Robust Replication of Volatility Derivatives

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April, 2003

We thank Peter Friz, Alireza Javaheri, Dilip Madan, Jeremy Staum, and Liuren Wu for helpful comments

Postscript/PDF files of these overheads can be downloaded from:
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`www.math.nyu.edu\research\carrp\papers`

Columbia University, Oxford University, Risk Europe, Mannheim University

Introduction

- It is widely recognized that delta-hedged options positions are a vehicle for trading volatility.
- In particular, under relatively weak conditions, a static position in European options maturing at T can be combined with dynamic trading in the underlying over $[t_0, T]$ to create a contract paying the realized variance over $[t_0, T]$:

$$\frac{1}{T}\int_{t_0}^{T} \sigma_t^2 dt \equiv \lim_{\triangle t \downarrow 0} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)^2 .$$

- The main conditions are frictionless markets, continuous positive price paths, continuous path monitoring, continuous trading in the underlying, and a continuum of option strikes. No assumptions are made regarding the dynamics of volatility.
- The ability to create the realized variance without requiring a model for volatility dynamics has been one of the reasons behind the emergence of an active over-the-counter market in variance swaps.
- See Neuberger (1990), Dupire(1992), Carr and Madan (1998) and Derman et. al. (1999) for more information on variance swaps.

Example of a Variance Swap

Bank of America Securities LLC **Indicative Terms**

(For Discussion Only) October 8, 1999

S&P 500 Index Realized Variance Swap

Equity Payer: Bank of America, N.A. ("BofA")

Equity Receiver: Merrill Lynch International

Trade Date: October 8, 1999

Maturity Date: May 7, 2003

Underlying Index: The Standard & Poor's 500 Composite Stock Price Index

Equity Calculations:

(a) "Initial Price" means 0.305

(b) "Final Price" means the actual realized index Variance defined in accordance with the following formula and definition:

$$\sqrt{\frac{\sum_{i=1}^{n-1}\left[\ln\left(\frac{P_{i+1}}{P_i}\right)\right]^2}{n-2}} \times \sqrt{52}$$

(c) "Natural Logarithm" means for any Daily Quotient, as determined by the Calculation Agent, the exponential number which equates 2.718281828 to such Daily Quotient;

(d) "n" means the total number of Valuation Dates;

(e) "P_i" means the closing level of the index on the ith valuation date (i.e.: P_1 is the closing level of the index on October 6, 1999, P_2 is the closing level of the index on the first Wednesday that is an Exchange Business Day following the Trade Date and P_n is the closing level of the index on the Final Valuation Date.

(f) "Valuation Dates" means, commencing on October 6, 1999, and each Wednesday thereafter up to and including the Final Valuation Date and if any such date is not an Exchange Business Day, the next following day that is an Exchange Business Day, subject to the Market Disruption Events as set forth in the 1996 ISDA Equity Derivatives Definitions.

(g) "$\sum_{t=1}^{n}$" means the summation from $t=1$ to $t=n$.

Notional: 111,230,666

Equity Payment: Notional * [Final Price^2 – Initial Price^2]
If the Equity Payment is a positive value, then the Equity Payer pays the Equity Receiver this value.
If the Equity Payment is a negative value, then the Equity Receiver pays the Equity Payer the absolute value of this number.

Credit Terms: n/a

Nonlinear Functions of Realized Variance

- Recall that the floating part of the payoff on a variance swap is:

$$\frac{1}{T}\int_{t_0}^{T} \sigma_t^2 dt \equiv \lim_{\triangle t \downarrow 0} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)^2 .$$

- In addition to variance swaps, there is also an active market for volatility swaps, i.e. swaps whose floating part is the square root of the realized variance.
- However, in contrast to a variance swap, no one has yet developed a hedge which bypasses the specification of a stochastic process for volatility.
- In this talk, we assume all of the conditions that lead to the robust replication of variance swaps.
- We show that by further allowing dynamic trading in the options and by modelling the correlation between volatility and returns, we can synthesize practically any function of final price and the final realized variance defined above.
- In particular, we can synthesize volatility swaps and European options on realized variance or volatility.
- Our work refutes the widely held notion that dynamic replication requires process restrictions.

Example of a Vol Swap

Merrill Lynch

S&P 500 INDEX VOLATILITY SWAP

Indicative Terms and Conditions

As of November 6 1997

22 26.5 31~ 53.1

Swap Party A:	Merrill Lynch International (MLI)
Swap Party B:	Investor
Underlying Index:	Standard & Poor's 500 Index (SPX)
Notional Amount:	$50 million
Maturity Date:	1 year
Valuation:	Closing prices
Party A Final Payment:	Notional * Max [0, (Vol - ActVol)]
Party B Final Payment:	Notional * Max [0, (ActVol - Vol)]
Vol:	22 %
ActVol:	Actually realized index volatility defined according to the following formula:

$$\sqrt{\frac{\sum_{i=1}^{n-1}\left[\ln\left(\frac{P_{i+1}}{P_i}\right) - Avg\right]^2}{n-2}} \times \sqrt{252}$$

where

n = number of business days from the Trade Date up to and including the Maturity Date

Pi = daily closing price of S&P 500 Index on the ith business day starting on Trade Date (i = 1)

Pn = closing price of the index on the Maturity Date

$$\text{Avg} = \frac{1}{n-1}\ln\left(\frac{P_n}{P_1}\right)$$

252

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Overview

- A review of static hedging of path independent payoffs (Breeden and Litzenberger (1978)).
- A review of variance swaps and similar contracts.
- Synthesizing volatility derivatives when returns and volatility are independent.
- Modelling correlation between returns and volatility.

