in [0, T]. \quad (2)$$

In words, the volatility when the spot is say twice the level of  $\bar{K}$  is the same as the volatility if the spot were instead half the level of  $\bar{K}$ . The simplest example of a volatility structure satisfying the symmetry condition (2) arises in the Black Scholes/Merton model where volatility is at most time-dependent. An example of a volatility structure satisfying (2) in which future levels of the underlying volatility depend on future levels of the underlying price is:

$$\sigma(S_t, t) = ae^{\alpha t} + be^{\beta t} \left[ \ln\left(\frac{S_t}{\bar{K}}\right) \right]^2, \quad t \in [0, T], \quad (3)$$

where  $\alpha$  and  $\beta$  are constants. Figure 1 shows a graph of this structure for  $a = .2, \alpha = .1, b = .1, \beta = 0, \bar{K} = 4$ , and  $T = 0.5$ . Thus, the volatility when the spot is 4 is initially 20% and grows exponentially over time at the rate of 10% per year to reach about 25% in half a year. Figure 1a shows that volatility is symmetric in the *log* of the spot price, with the axis of symmetry at  $\ln(\bar{K}) = \ln(4) = 1.39$ . Figure 1b indicates that volatility is asymmetric when graphed against the price itself, with greater volatility to the downside for spot levels equidistant from  $\bar{K} = 4$ . This volatility structure can be used to generate results when comparing the value of a call struck at 16 if the spot price is 8 in three months with the value of a put struck at 1 if the spot value were instead 2 in three months.

It is also possible to generate symmetric volatility structures permitting comparisons between call and put values at different times. The easier case is if the comparison dates differ by  $T$  or more. For example, a 3 month call issued at the beginning of this year may be compared with a 3 month put issued at the beginning of next year. In this case, the symmetry condition (2) would require that a volatility structure such as (3) reigned over the first quarter of each year. The more difficult case is if the comparison dates differ by less than  $T$ . For example, a 3 year call issued at the beginning of this year may be compared with a 3 year put issued at the beginning of next year. In this case, the symmetry condition (2) would require an annual seasonality in the volatility structure.

The log-symmetric volatility structure defined by (2) permits volatility frowns or even more complex<sup>2</sup> patterns with inflection points. Log-symmetry in volatility is roughly observed<sup>3</sup> in currencies and some commodities<sup>4</sup>. However, given the complexities of the modern marketplace, we would be naive to assert that actual volatility structures exactly display the mathematical regularity we assume. Nonetheless, just as practitioners frequently use the Black Scholes model as a first approximation, we propose that our slightly more general log-symmetric volatility structure may be a convenient second approximation in certain markets<sup>5</sup>.

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<sup>2</sup>Mathematically, volatility can be any positive even function of  $\ln(S/\bar{K})$ . If it is analytic, this is equivalent to having a Taylor series expansion about zero with only even terms. Obviously, this includes a large class of functions.

<sup>3</sup>Note that the symmetry is in local volatility not implied.

<sup>4</sup>While equity index volatilities generally display a downward sloping pattern (see French, Schwert, and Stambaugh[8] or Rubinstein[15]), it is possible that the structure slopes upwards at higher unobserved spot levels.

<sup>5</sup>We thank a referee for this observation.

### III American Put Call Symmetry

Let  $C(S_c, K_c; \delta, r)$  denote the value of an American call when the underlying stock price is at level  $S_c > 0$ , the strike price is at  $K_c > 0$ , the dividend yield is at  $\delta > 0$ , and the riskless rate is at  $r > 0$ . Similarly, let  $P(S_p, K_p; \delta, r)$  denote the value of an American put when the underlying stock price is at level  $S_p > 0$ , the strike price is at  $K_p > 0$ , the dividend yield is at  $\delta > 0$ , and the riskless rate is at  $r > 0$ . Implicitly, the American call and put are written on the same underlying asset and have the same time to maturity. Compare an American call and put which have equal moneyness, i.e.:

$$\frac{S_c}{K_c} = \frac{K_p}{S_p}. \quad (4)$$

Our American Put Call Symmetry relates the values of two options with the same moneyness as:

$$\frac{C(S_c, K_c; \delta, r)}{\sqrt{S_c K_c}} = \frac{P(S_p, K_p; r, \delta)}{\sqrt{S_p K_p}}. \quad (5)$$

The result equates the normalized value of an American call struck at  $K_c$  in an economy where the underlying price is  $S_c$ , the continuous dividend yield is  $\delta$ , and the riskless rate is  $r$  to the normalized value of an American put in another economy where the underlying price is  $S_p$ , the continuous<sup>6</sup> dividend yield is  $r$ , and the riskless rate is  $\delta$ . A minor modification of the proof in the Appendix<sup>7</sup> shows that the result holds for European options as well.

