

Randomization and the American Put

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Introduction

- Since the discovery of the Black/Scholes European option formulas in 1973, researchers have labored to uncover the corresponding exact solution for the value of an American put.
- Despite the profusion of papers on the subject, no completely satisfactory analytic solution has been found.
- However, one can decompose the American put into the corresponding European put and an early exercise premium (see Kim, Jacka, or Carr, Jarrow, and Myneni).
- In this representation, the early exercise premium is expressed in terms of the unknown early exercise boundary.

Analytic Valuation and the Exercise Boundary

- Although the American put can be explicitly valued in terms of the exercise boundary, the exercise boundary itself must be determined numerically.
- This paper presents a new approach for valuing options called “randomization”, which can be applied to a wide variety of contingent claims.
- When applied to American options, the randomization approach yields an analytic approximation expressing the American option value explicitly in terms of the exercise boundary.
- The main advantage of this approach when compared to most other American option approximations is that it also generates an explicit analytic approximation of the exercise boundary.
- The put value formula does not use any special functions such as the normal distribution function. This speeds up the valuation of American puts significantly as we will demonstrate.

Overview of Talk

1. Valuation of American puts in the Black Scholes model
2. Some analytic approximations
3. Some philosophy about modeling
4. Randomization
5. Implementation
6. Speed/Accuracy Comparison

American Put Valuation in the Black Scholes Model

- Figure 1 graphs the Black-Scholes value of an American put against the current stock price.
- The critical stock price is the highest stock price at which the alive value matches the exercise value.
- It can be shown that the slope (delta) is continuous at the critical stock price, while the curvature (gamma) is not.
- As time evolves, the alive American put value falls, while the exercise value remains constant. The passage of time therefore raises the critical stock price (see Figure 2).
- The initial put value P_0 is given by taking the supremum over all exercise boundaries of the expected discounted payoff at exercise:

$$P_0 = \sup_{B(t); t \in [0, T]} E_{0, S} \{ e^{-r(\tau_B \wedge T)} [K - S_{\tau_B \wedge T}]^+ \},$$

where τ_B is the first passage time from S to an exercise boundary $B(t)$, $t \in [0, T]$.

- Analytic lower bounds to American option values can be obtained by restricting the form of the exercise boundaries considered.

Some Analytic Approximations

- Geske and Johnson (GJ) effectively consider boundaries which are zero except at a set of times at which exercise can occur. In the simplest case, exercise can either occur immediately or at maturity. In this case, the American put value is just the European value above the critical price H and is the exercise value otherwise:

$$P_{gj} = \begin{cases} PV(K)N(c_0 + c_1 \ln(K/S)) - SN(c_2 + c_3 \ln(K/S)), & \text{if } S > H, \\ K - S & \text{if } S < H, \end{cases}$$

where the c_i are constants.

- A similar approach considered by Broadie and Detemple (BD) pretends that the exercise boundary is flat. The American put is then approximated by a down-and-out put with a rebate equal to the exercise value:

$$\begin{aligned} P_{bd} = & PV(K)[N(c_0 + c_1 \ln(K/S)) - N(c_2 + c_3 \ln(H/S))] \\ & - PV(K)(H/S)^{p_1}[N(c_4 + c_5 \ln(SK/H^2)) - N(c_6 + c_7 \ln(S/H))] \\ & - S[N(c_8 + c_9 \ln(K/S)) - N(c_{10} + c_{11} \ln(H/S))] \\ & + H(H/S)^{p_2}[N(c_{12} + c_{13} \ln(SK/H^2)) - N(c_{14} + c_{15} \ln(H/S))] \\ & + (K - H)(H/S)^{p_3} N(c_{16} + c_{17} \ln(SK/H^2)) \\ & + (K - H)(H/S)^{p_4} N(c_{18} + c_{19} \ln(H/S)), \end{aligned}$$

where the powers p_i are also constants.

- In both cases, maximizing over H yields a first order condition which must be solved numerically.
- Generalizing GJ to multiple exercise points or BD to piecewise constant barrier levels leads to valuation formulas involving multivariate normal distribution functions.

- Note also that in both approaches, the slope continuity condition does not hold at any time step.

Some Philosophy about Modelling

- At this juncture, one cannot resist the opportunity to have a little fun at a competitor's expense.
- In his paper “Valuing Models and Modeling Value”, Emanuel Derman of Goldman Sachs writes:

Certainty is Easier than Uncertainty

The most amenable problems are those that involve no uncertainty...