Static Hedging of Path Indep. Payoffs

- Appendix 1 proves that for any generalized function $f(S), S > 0$ and any expansion point $\kappa \geq 0$:

$$f(S) = f(\kappa) + f'(\kappa)(S - \kappa) + \int_{\kappa}^{\infty} f''(K)(S - K)^+ dK + \int_{0}^{\kappa} f''(K)(K - S)^+ dK.$$

- This decomposition may be interpreted as a Taylor series expansion with remainder of the final payoff $f(\cdot)$ about the expansion point κ.

- The first two terms give the tangent to the payoff at κ; the last two terms continuously bend this tangent so it conforms to the nonlinear payoff.

- The payoff from an arbitrary path-independent claim has been decomposed into the payoff from $f(\kappa)$ bonds, $f'(\kappa)$ forward contracts with delivery price κ, $f''(K)dK$ calls of all strikes $K > \kappa$, and $f''(K)dK$ puts of all strikes $K < \kappa$.

From Payoffs to Prices

- Recall the decomposition of the payoff function $f(S)$ into payoffs from bonds, forwards, and options:

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) \\ &\quad + \int_0^{\kappa} f''(K)(K-S)^+ dK + \int_{\kappa}^{\infty} f''(K)(S-K)^+ dK. \end{aligned}$$

- Assume the existence of a pure discount bond and *all* European options of maturity T.

- Assume no arbitrage and hence the existence of a martingale measure Q equivalent to the physical probability measure.

- Then the initial value $V_0[f]$ of the continuous payoff $f(\cdot)$ can be expressed in terms of the initial prices of the bond B_0, calls $C_0(K)$, and puts $P_0(K)$ respectively:

$$\begin{aligned} V_0[f] &= f(\kappa)B_0 + f'(\kappa)[C_0(\kappa) - P_0(\kappa)] \\ &\quad + \int_0^{\kappa} f''(K)P_0(K)dK + \int_{\kappa}^{\infty} f''(K)C_0(K)dK. \end{aligned}$$

- Note that we did not restrict the underlying price process in any way!

Example 1: Power Plays

- Consider the risk-neutral moment generating function of X_T:

$$M_0(p,T) \equiv E_0^Q e^{pX_T}, \quad p \in \Re.$$

- Since $X_T \equiv \ln(F_T/F_0)$, $M_0(p,T)$ is just the forward price of a claim whose payoff $e^{pX_T} = \left(\frac{F_T}{F_0}\right)^p$, i.e. a power of F_T.
- Taylor expand the function $f(F) \equiv \left(\frac{F}{F_0}\right)^p$ about $F = F_0$:

$$\left(\frac{F_T}{F_0}\right)^p = 1 + \tfrac{p}{F_0}(F_T - F_0) + \int_0^{F_0} p(p-1)\left(\frac{K}{F_0}\right)^{p-2}(K-F_T)^+ dK + \int_{F_0}^{\infty} p(p-1)\left(\frac{K}{F_0}\right)^{p-2}(F_T-K)^+ dK.$$

- Hence the initial value is given by:

$$V_0\left[\left(\frac{F_T}{F_0}\right)^p\right] = B_0 + \int_0^{F_0} p(p-1)\left(\frac{K}{F_0}\right)^{p-2} P_0(K)dK + \int_{F_0}^{\infty} p(p-1)\left(\frac{K}{F_0}\right)^{p-2} C_0(K)dK.$$

- The RHS is the initial cost of a static position in bonds and OTM options maturing at T. Its forward price is the MGF:

$$M_0(p,T) = \frac{V_0\left[\left(\frac{F_T}{F_0}\right)^p\right]}{B_0}.$$

Example 2: Complex Exponential

- For p real, the MGF $M_0(p,T)$ of $X_T \equiv \ln(F_T/F_0)$ might not exist. Now suppose that p is complex:

$$M_0(p,T) \equiv E_0^Q e^{pX_T}, \quad p \in C.$$

- Let $p_r \equiv \mathbf{Re}(p)$ and $p_i \equiv \mathbf{Im}(p)$, so $p = p_r + ip_i$. From Euler:

$$\begin{aligned} e^{pX_T} &= e^{p_r X_T}\cos(p_i X_T) + ie^{p_r X_T}\sin(p_i X_T) \\ &= \left(\frac{F_T}{F_0}\right)^{p_r}\cos(p_i \ln(F_T/F_0)) + i\left(\frac{F_T}{F_0}\right)^{p_r}\sin(p_i \ln(F_T/F_0)). \end{aligned}$$

- We can separately create the real part $\left(\frac{F_T}{F_0}\right)^{p_r}\cos(p_i \ln(F_T/F_0))$ and the imaginary part $\left(\frac{F_T}{F_0}\right)^{p_r}\sin(p_i \ln(F_T/F_0))$ using bonds and options.
- We refer to the real part of the payoff as the cosine claim and we refer to the imaginary part of the payoff as the sine claim.
- The initial cost of creating the payoff on the cosine claim is the real part of the complex initial value. Likewise, the initial cost of creating the sine claim is the imaginary part of the complex initial value.
- We will later find it useful to allow complex payoffs and values.