Before indicating the utility of this result, we first delineate the corresponding result for exercise boundaries. Let  $\bar{S}(K_c; \delta, r)$  denote the critical stock price of an American call struck at  $K_c > 0$ , when the dividend yield is  $\delta > 0$ , and the riskless rate is  $r > 0$ . Similarly, let  $\underline{S}(K_p; \delta, r)$  denote the critical stock price of an American put struck at  $K_p$ . Then our American Put Call Symmetry result for the exercise boundaries is:

$$\sqrt{\bar{S}(K_c; \delta, r)\underline{S}(K_p; r, \delta)} = \bar{K} \equiv \sqrt{K_c K_p}. \quad (6)$$

In words, the geometric mean of the two critical prices is always equal to the geometric mean of the strikes, when the cost of carrying the underlying are of opposite sign.

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<sup>6</sup>If dividends in the first economy are discrete, then the interest rate in the second economy would have to jump at the ex-dividend date. Thus our results are of limited utility for valuing options on dividend-paying stocks.

<sup>7</sup>The proof is based on uniqueness results for solutions to partial differential equations. Please contact the first author for an alternative proof based on probability theory.

These results are primarily useful for transferring knowledge about the fair pricing and optimal exercise of one type of option to the other. For example, it is well-known that an American call is not exercised early if the underlying asset has no dividend yield ( $\delta = 0$ ). An implication of (6) combined with this standard result is that an American put is not exercised early when the riskless rate is zero. Furthermore, just as raising the dividend yield promotes early exercise of American calls, raising the interest rate promotes early exercise of American puts<sup>8</sup>.

Using (5), any procedure for pricing an American put may be used to instead price an American call or vice versa. Alternatively, the integrity of a method used to price an American call may be checked against a secure method for pricing an American put or vice versa. Similarly using (6), the critical stock price of an American call or put may be easily obtained or verified from that of its counterpart.

An important special case arises when the options are both written on the futures price of the same underlying asset. This case may be obtained from the above by equating the dividend yield and riskless rate ( $\delta = r$ ). If we further equate the underlying futures prices ( $S_c = S_p = F$ ), then equal moneyness of the call and put as in (4) implies that the geometric mean of the call strike and the put strike is the current futures price:

$$\bar{K} \equiv \sqrt{K_c K_p} = F. \quad (7)$$

For two such options, we have the following futures option version of American Put Call Symmetry:

$$\frac{C(F, K_c)}{\sqrt{K_c}} = \frac{P(F, K_p)}{\sqrt{K_p}}. \quad (8)$$

When the two strikes are equal, we obtain a generalization to American options of the standard result that European options struck at-the-money forward have the same value. However, we should caution the reader that this generalization rests on the assumed symmetry of volatility.

In contrast to (5), equation (8) holds for two options in the same economy. Thus, given the symmetry condition (2) on the volatility, any violation of (8) implies the existence of an arbitrage opportunity, independent of the specific form of the volatility function. Note that while the *existence* of this arbitrage is independent of the specific form, the number of futures contracts opened in order to exploit the opportunity

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<sup>8</sup>Similarly, when the dividend yield is positive, lowering the interest rate promotes early exercise of American calls and lowering the dividend yield promotes early exercise of American puts.

would depend on the specific form of the volatility function. Nonetheless, the result may be used by a researcher to test for the efficiency of a futures option market in detecting arbitrage opportunities, under the usual assumption that market participants know the volatility function.