If you must live with modeling uncertainty, a model with only one uncertain factor is best...

So move to two or more factors only when you can get no farther with one.

Randomization

- The randomization approach assumes quite incorrectly that a second random factor is the derivative security's expiration date, which is always assumed to be independent of the stock price.
- Thus, the owner of this random maturity American put can exercise at any time up to and including some random maturity date.
- In order to differentiate random maturity American options from standard American ones, we christen this option as “Canadian”.
- Thus, a Canadian option is like an American one, except that a Canadian option doesn't know its own maturity.

First Approximation

- We initially assume that the maturity date of the Canadian put is exponentially distributed with a mean equal to the actual maturity T :

$$\text{Prob}\{\tau \in dt\} = \lambda e^{-\lambda t} dt,$$

where $\lambda = \frac{1}{T}$.

- In effect, the Canadian put can be exercised at any time up to and including the first jump time of a Poisson process.
- Because of the memoryless property of the exponential distribution, the exercise boundary of a Canadian put is completely flat!
- See Figure 3 for a graph of the optimal flat exercise boundary \underline{S}_1 for a Canadian put with a mean maturity of one year. The graph reflects a realized maturity of 1.23 years, at which time the Canadian option value jumps down to intrinsic value $(K - S)^+$.
- Thus, one can think of the pent up time decay of the option as being released at the jump time. This release causes the exercise boundary to jump up from \underline{S}_1 to K , crudely approximating the behavior of the true exercise boundary graphed in Figure 2.

Exponential Maturity Put and Barrier Options

- Recall the formula for the value P_0 of a fixed maturity American put:

$$P_0 = \sup_{B(t); t \in [0, T]} E_{0, S} \{ e^{-r(\tau_B \wedge T)} [K - S_{\tau_B \wedge T}]^+ \},$$

where τ_B is the first passage time from S to an exercise boundary $B(t)$, $t \in [0, T]$.

- The analogous expression for a Canadian put with an exponentially distributed maturity τ is:

$$P^{(1)}(S) = \sup_B E_{0, S} \{ e^{-r(\tau_B \wedge \tau)} [K - S_{\tau_B \wedge \tau}]^+ \}.$$

- Since the exercise boundary is flat, our Canadian put is just the maximized value of a down-and-out put with a random maturity and a fixed rebate given by the difference between the strike K and the flat barrier B :

$$P^{(1)}(S) = \sup_B D^{(1)}(S; B).$$

Valuation of Exponential Maturity Put

- The exponential maturity assumption gives very simple valuation formulas. For example, the in-the-money value of a down-and-out put has the simple form:

$$D^{(1)}(S; B) = c_0 + c_1S + c_2S^{p_1} + c_3S^{p_2}, \quad S \in (B, K),$$

where the constants c_i depend on the barrier B .

- The simple form arises because valuing a derivative security with an exponentially distributed maturity is in fact equivalent to taking the Laplace (Carson) transform of the fixed maturity value:

$$D^{(1)}(S) = \lambda \int_0^\infty e^{-\lambda t} D(0, S; t) dt,$$

where $D(0, S; T)$ is the initial value of a down-and-out put with fixed maturity T .

- Recall that the Canadian put value $P^{(1)}(S)$ is related to the random maturity down-and-out put value $D^{(1)}(S; B)$ by:

$$P^{(1)}(S) = \sup_B D^{(1)}(S; B).$$

- Optimizing the top equation over the barrier gives an explicit expression for the critical stock price \underline{S}_1 .
- The Canadian put value has the same simple form as in the top equation. See Figure 4 for a graph of this first step solution of the put value against the stock price.

Better Approximation

- While these first approximations to the put value and exercise boundary are simple and explicit, numerical valuation indicates substantial approximation error.
- In particular, since the value of an American put is a concave function of its maturity (see Figure 5), randomizing the maturity leads to undervaluation by Jensen's inequality.
- The error can be reduced by changing the distribution governing maturity to one with lower variance.
- A simple and standard way to reduce variance is to not put all our eggs in one basket.