Eg. 3: Continuously Compounded Return

- Finally, suppose the payoff of interest is just $X_T \equiv \ln(F_T/F_0)$.
- Taylor expand the function $f(F) \equiv \ln(F/F_0)$ about $F = F_0$:

$$\ln\left(\frac{F_T}{F_0}\right) = \tfrac{1}{F_0}(F_T - F_0) - \int_0^{F_0} \frac{1}{K^2}(K - F_T)^+ dK - \int_{F_0}^{\infty} \frac{1}{K^2}(F_T - K)^+ dK.$$

- Hence the initial value is given by:

$$V_0\left[\ln\left(\frac{F}{F_0}\right)\right] = -\int_0^{F_0} \frac{1}{K^2} P_0(K) dK - \int_{F_0}^{\infty} \frac{1}{K^2} C_0(K) dK.$$

- The RHS is just the initial cost of a static position in OTM options maturing at T.
- A variance swap is a contract paying $\int_{t_0}^{T} \sigma_{t-}^2 dt - \bar{v}$ at T where the fixed payment $\bar{v}$ is chosen so that the swap has zero cost to enter.
- The ability to create the log price relative using options is the key component in synthesizing a variance swap, as we illustrate next.

(Realized) Variance Swaps

- Now assume deterministic interest rates, no jumps in the futures price F, and continuous marking-to-market.

- No arbitrage implies the existence of a risk-neutral probability measure Q under which the futures price dynamics are:
$$\frac{dF_t}{F_t} = \sigma_{t-} dW_t, \quad t \in [t_0, T],$$
where σ_{t-} is the pre-jump volatility level at time t.

- Let $X_t \equiv \ln\left(\frac{F_t}{F_0}\right)$ be the return over $[t_0, t]$. Itô's lemma implies:
$$X_T = \int_{t_0}^{T} \frac{1}{F_t} dF_t - \frac{1}{2} \int_{t_0}^{T} \sigma_{t-}^2 dt.$$

- Re-arranging implies that the realized variance is just the sum of the payoffs from a static options position and a dynamic trading strategy in futures:
$$\int_{t_0}^{T} \sigma_{t-}^2 dt = -2X_T + \int_{t_0}^{T} \frac{2}{F_t} dF_t.$$

- Since futures are costless, the fair fixed payment to charge on a variance swap is:
$$\frac{V_0\left[-2X_T\right]}{B_0} = \int_0^{F_0} \frac{2}{K^2} \frac{P_0(K)}{B_0} dK + \int_{F_0}^{\infty} \frac{2}{K^2} \frac{C_0(K)}{B_0} dK.$$

Localizing in Space

- We replicated the quadratic variation of returns $X_t \equiv \ln\left(\frac{F_t}{F_0}\right)$:
$$\langle X \rangle_T \equiv \int_{t_0}^{T} \sigma_{t-}^2 dt = -2\ln(F_T/F_0) + \int_{t_0}^{T} \frac{2}{F_t} dF_t.$$

- One can add and subtract the linear function $2\frac{F_T - F_0}{K}$:
$$\langle X \rangle_T = 2\left[\frac{F_T - F_0}{K} - \ln\left(\frac{F_T}{F_0}\right)\right] + 2\int_{t_0}^{T} \left[\frac{1}{F_t} - \frac{1}{K}\right] dF_t.$$

- If $K = F_0$, the first term is just twice the difference between the return compounded discretely versus continuously.

- More generally, the quadratic variation of returns realized when $F_t \in (K - \triangle K, K + \triangle K)$ can also be replicated:
$$\int_{t_0}^{T} \sigma_{t-}^2 1(F_t \in (K - \triangle K, K + \triangle K))dt$$
$$= 2\left[\frac{F_T - F_0}{K} + \frac{F_0}{\bar{\underline{F}}_0} - \frac{F_T}{\bar{\underline{F}}_T} - \ln\left(\frac{\bar{\underline{F}}_T}{K}\right)\right] + 2\int_{t_0}^{T} \left[\frac{1}{\bar{\underline{F}}_t} - \frac{1}{K}\right] dF_t,$$
where $\bar{\underline{F}}_t \equiv \max[K - \triangle K, \min(F_t, K + \triangle K)]$.

- Dividing by the width of the corridor and letting it approach zero, one can localize on the underlying futures price:
$$\int_{t_0}^{T} \sigma_{t-}^2 \delta(F_t - K)dt = \frac{2}{K^2}\left[(F_T - K)^+ - \int_{t_0}^{T} 1(F_t > K)dF_t\right].$$

Example of a Corridor Variance Swap

Equity Financial Products

Indicative Terms as of March 18, 2001

Banc of America Securities

Please call 212-583-8373 for questions

For Discussion Only: Proposed transaction with Bank of America N.A. All terms are subject to change and all prices are strictly illustrative.

Realized "Corridor" Variance Swap

Indicative Terms and Conditions

Variance Payer:	TBD
Variance Receiver:	Bank of America, N.A. ("BofA")
Trade Date:	TBD
Maturity Date:	1/18/02
Underlying Index:	TBD
Lower Corridor:	50% of the closing level of the Underlying Index on the Trade Date
Upper Corridor:	80% of the closing level of the Underlying Index on the Trade Date
Corridor Valuation Date:	Any Valuation Date for which the closing level of the Underlying Index on both a given Valuation Date and the immediately preceding Valuation Date are between the Lower Corridor and Upper Corridor.
Non-Corridor Valuation Date:	Any Valuation Date for which the closing level of the Underlying Index on both a given Valuation Date and the immediately preceding Valuation Date are outside the Lower Corridor and Upper Corridor.
Cross-over Valuation Date:	Any Valuation Date for which either: (1) the closing level of the Underlying Index on a given Valuation Date is between the Lower Corridor and the Upper Corridor and the closing level of the Underlying Index on the immediately preceeding Valuation Date is outside of the Lower Corridor and Upper Corridor; <u>or</u> (2) the closing level of the Underlying Index on a given Valuation Date is outside of the Lower Corridor and Upper Corridor and the closing level of the Underlying Index on the immediately preceding Valuation Date is between the Lower Corridor and Upper Corridor.
Corridor Variance:	σ_R^2 is the Corridor Variance of the Underlying Index from Trade Date until the Maturity Date as calculated below:

$$\sigma_R^2 = \sum_{t=1}^{n} \text{Return}_i^2 \times \frac{252}{n} \times 10{,}000 \quad \text{where:}$$

(a) "Return$_i$" means the Natural Logarithm of the Daily Quotient for a given Valuation Date t;

(b) "Daily Quotient" means, for a Corridor Valuation Date, the closing level of the Underlying Index for such Valuation Date divided by the Closing Level on the immediately preceding Exchange Business Day.