The futures version of American Put Call Symmetry may be used to generate static replicating portfolios for certain American<sup>9</sup> exotic options. For example, consider the valuation of a down-and-in American call struck above its in-barrier. If the underlying futures price touches the barrier  $H$ , then (8) implies that the nascent American call has the same value at this time as some American puts:

$$C(H, K_c) = \sqrt{\frac{K_c}{K_p}} P(H, K_p), \quad (9)$$

where  $\sqrt{K_c K_p} = H$ . Since  $K_c > H$  by assumption, the puts are struck below the barrier implying no payoff in the event that the futures price remains above the barrier over the in-option's life. Thus, a static position in  $\frac{K_c}{K_p}$  American puts struck at  $K_p = \frac{H^2}{K_c}$  which is held until the earlier of the first passage time to the barrier and maturity will replicate the payoffs to a down-and-in American call. Absence of arbitrage implies that the down-and-in American call is initially valued at the cost of purchasing these puts:

$$DIAC(F, K_c; H) = \frac{K_c}{H} P\left(F, \frac{H^2}{K_c}\right), F > H, K_c > H.$$

The implications of American Put Call Symmetry for critical futures prices is:

$$\sqrt{\bar{F}(K_c) \underline{F}(K_p)} = \bar{K}. \quad (10)$$

In words, for any futures price, the geometric mean of the two critical prices is the geometric mean of the two strikes .

## IV Summary and Future Research

We derived a simple relationship between the values and exercise boundaries of American calls and puts with the same moneyness. The result holds in both the Black-Scholes/Merton model, and in a more general diffusion setting in which the volatility depends on the level of the underlying in a log-symmetric manner. The results may be used to identify or verify the value and exercise boundary of one type of option, given

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<sup>9</sup>See Carr, Ellis, and Gupta[5] for the corresponding results for European options.

the corresponding result for the other. For futures options, the results may also be used by a researcher to test for market efficiency.

Future research should focus on developing alternative sufficient conditions for the symmetry result. For example, we anticipate that similar results hold when the underlying price can jump so long as the jump distribution is log-symmetric. We also anticipate that similar results exist for many exotic options. For example, an American down-and-out call may be related to an American up-and-out put so long as the strikes and barriers have the same geometric mean. In the interests of brevity, these extensions are left for future research.

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## Appendix: Proof of American Put Call Symmetry

Let  $C(S, \tau; K_c, \delta, r)$  denote the American call value as a function of the underlying security price  $S > 0$  and the time to maturity  $\tau > 0$ . The notation also explicitly indicates the dependence on the strike price  $K_c > 0$ , the dividend yield  $\delta > 0$ , and the riskfree rate  $r > 0$ . In contrast, the notation suppresses the dependence on the current time  $t$  and the volatility function  $\sigma(S, t)$ . Similarly, let  $P(S, \tau; K_p, \delta, r)$  denote the American put value as a function of the positive independent variables  $S$  and  $\tau$ , and the positive parameters  $K_p$ ,  $\delta$ , and  $r$ . Consider an American call and put with the same time to maturity  $T$  and the same underlying asset, whose volatility satisfies the log-symmetry condition (2). Note that the calendar times at which we are comparing values need not be the same. Even if they are the same, the underlying spot prices at these times  $S_c$  and  $S_p$  need not be the same. This appendix proves the following American Put Call Symmetry for values:

$$\frac{C(S_c, T; K_c, \delta, r)}{\sqrt{S_c K_c}} = \frac{P(S_p, T; K_p, r, \delta)}{\sqrt{S_p K_p}}, \quad (11)$$

where the two options have the same moneyness:

$$\frac{S_c}{K_c} = \frac{K_p}{S_p}. \quad (12)$$

Let  $\bar{S}(\tau; K_c, \delta, r)$  and  $\underline{S}(\tau; K_p, \delta, r)$  denote respective American call and put exercise boundaries as functions of the time to maturity  $\tau$  and the parameters  $K_c$  and  $K_p$ ,  $\delta$  and  $r$ . Given two options with the same time to maturity  $T$  and the log-symmetry condition (2), this appendix also proves the following American Put Call Symmetry for these boundaries:

$$\sqrt{\bar{S}(T; K_c, \delta, r)\underline{S}(T; K_p, r, \delta)} = \sqrt{K_c K_p}. \quad (13)$$

Note that (11) holds for *any* pair of positive levels  $S_c$  and  $S_p$  satisfying (12). We will first prove that (11) holds for alive American options and then prove that it also holds for dead (exercised) values.