Gamma Maturity

- Suppose that we break up the Canadian put's life into two half lives. If each half-life has mean $T/2$, then the sum of the two half-lives has mean T .
- Furthermore, if the two half lives are independent and exponentially distributed, then the whole life is gamma distributed with only half the variance of the exponential maturity case.
- More generally, the sum of n independent exponential time steps, each of mean T/n will have the same mean of T but only $\frac{1}{n}$ the variance of the original exponential maturity. Thus, if we can approximate the American put value by assuming that the maturity is gamma distributed with mean T , then our approximation should be good for n large.
- In fact, as n approaches infinity, the density function describing our random maturity approaches a delta function centered at T . Figure 6 shows three gamma density functions, with each corresponding to a maturity of mean $T = 1$ year, and with variances of 1, $1/2$, and $1/3$ respectively. The densities are converging to a Dirac delta function centered at $T = 1$ year.
- As we break up the Canadian put's life into more and more subperiods, the Canadian option value should converge to the desired American value.

Valuing Canadian Puts with Gamma Maturity

- For simplicity, we start with the simplest case of two half lives, each i.i.d. exponential with mean $\frac{T}{2}$. In other words, our second approximation values a put which can be exercised at any time up to and including the second jump time of a Poisson process.
- Working backwards from expiration as usual, the value of the Canadian put one (random) time step before maturity is determined as before.
- The value of the Canadian put at the valuation date can also be solved as a single period problem, using the optimized value determined above as the payoff at the end of the first (random) step.
- Figure 7 shows a graph of this second step solution against the stock price for a realization in which the first jump time occurred at 0.53, while the put matured at the second jump time of 0.93. Figure 8 graphs our solution against the stock price and calendar time.
- Our approximation is in fact the optimized value of the (Carson) Laplace transform of a fixed maturity down-and-out put with a step function barrier (see Figure 9) and rebates give by the difference between strike and barrier.
- The height of the stairs have been chosen to maximize the value of this put.

The n step Formula

- The paper gives an explicit formula for the Canadian put value using an arbitrary number n of exponential time steps.
- This formula can be derived using risk-neutral valuation. Madan and Seneta have shown that a Brownian motion evaluated at a gamma time has the same distribution as the difference of two independent gamma random variables.
- Importantly, in our setting, the distribution function integrates out to give a simple closed form (in contrast to the normal distribution function.)
- Unfortunately, the formula involves a triple sum. For n large, this would be computationally intensive.
- Fortunately, Richardson extrapolation can be used to enhance convergence.

Richardson Extrapolation

- Figures 10 and 11 illustrate the idea behind a 3 step extrapolation.
- Marchuk and Shaidurov prove that an N point Richardson extrapolation is the following weighted average of our N put values:

$$P^{1:N}(S) \equiv \sum_{n=1}^N \frac{(-1)^{N-n} n^N}{n!(N-n)!} P^{(n)}(S).$$

- The table below shows that by using Richardson extrapolation, accurate American put values can be obtained using just a few time steps.

Convergence with and without Richardson Extrapolation

$$S = 100, K = 100, T = 1, r = 0.1, \delta = 0, \sigma = 0.3$$

Number of Steps n or Points N	Unextrapolated Put Value $P^{(n)}$	Extrapolated Put Value $P^{1:N}$
1	7.0405	7.0405
2	7.6175	8.1946
3	7.8353	8.3089
4	7.9505	8.3257
5	8.0220	8.3311
6	8.0709	8.3333
7	8.1065	8.3345
8	8.1335	8.3353
9	8.1548	8.3358
10	8.1720	8.3362
11	8.1862	8.3365
12	8.1981	8.3367
13	8.2082	8.3369
14	8.2169	8.3370
15	8.2246	8.3371

Speed-Accuracy Comparison

- We compared most of the existing approaches for valuing American puts in terms of speed and accuracy.
- The randomization approach (labelled ML for Method of Lines) is on the speed-accuracy frontier, along with other analytic approximations by Johnson, Omberg, and Broadie and Detemple.
- Broadie and Detemple (RFS forthcoming) also compare the speed and accuracy of most approaches and obtain broadly similar results.

Other Extensions

- The paper also extends our randomization approach to American calls on dividend paying stocks.
- Although not in the paper, maturity randomization has been successfully applied to single and double time-varying barrier options and to lookback options.
- In joint work with Yuri Boykov (now at Carnegie Mellon), the approach has also been applied to Asian options.
- Maturity randomization has also been extended to allow for volatility smiles.
- One can alternatively randomize spatial variables, such as the initial underlying, the strike price, or the barrier. This has been left for future play!