Localizing in Time

- Recall that the quadratic variation along a strike (local time) can be replicated:

$$\int_{t_0}^{T} \sigma_{t-}^2 \delta(F_t - K)dt = \frac{2}{K^2}\left[(F_T - K)^+ - \int_{t_0}^{T} 1(F_t > K)dF_t\right].$$

- By differentiating w.r.t. T, one can localize the payoff in time:

$$\sigma_{T-}^2 \delta(F_T - K) = \frac{2}{K^2}\left[\frac{\partial (F_T - K)^+}{\partial T} - 1(F_T > K)dF_T\right].$$

- We'd like to further localize on quadratic variation without having to specify a stochastic process for volatility:

$$\sigma_{T-}^2 \delta(F_T - K)\delta(\langle X\rangle_T - L) = ?$$

- The ability to create and price the LHS allows one to create and price integrals of the form $\int_{t_0}^{T} f(X_t, \langle X\rangle_t)d\langle X\rangle_t$.

- These integrals are needed to create and price payoffs such as the floating part of a volatility swap $\sqrt{\langle X\rangle_T}$ or an option on variance $(\langle X\rangle_T - k)^+$.

- We turn to this problem shortly.

Literature on Volatility Derivatives

- Grünbuchler and Longstaff (1993) assume the Heston (1993) SV model to value options on the instantaneous variance.
- Brockhaus & Long (1999) use Heston to value volatility swaps.
- Heston and Nandi (2000) value both volatility options and volatility swaps using a discrete time GARCH process.
- Javaheri, Wilmott, and Haug (2002) value volatility swaps using Nelson's continuous time limit of the GARCH (1,1) process.
- Brenner, Ou, and Zhang (2001) use Stein and Stein (1991).
- Howison, Rafailidis, and Rasmussen (2002) value both volatility average and variance swaps in an SV model where the instantanous volatility follows a mean-reverting lognormal process.
- Matytsin (2000) values volatility swaps and options on variance using a jump diffusion model with stochastic volatility.
- Finally, Detemple and Osakwe (1999) value volatility options in a general equilibrium framework.

Pros and Cons of SV Models

- All of the literature on volatility derivatives employs some kind of stochastic volatility (SV) model.
- Furthermore, the models typically (but not always) assume that the price process and the volatility process are both continuous over time.
- To the extent that these assumptions are correct and that the particular process specification is correct, one can use dynamic trading in one option and its underlying to hedge.
- Most of the SV models used further assume that prices and instantaneous volatility both diffuse, leading to a simple and parsimonious world view.
- However, simple SV models cannot simultaneously fit option prices at long and short maturities. Furthermore, since instantaneous volatility is not directly observable, the assumption that the diffusion coefficients of the volatility process are known is debatable.
- Indeed, the SV diffusion parameters implied from time series typically differ from the risk-neutral parameters, contradicting the implications of Girsanov's theorem.

Pros and Cons of SV Models

The great tragedy of science - the slaying of a beautiful hypothesis by an ugly fact – Thomas Henry Huxley.

- Complicating the pricing and hedging of volatility derivatives is the ugly fact that prices jump.
- Furthermore, when prices jump by a large amount, it is widely believed that expectations of future realized volatility jump as well. The high levels of mean reversion and vol vol implied from option prices further suggests that volatility jumps.
- When price and/or volatility can jump from one level to any other, then in a model with two or more sources of uncertainty, perfect replication typically requires dynamic trading in a portfolio of options.
- While option bid/ask spreads typically render this strategy as prohibitively expensive at the individual contract level, knowledge of desired option hedges can be enacted at the aggregate portfolio level.
- Can we use dynamic trading in a portfolio of options to develop a theory which does not require a complete specification of the stochastic process for volatility?

Space Time Harmonic Payoffs

- For the rest of my talk, assume zero interest rates for simplicity. We now temporarily drop the assumption that options trade.

- Consider some $C^{2,1}$ function $u(x,q)$. Note that by Itô's lemma:
$$u(X_T, \langle X\rangle_T) \;=\; u(0,0) + \int_{t_0}^{T} u_x(X_t, \langle X\rangle_t) dX_t$$
$$+ \int_{t_0}^{T} \left[\frac{u_{xx}(X_t, \langle X\rangle_t)}{2} + u_q(X_t, \langle X\rangle_t)\right] d\langle X\rangle_t,$$
where recall $X_t \equiv \ln(F_t/F_0)$ and hence $dX_t = \frac{dF_t}{F_t} - \frac{1}{2} d\langle X\rangle_t$.