It is well-known that an alive American call's value  $C(S, \tau; K_c, \delta, r)$  and its exercise boundary  $\bar{S}(\tau; K_c, \delta, r)$  uniquely solve a boundary value problem (BVP) consisting of the following p.d.e.:

$$\frac{v^2(S/\bar{K}, \tau)S^2}{2} \frac{\partial^2 C}{\partial S^2}(S, \tau) + (r - \delta)S \frac{\partial C}{\partial S}(S, \tau) - rC(S, \tau) = \frac{\partial C}{\partial \tau}(S, \tau), \quad S \in (0, \bar{S}(\tau)), \tau \in (0, T], \quad (14)$$

and the following boundary conditions:

$$\lim_{\tau \downarrow 0} C(S, \tau) = \max[0, S - K_c], \quad S \in (0, \bar{S}(0))$$

$$\lim_{S \downarrow 0} C(S, \tau) = 0, \quad \tau \in (0, T]$$

$$\lim_{S \uparrow \bar{S}(\tau)} C(S, \tau) = \bar{S}(\tau) - K_c, \quad \tau \in (0, T]$$

$$\lim_{S \uparrow \bar{S}(\tau)} \frac{\partial C}{\partial S}(S, \tau) = 1, \quad \tau \in (0, T].$$

Consider the change of variables:

$$u_c(x_c, \tau) = \frac{C(S, \tau; K_c, \delta, r)}{\sqrt{SK_c}} \text{ where } x_c = \ln\left(\frac{S}{K_c}\right), \quad (15)$$

and:

$$\bar{x}_c(\tau) = \ln\left(\frac{\bar{S}(\tau; K_c, \delta, r)}{K_c}\right). \quad (16)$$

Then  $u_c(x_c, \tau)$  and  $\bar{x}_c(\tau)$  uniquely solve the BVP consisting of the following p.d.e.:

$$\frac{1}{2}v^2 \left( \sqrt{\frac{K_c}{K_p}} e^{x_c}, \tau \right) \frac{\partial^2 u_c}{\partial x_c^2}(x_c, \tau) + (r - \delta) \frac{\partial u_c}{\partial x_c}(x_c, \tau) - \frac{1}{2} \left[ r + \delta + \frac{1}{4}v^2 \left( \sqrt{\frac{K_c}{K_p}} e^{x_c}, \tau \right) \right] u_c(x_c, \tau) = \frac{\partial u_c}{\partial \tau}(x_c, \tau), \quad (17)$$

for  $x_c \in (-\infty, \bar{x}_c(\tau)), \tau \in (0, T]$  and the following boundary conditions:

$$\lim_{\tau \downarrow 0} u_c(x_c, \tau) = \max[0, e^{x_c/2} - e^{-x_c/2}], \quad x_c \in (-\infty, \bar{x}_c(0))$$

$$\lim_{x_c \downarrow -\infty} u_c(x_c, \tau) = 0, \quad \tau \in (0, T]$$

$$\lim_{x_c \uparrow \bar{x}_c(\tau)} u_c(x_c, \tau) = e^{\bar{x}_c(\tau)/2} - e^{-\bar{x}_c(\tau)/2}, \quad \tau \in (0, T]$$

$$\lim_{x_c \uparrow \bar{x}_c(\tau)} \frac{\partial u_c}{\partial x_c}(x_c, \tau) = \frac{e^{\bar{x}_c(\tau)/2} + e^{-\bar{x}_c(\tau)/2}}{2}, \quad \tau \in (0, T].$$

An alive American put's value  $P(S, \tau; K_p, \delta, r)$  solves the same p.d.e. (14) as alive American calls, but with a different domain,  $S > \underline{S}(\tau; K_p, \delta, r), \tau \in (0, T]$ , and different boundary conditions:

$$\lim_{\tau \downarrow 0} P(S, \tau) = \max[0, K_p - S], \quad S > \underline{S}(0) \quad (18)$$

$$\lim_{S \uparrow \infty} P(S, \tau) = 0, \quad \tau \in (0, T] \quad (19)$$

$$\lim_{S \downarrow \underline{S}(\tau)} P(S, \tau) = K_p - \underline{S}(\tau), \quad \tau \in (0, T] \quad (20)$$

$$\lim_{S \downarrow \underline{S}(\tau)} \frac{\partial P}{\partial S}(S, \tau) = -1, \quad \tau \in (0, T]. \quad (21)$$