- Substituting into the top equation implies:
$$u(X_T, \langle X\rangle_T) = u(0,0) + \int_{t_0}^{T} u_x(X_t, \langle X\rangle_t) \frac{dF_t}{F_t}$$
$$+ \int_{t_0}^{T} \left[\frac{u_{xx}(X_t, \langle X\rangle_t)}{2} - \frac{u_x(X_t, \langle X\rangle_t)}{2} + u_q(X_t, \langle X\rangle_t)\right] d\langle X\rangle_t.$$

- Now suppose that the function $u(x,q)$ solves the PDE:
$$\frac{1}{2} u_{xx}(x,q) - \frac{1}{2} u_x(x,q) + u_q(x,q) = 0.$$

- Then $u(X_T, \langle X\rangle_T)$ is just the sum of the payoffs from a static bond position and a dynamic futures position:
$$u(X_T, \langle X\rangle_T) = u(0,0) + \int_{t_0}^{T} \frac{u_x(X_t, \langle X\rangle_t)}{F_t} dF_t.$$

Example: Exponential Payoffs

- Recall that if $u(x,q)$ solves the PDE:

$$\frac{1}{2}u_{xx}(x,q) - \frac{1}{2}u_x(x,q) + u_q(x,q) = 0,$$

then the payoff $u(X_T, \langle X \rangle_T)$ is just the sum of the payoffs from a static bond position and a dynamic futures position:

$$u(X_T, \langle X \rangle_T) = u(0,0) + \int_{t_0}^{T} \frac{u_x(X_t, \langle X \rangle_t)}{F_t} dF_t.$$

- For example, $G(x,q) \equiv e^{p_\pm(\lambda)x - \lambda q}$ solves the PDE, where:

$$p_\pm(\lambda) \equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}.$$

- Hence $e^{p_\pm(\lambda)X_T - \lambda \langle X \rangle_T} = 1 + \int_{t_0}^{T} \frac{p_\pm(\lambda) G_t}{F_t} dF_t$ as $X_0 = \langle X \rangle_0 = 0$.

- Thus, the process $G_t \equiv e^{p_\pm(\lambda)X_t - \lambda \langle X \rangle_t}$ is a Q martingale started at one. For $\lambda > -\frac{1}{8}$, $G_T > 0$, so it is also a likelihood ratio.

- If $\lambda < -\frac{1}{8}$, then G is a complex-valued martingale, but one need only take real or imaginary parts to obtain real results.

- If we can break the link between p and λ, then we can determine the joint MGF of X_T and $\langle X \rangle_T$, and we are done.

MGF of Quadratic Variation

- Recall spanning an exponential payoff with bonds and futures:

$$G_T \equiv e^{p_\pm(\lambda)X_T - \lambda\langle X\rangle_T} = 1 + \int_{t_0}^T \frac{p_\pm(\lambda)G_t}{F_t} dF_t, \quad \forall t \in [t_0, T].$$

- Despite the dependence of the payoff on $\langle X\rangle_T$, increments in G depend only on increments in F:

$$\frac{dG_t}{G_t} = p_\pm(\lambda)\frac{dF_t}{F_t}.$$

- Let $M_t \equiv E_t^Q e^{\lambda\langle X\rangle_T}$. Since it is a conditional expectation of a terminal random variable, it is a martingale under Q.
- For now, we assume that the martingale F and the martingale M are orthogonal:

$$\langle F, M\rangle_t = 0, \quad \forall t \in [t_0, T].$$

- We will relax this arguable assumption later, but some modelling of the co-movement of F and M will always be required.
- Since increments in G depend only on increments in F, the martingale $G_t \equiv e^{p_\pm(\lambda)X_t - \lambda\langle X\rangle_t}$ and the martingale M are also orthogonal:

$$\langle G, M\rangle_t = 0, \quad \forall t \in [t_0, T].$$

Aside on Conditional Independence

- Our orthogonality assumption implies that $dM_t = dE_t^Q[e^{\lambda\langle X\rangle_T}]$ and dF_t are *conditionally* independent for each $t \in [0, T]$.
- To illustrate, consider the following bivariate diffusion:
$$\begin{aligned} \frac{dF_t}{F_t} &= \sqrt{v_t}dW_{1t} \qquad (1) \\ dv_t &= \mu(v_t, t)dt + \omega(v_t, t)dW_{2t}, \end{aligned}$$
where W_1 & W_2 are independent standard Brownian motions.
- We assume that F_t and v_t are adapted to the filtration $\mathcal{F}$.
- (1) $\Rightarrow d\langle X\rangle_t = v_t dt$ so $\langle X\rangle_T = \int_0^T v_t dt$ and $e^{\lambda\langle X\rangle_T} = e^{\lambda\int_0^T v_t dt}$.
- In this model, the pair $(\langle X\rangle_t, v_t)$ is Markov in itself, and hence $E^Q[e^{\lambda\langle X\rangle_T}|\mathcal{F}_t] = m(\langle X\rangle_t, v_t, t)$ for some function $m(q, v, t)$.
- By Itô's lemma and the fact that $E_t^Q[e^{\lambda\langle X\rangle_T}]$ is a martingale:
$$dE_t^Q[e^{\lambda\langle X\rangle_T}] = \frac{\partial}{\partial v}m(\langle X\rangle_t, v_t, t)\omega(v_t, t)dW_{2t}. \qquad (2)$$
- Since W_1 and W_2 are independent, increments in $E_t^Q[e^{\lambda\langle X\rangle_T}]$ and in F are conditionally independent. While $dE_t^Q[e^{\lambda\langle X\rangle_T}]$ and dF_t both depend on v_t, this variable is part of the information set that we are conditioning on, and hence they are conditionally independent.
- This ends the example.

Determining the MGF

- Recall that $M_t \equiv E_t^Q e^{\lambda \langle X \rangle_T}$ and $G_t \equiv e^{p_\pm(\lambda) X_t - \lambda \langle X \rangle_t}$ are both Q martingales.
- Now, $\langle G, M \rangle_t$ is defined as the predictable process which one subtracts from $M_t G_t$ to get a Q martingale.
- Since $\langle G, M \rangle_t = 0$, $M_t G_t$ is a Q martingale and hence:
$$E_t^Q[M_T G_T] = M_t G_t, \quad t \in [t_0, T].$$
- Since $M_T \equiv e^{\lambda \langle X \rangle_T}$ and $G_T \equiv e^{p_\pm(\lambda) X_T - \lambda \langle X \rangle_T}$:
$$M_t G_t = E_t^Q[e^{p_\pm(\lambda) X_T}] \equiv P_t, \quad t \in [t_0, T].$$
- Since the payoff $e^{p_\pm(\lambda) X_T}$ depends only on the futures price F_T, we re-introduce the existence of options of all strikes maturing at T. P_t is the observable price at time t of the static portfolio of options that spans the payoff $e^{p_\pm(\lambda) X_T}$ at T.
- We allow the payoff, $e^{p_\pm(\lambda) X_T}$, and hence its price, P_t, to have imaginary parts. Hence, M_t, G_t, and P_t can all be complex.
- The desired MGF is just the ratio of two observables:
$$M_t = \frac{P_t}{G_t}, \quad t \in [t_0, T].$$
- In particular, $M_0 = P_0 = E_0^Q e^{p_\pm(\lambda) X_T}$, since $G_0 = 1$.

Replicating the Exponential of QV

- Recall our pricing formula for $M_t \equiv E_t^Q e^{\lambda \langle X \rangle_T}$:
$$M_t = \frac{P_t}{G_t}, \quad t \in [t_0, T],$$
where $G_t \equiv e^{p_\pm(\lambda) X_t - \lambda \langle X \rangle_t}$, $P_t \equiv E_t^Q [e^{p_\pm(\lambda) X_T}]$, $t \in [t_0, T]$.
- Taking the total derivative: $dM_t = \frac{1}{G_t} dP_t - \frac{P_t}{G_t^2} dG_t$, where higher order terms vanish because M is a martingale.
- Recall that G is driven by just F, so $\frac{dG_t}{G_t} = p_\pm(\lambda) \frac{dF_t}{F_t}$.
- Hence, dM_t can be represented in terms of dP_t and dF_t:
$$dM_t = \frac{1}{G_t} dP_t - \frac{P_t p_\pm(\lambda)}{G_t F_t} dF_t.$$
- Integrating over time:
$$e^{\lambda \langle X \rangle_T} = E_0^Q e^{p_\pm(\lambda) X_T} + \int_{t_0}^T \frac{1}{G_t} dP_t - \int_{t_0}^T \frac{P_t p_\pm(\lambda)}{F_t G_t} dF_t.$$
- If everything is real-valued, then the payoff $e^{\lambda \langle X \rangle_T}$ is constructed by charging a premium of $E_0^Q e^{p_\pm(\lambda) X_T}$ dollars, and holding $\frac{1}{G_t}$ plays and $-P_t \frac{p_\pm(\lambda)}{F_t G_t}$ futures at each $t \in [t_0, T]$. The premium is used to finance the initial purchase of $\frac{1}{G_0} = 1$ play.
- Thus, the exhibited dynamic strategy in power plays and futures is self-financing, non-anticipating, and replicating.

Complex Replication

- Recall that the exponential function of $\langle X\rangle_T$ is replicated as:

$$e^{\lambda\langle X\rangle_T} = E_0^Q e^{p_\pm(\lambda)X_T} + \int_{t_0}^T \frac{1}{G_t} dP_t - \int_{t_0}^T \frac{P_t p_\pm(\lambda)}{F_t G_t} dF_t.$$

- If M,G, and P have imaginary parts, then one must take the real part of both sides to determine the trading strategy required to create the real part of M.
- Likewise, one takes the imaginary part of both sides to determine the trading strategy which creates the imaginary part of M.
- If P has an imaginary part, then the real part of P is given by cosine claims, while the imaginary part is given by a sine claim.
- Our results bear out Hadamard's famous dictum that:

  The shortest path between two results in the real domain passes through the complex domain.

- If λ is real, then the Post Widder algorithm can be used to invert the Laplace transform. If λ is complex, then efficient Laplace Transform inversion algorithms such as Abate Whitt can be used. In either case, a wide class of functions of $\langle X\rangle_T$ can be created including $\sqrt{\langle X\rangle_T}$ and $(\langle X\rangle_T - k)^+$.

Volatility Swaps

- The floating part of a volatility swap has payoff $\sqrt{\langle X\rangle_T}$.
- The following identity is proved in Appendix 2:

$$\sqrt{q} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1-e^{-sq}}{s^{\frac{3}{2}}} ds \text{ for all } q \geq 0. \tag{3}$$

- Evaluating at $q = \langle X\rangle_T$ and taking risk-neutral expectations, the fixed part of a volatility swap is:

$$\begin{aligned} E_0^Q \sqrt{\langle X\rangle_T} &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1-E_0^Q e^{-s\langle X\rangle_T}}{s^{\frac{3}{2}}} ds \\ &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1-E_0^Q e^{p_\pm(-s)X_T}}{s^{\frac{3}{2}}} ds \end{aligned}$$

where recall $p_\pm(-s) \equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - 2s}$.

- For $s \in (0, \frac{1}{8})$, p is real, while for $s > \frac{1}{8}$, p is complex:

$$\begin{aligned} E_0^Q \sqrt{\langle X\rangle_T} &= \frac{1}{2\sqrt{\pi}} \int_0^{\frac{1}{8}} \frac{1-E_0^Q e^{\left(\frac{1}{2}\pm\sqrt{\frac{1}{4}-2s}\right)X_T}}{s^{\frac{3}{2}}} ds \\ &+ \frac{1}{2\sqrt{\pi}} \int_{\frac{1}{8}}^\infty \frac{1-E_0^Q e^{\frac{1}{2}X_T} \cos(\sqrt{2s-\frac{1}{4}}X_T)}{s^{\frac{3}{2}}} ds. \end{aligned}$$

- We need to numerically check the robustness of this inversion.