It follows that  $P(S, \tau; K_p, r, \delta)$  and  $\underline{S}(\tau; K_p, r, \delta)$  uniquely solve the following p.d.e.:

$$\frac{v^2(S/\bar{K}, \tau)S^2}{2} \frac{\partial^2 P}{\partial S^2}(S, \tau) - (r - \delta)S \frac{\partial P}{\partial S}(S, \tau) - \delta P(S, \tau) = \frac{\partial P}{\partial \tau}(S, \tau), \quad S \in (\underline{S}(\tau), \infty), \tau \in (0, T], \quad (22)$$

subject to the boundary conditions in (18) to (21).

Consider the change of variables:

$$u_p(x_p, \tau) = \frac{P(S, \tau; K_p, r, \delta)}{\sqrt{SK_p}}, \quad (23)$$

where:

$$x_p = \ln \left( \frac{K_p}{S} \right), \quad (24)$$

and:

$$\underline{x}_p(\tau) = \ln \left( \frac{K_p}{\underline{S}(\tau; K_p, r, \delta)} \right). \quad (25)$$

For future use, note that from (23) and (24):

$$u_p(x_p, \tau) = \frac{P(K_p e^{-x_p}, \tau; K_p, r, \delta)}{K_p e^{-x_p/2}} \text{ for all } x_p \in (-\infty, \underline{x}_p(\tau)). \quad (26)$$

The functions  $u_p(x_p, \tau)$  and  $\underline{x}_p(\tau)$  uniquely solve the BVP consisting of the following p.d.e.:

$$\frac{1}{2}v^2 \left( \sqrt{\frac{K_p}{K_c}} e^{-x_p}, \tau \right) \frac{\partial^2 u_p}{\partial x_p^2}(x_p, \tau) + (r - \delta) \frac{\partial u_p}{\partial x_p}(x_p, \tau) - \frac{1}{2} \left[ r + \delta + \frac{1}{4}v^2 \left( \sqrt{\frac{K_p}{K_c}} e^{-x_p}, \tau \right) \right] u_p(x_p, \tau) = \frac{\partial u_p}{\partial \tau}(x_p, \tau), \quad (27)$$

for  $x_p \in (-\infty, \underline{x}_p(\tau))$ ,  $\tau \in (0, T]$ , and the following boundary conditions:

$$\lim_{\tau \downarrow 0} u_p(x_p, \tau) = \max[0, e^{x_p/2} - e^{-x_p/2}], \quad x_p \in (-\infty, \underline{x}_p(\tau))$$

$$\lim_{x_p \downarrow -\infty} u_p(x_p, \tau) = 0, \quad \tau \in (0, T]$$

$$\lim_{x_p \uparrow \underline{x}_p(\tau)} u_p(x_p, \tau) = e^{\underline{x}_p(\tau)/2} - e^{-\underline{x}_p(\tau)/2}, \quad \tau \in (0, T]$$

$$\lim_{x_p \uparrow \underline{x}_p(\tau)} \frac{\partial u_p}{\partial x_p}(x_p, \tau) = \frac{e^{-\underline{x}_p(\tau)/2} + e^{\underline{x}_p(\tau)/2}}{2}, \quad \tau \in (0, T].$$

The log-symmetry condition (2), implies that  $v^2\left(\sqrt{\frac{K_p}{K_c}}e^{-x_p}, \tau\right) = v^2\left(\sqrt{\frac{K_c}{K_p}}e^{x_p}, \tau\right)$ . Consequently, the pair  $u_p(x_p, \tau), \underline{x}_p(\tau)$  solves the same BVP (17) as the pair  $u_c(x_c, \tau), \bar{x}_c(\tau)$  and are therefore equal:

$$\bar{x}_c(\tau) = \underline{x}_p(\tau), \quad \tau \in (0, T] \quad (28)$$

$$u_c(x_c, \tau) = u_p(x_p, \tau), \quad x_c, x_p \in (-\infty, \bar{x}_c(\tau)), \tau \in (0, T]. \quad (29)$$

In the original call valuation problem (14), suppose that when the time to maturity is  $T$ , the spot price  $S$  is at some level  $S_c$  such that the call is optimally held alive:

$$S_c < \bar{S}(T; K_c, \delta, r). \quad (30)$$

Then (15) implies that  $x_c$  is fixed at  $\ln\left(\frac{S_c}{K_c}\right) < \bar{x}_c(T)$  from (16). Similarly, in the original put valuation problem distinguished by the boundary conditions (18) to (21), suppose that when the time to maturity is  $T$ , the spot price  $S$  is at  $S_p = \frac{K_c K_p}{S_c}$  from (12). Then (24) implies that  $x_p$  is fixed at the same level as  $x_c$ :

$$x_p = \ln\left(\frac{K_p}{S_p}\right) = \ln\left(\frac{S_c}{K_c}\right). \quad (31)$$

Consequently, from (29) with  $\tau = T$ , we have:

$$u_c\left(\ln\left(\frac{S_c}{K_c}\right), T; \delta, r\right) = u_p\left(\ln\left(\frac{S_c}{K_c}\right), T; r, \delta\right). \quad (32)$$

Substituting  $x_p = \ln\left(\frac{S_c}{K_c}\right)$  in (26) implies that:

$$u_p\left(\ln\left(\frac{S_c}{K_c}\right), T\right) = \sqrt{\frac{S_c}{K_c}} \frac{P\left(\frac{K_p K_c}{S_c}, T; K_p, r, \delta\right)}{K_p} = \frac{P(S_p, T; K_p, r, \delta)}{\sqrt{S_p K_p}},$$

from (12). Substituting this result and (15) into (32) shows that (11) holds for alive American values.

Similarly, substituting (16) and (25) into (28) implies:

$$\ln\left(\frac{\bar{S}(T; K_c, \delta, r)}{K_c}\right) = \ln\left(\frac{K_p}{\underline{S}(T; K_p, r, \delta)}\right). \quad (33)$$

Exponentiating and re-arranging gives the desired result (13) for exercise boundaries.

To show that (11) also holds for dead (exercised) values, note that these values satisfy:

$$C(S, \tau; K_c, \delta, r) = S - K_c \quad \text{for } S \geq \bar{S}(\tau; K_c, \delta, r) \quad (34)$$

$$P(S, \tau; K_p, \delta, r) = K_p - S \quad \text{for } S \leq \underline{S}(\tau; K_p, \delta, r). \quad (35)$$

Suppose that when the time to maturity is  $T$ , the spot price is at some level  $S_c$  at which the call is optimally exercised:

$$S_c \geq \bar{S}(T; K_c, \delta, r). \quad (36)$$

From (34), the left hand side of (11) is given by:

$$\frac{C(S_c, T; K_c, \delta, r)}{\sqrt{K_c S_c}} = \sqrt{\frac{S_c}{K_c}} - \sqrt{\frac{K_c}{S_c}}.$$

Switching  $\delta$  and  $r$  in (35) implies:

$$P(S, \tau; K_p, r, \delta) = K_p - S \text{ for } S \leq \underline{S}(\tau; K_p, r, \delta). \quad (37)$$

For  $S_p = \frac{K_c K_p}{S_c}$ , (36) and (13) imply:

$$S_p \leq \underline{S}(T; K_p, r, \delta). \quad (38)$$

Thus, from (37), the right hand side of (11) is given by:

$$\begin{aligned} \frac{P(S_p, T; K_p, r, \delta)}{\sqrt{K_p S_p}} &= \sqrt{\frac{K_p}{S_p}} - \sqrt{\frac{S_p}{K_p}} \\ &= \sqrt{\frac{S_c}{K_c}} - \sqrt{\frac{K_c}{S_c}}, \text{ from (12)} \\ &= \frac{C(S_c, T; K_c, \delta, r)}{\sqrt{K_c S_c}}, \end{aligned}$$

as desired. **Q.E.D.**