A Recap

- Under our assumptions, claims on quadratic variation $\langle X \rangle_T$ maturing at T can be priced relative to the initial futures price and the initial prices of all European options maturing at T.
- This information is the same as for a variance swap. However, robust replication of a nonlinear function of $\langle X \rangle_T$ requires dynamic trading in futures *and* all options of maturity T.
- Notice that we have not assumed a Markov process for the state variables in the problem. It may well be for example that the whole surface of implied volatilities is following some unknown stochastic process.
- However, if we do assume some Markov process eg. that time, price, and the instantaneous variance rate v_t are Markov in themselves, then note that we have not assumed that v_t is continuous over time, i.e. v_t is just an arbitrary process with unknown characteristics.
- Even if we do assume a bivariate diffusion for returns X and its variance rate v_t, we do not need to specify the drift or diffusion coefficients of the SDE governing v_t.
- In contrast to the variance swap, we assumed that returns and the MGF of $\langle X \rangle_T$ are orthogonal. We relax this assumption next.

Modelling Correlation

- Recall that we assumed that $\langle X, M\rangle_t = 0$, where $X_t \equiv \ln(F_t/F_0)$ and $M_t \equiv E_t^Q e^{\lambda\langle X\rangle_T}$.
- We now redefine M as $M_t(\mu,\lambda) \equiv E_t^Q e^{\mu X_T + \lambda\langle X\rangle_T}$, $\mu, \lambda \in C$.
- The joint MGF $M_t(\mu,\lambda)$ is still a Q martingale and it reverts to its former definition if $\mu = 0$.
- To relax the orthogonality assumption, suppose instead:
$$\frac{dM_t}{M_t} = \beta(X_t,\mu,\lambda)dX_t + dN_t,$$
where X and the noise N are orthogonal, i.e. $\langle X, N\rangle_t = 0$.
- The function $\beta(x,\mu,\lambda)$ is known and we allow $\beta(x,0,\lambda) \neq 0$.
- Under our new assumption, $\frac{d\langle X,M\rangle_t}{M_t} = \beta(X_t,\mu,\lambda)d\langle X\rangle_t$.
- Extend $G_t \equiv C(X_t)e^{-\lambda\langle X\rangle_t}$, where $C(x)$ solves the ODE:
$$\frac{1}{2}C_{xx}(x) + \left(\beta(x,\mu,\lambda) - \frac{1}{2}\right)C_x(x) - \lambda C(x) = 0.$$
- If β is independent of x, then $C(x) = e^{p_\pm(\lambda,\beta)x}$ where:
$$p_\pm(\lambda,\beta) \equiv \frac{1}{2} - \beta(\mu,\lambda) \pm \sqrt{\left(\frac{1}{2} - \beta(\mu,\lambda)\right)^2 + 2\lambda}.$$
- If $\beta(x,\mu,\lambda) = 0$, then G reverts to its previous definition.

Multiplication Makes Martingale

- Recall $G_t \equiv C(X_t)e^{-\lambda\langle X\rangle_t}$ where $C(x)$ solves:

$$\frac{1}{2}C_{xx}(x) + \left(\beta(x,\mu,\lambda) - \frac{1}{2}\right)C_x(x) - \lambda C(x) = 0.$$

- From Itô's lemma:

$$\begin{aligned} dG_t &= G_x \frac{dF_t}{F_t} + \left(-\frac{1}{2}C_x + \frac{1}{2}C_{xx} - \lambda C\right) e^{-\lambda\langle X\rangle_t} d\langle X\rangle_t \\ &= G_x \frac{dF_t}{F_t} - \beta(x,\mu,\lambda)C_x e^{-\lambda\langle X\rangle_t} d\langle X\rangle_t \\ &= G_x \frac{dF_t}{F_t} - G_x \beta(x,\mu,\lambda) d\langle X\rangle_t. \end{aligned}$$

- Note that G is no longer a Q martingale. Furthermore, the covariation of M with X induces covariation of M with G:

$$\frac{d\langle G, M\rangle_t}{M_t} = G_x \frac{d\langle X, M\rangle_t}{M_t} = G_x \beta(x,\mu,\lambda) d\langle X\rangle_t.$$

- From integration by parts:

$$\begin{aligned} d(GM)_t &= G_t dM_t + M_t \left(dG_t + \frac{d\langle G, M\rangle_t}{M_t}\right) \\ &= G_t dM_t + M_t G_x \frac{dF_t}{F_t}, \end{aligned}$$

  and hence MG is still a Q martingale.

Determining the MGF

- Recall $G_t \equiv C(X_t)e^{-\lambda\langle X\rangle_t}$ where $C(x)$ solves an ODE and $M_t \equiv E_t^Q e^{\mu X_T+\lambda\langle X\rangle_T}$.
- Also recall that MG is still a Q martingale and hence:
$$E_t^Q[M_T G_T] = M_t G_t, \quad t \in [t_0, T].$$
- Since $M_T \equiv e^{\mu X_T+\lambda\langle X\rangle_T}$ and $G_T \equiv C(X_T)e^{-\lambda\langle X\rangle_T}$:
$$M_t G_t = E_t^Q[C(X_T)e^{\mu X_T}] \equiv P_t, \quad t \in [t_0, T].$$
- P_t is now the observable price at time t of the static portfolio of options that spans the payoff $C(X_T)e^{\mu X_T}$ at T.
- If the payoff has an imaginary part, so does P_t.
- The desired MGF is again just the ratio of two observables:
$$M_t = \frac{P_t}{G_t}.$$
- Replication proceeds as before. Hence we can model some covariation of returns with volatility and still obtain robust replication.
- If the desired option maturities are not available in the market initially, then one can let price and quadratic variation be Markov state variables and use our results to roll over shorter maturity options (introducing more model risk).

A Problem

- Recall that for the joint MGF $M_t \equiv E_t^Q e^{\mu X_T + \lambda \langle X \rangle_T}$ and the return $X_t \equiv \ln(F_t/F_0)$, we assumed that:

$$\frac{dM_t}{M_t} = \beta(X_t, \mu, \lambda) dX_t + dN_t,$$

where $\langle X, N \rangle_t = 0$ and $\beta(x, \mu, \lambda)$ is known.

- If $\mu = 0$, then as time increases, the MGF M of $\langle X \rangle_T$ should vary less as more of its payoff gets determined.
- In fact, as $t \uparrow T$, the covariation of M with X should tend towards 0.
- However, β is independent of time, so we can't capture this effect.
- Future research needs to address this issue, perhaps by trading in options of all maturities as well as all strikes.

Summary and Extensions

- Just as a linear payoff on price can be robustly synthesized by a static position in stock and bond, a linear payoff on quadratic variation such as a variance swap can be robustly synthesized using a static position in options on price.
- However, just as a nonlinear payoff on price usually requires dynamic trading in the stock, robust replication of a nonlinear payoff on quadratic variation seems to require dynamic trading in options on price.
- The claims we considered mature at a fixed time. For continuous processes, it is actually easier to have the maturity occur at the first time that quadratic variation or price crosses a level, since one random variable is determined at the payoff time.
- We have extended this work to $X = g(F)$ and to jumps.
- It would be interesting to extend this work to the multivariate setting.
- It would also be interesting to try to replicate claims on other notions of volatility, such as local volatility or exponentially weighted quadratic variation.
- In the interests of brevity, these extensions are best left for future research.

App. 1: Replicating with Bonds & Options

- For any fixed κ, the fundamental theorem of calculus implies:

$$\begin{aligned} f(S) &= f(\kappa) + 1_{S>\kappa} \int_\kappa^S f'(u)du - 1_{S<\kappa} \int_S^\kappa f'(u)du \\ &= f(\kappa) + 1_{S>\kappa} \int_\kappa^S \left[f'(\kappa) + \int_\kappa^u f''(v)dv\right] du \\ &\quad -1_{S<\kappa} \int_S^\kappa \left[f'(\kappa) - \int_u^\kappa f''(v)dv\right] du. \end{aligned}$$

- Noting that $f'(\kappa)$ is independent of u, Fubini's theorem implies:

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) + 1_{S>\kappa} \int_\kappa^S \int_v^S f''(v)dudv \\ &\quad +1_{S<\kappa} \int_S^\kappa \int_S^v f''(v)dudv. \end{aligned}$$

- Integrating over u yields:

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) + 1_{S>\kappa} \int_\kappa^S f''(v)(S-v)dv \\ &\quad +1_{S<\kappa} \int_S^\kappa f''(v)(v-S)dv \\ &= f(\kappa) + f'(\kappa)(S-\kappa) + \int_\kappa^\infty f''(v)(S-v)^+ dv \\ &\quad + \int_0^\kappa f''(v)(v-S)^+ dv. \end{aligned}$$

- Q.E.D. (quite easily done).

App. 2: The Square Root Function

Let $\Gamma(\alpha) \equiv \int_0^\infty t^{\alpha-1}e^{-t}dt$ be the gamma function with α a positive real. Then it is well known that:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{4}$$

i.e.

$$\sqrt{\pi} = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}dt. \tag{5}$$

Consider the change of variables $s = \frac{t}{q}$ for $q > 0$. Then $t = sq$, $dt = qds$ and hence:

$$\sqrt{\pi} = \int_0^\infty \frac{e^{-sq}}{\sqrt{sq}}qds = \sqrt{q}\int_0^\infty \frac{e^{-sq}}{\sqrt{s}}ds.$$

Solving for $\sqrt{q}$ gives one representation:

$$\sqrt{q} = \frac{\sqrt{\pi}}{\int_0^\infty \frac{e^{-sq}}{\sqrt{s}}ds}. \tag{6}$$

Integrating (5) by parts, let:

$$\begin{aligned} u &= \frac{1}{\sqrt{t}} \qquad & dv &= e^{-t}dt \\ du &= -\frac{1}{2t^{3/2}}dt \qquad & v &= 1 - e^{-t} \end{aligned} \tag{7}$$

Hence:

$$\sqrt{\pi} = \left.\frac{1 - e^{-t}}{\sqrt{t}}\right|_{t=0}^{t=\infty} + \frac{1}{2}\int_0^\infty \frac{1 - e^{-t}}{t^{3/2}}dt \tag{8}$$

or

$$2\sqrt{\pi} = \int_0^\infty \frac{1 - e^{-t}}{t^{3/2}}dt. \tag{9}$$

Again consider the change of variables $s = \frac{t}{q}$ for $q > 0$:

$$2\sqrt{\pi} = \int_0^\infty \frac{1 - e^{-sq}}{(sq)^{3/2}}qds = \frac{1}{\sqrt{q}}\int_0^\infty \frac{1 - e^{-sq}}{s^{3/2}}ds. \tag{10}$$

Solving for $\sqrt{q}$ gives a second representation:

$$\sqrt{q} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1 - e^{-sq}}{s^{\frac{3}{2}}}ds. \tag{11}$$

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