

Determining Volatility Surfaces  
and Option Values  
From an Implied Volatility Smile

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## Overview

- Introduction
- Stock Pricing Function
- Volatility Surface
- Option Pricing
- Summary and Extensions

# Part I

## Introduction

## The Standard Model

- Recall the assumptions of the standard Black-Scholes/Merton (BSM) model:
  - frictionless markets
  - no arbitrage
  - constant interest rate  $r$
  - constant dividend yield  $q$
  - stock price  $S_t$  obeys:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma dW_t, \quad t > 0,$$

where  $\mu_t$  is the drift process,  $\sigma$  is the constant volatility rate, and  $W_t$  is standard Brownian motion (SBM).

- In this model, all standard and exotic options are priced using only the single parameter  $\sigma$  for each underlying.

## However...

- It appears that more parameters are needed to describe listed option prices.
- Dupire's approach generalizes the stock price process in BSM to:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(S, t) dW_t, \quad t > 0,$$

where  $\sigma(S, t)$  is the local volatility function.

- Dupire shows how to obtain  $\sigma(S, t)$ ,  $S > 0$ ,  $t \in (0, T)$  from option prices of all strikes  $K > 0$  and maturities  $M \in (0, T)$ .
- For  $T'$  arbitrarily large, this paper shows how to obtain  $\sigma(S, t)$ ,  $S > 0$ ,  $t \in (0, T')$  from option prices of all strikes  $K > 0$  and a single maturity  $T \leq T'$ .
- Equivalently, the initial data can be an implied volatility smile since the smile of maturity  $T$  is mathematically equivalent to the strike structure at maturity  $T$ .

## How?

- In the BSM model, a unique risk-neutral measure  $Q$  is defined such that:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t, \quad t > 0,$$

- The stochastic differential equation for the stock price can be solved:

$$S_t = S_0 e^{(r - q - \frac{\sigma^2}{2})t + \sigma W_t}, \quad t > 0.$$

- Notice that the stock price at any future date  $t$  depends on the time  $t$  and on the SBM  $W_t$ , but does not depend on the path of the SBM (path-independence).
- To determine the entire local volatility surface, assume path-independence:

$$S_t = s(W_t, t), \quad t \in (0, T'),$$

where  $T' \geq T$  is the arbitrarily distant horizon.

- In contrast to BSM, we infer this spot pricing function as a consequence of our three primary assumptions, namely path-independence, no arbitrage, and the liquidity of the  $T$ -maturity options.

## Bankruptcy

- In contrast to BSM, stock prices sometimes hit zero.
- For analytical tractability, we assume that the stock price first hits zero when the driving SBM first hits a lower liquidation level  $L < 0$ .
- For prices which can't vanish,  $L = -\infty$ .
- Given our path-independence assumption, both the stock pricing function and the constant absorbing boundary (if any) can be inferred from the option prices.
- Once the stock price becomes a known function of the driving SBM, the stock price process inherits the analytical tractability of this SBM. For example, the probability that the stock vanishes over  $[0, T')$  is just the probability that the SBM hits  $L$  over  $[0, T')$ .
- The tight link to SBM permits analytic evaluation of the risk-neutral stock price process, the volatility surface, and option valuation formulas.

## Part II

# The Stock Pricing Function

## PDE For The Stock Pricing Function

- Assume that under the risk-neutral measure  $Q$ :

$$dS_t = (r - q)S_t dt + \sigma(S_t, t)S_t dW_t^a, \quad t \in [0, T'),$$

where  $W_t^a$  is a  $Q$ -SBM absorbing at  $L < 0$ .

- Also assume path-independence:

$$S_t = s(W_t^a, t), \quad t \in [0, T'),$$

where  $s(x, t)$  is a  $C^{2,1}$  function.

- By Itô's lemma:

$$dS_t = \left[ \frac{\partial s}{\partial t}(W_t^a, t) + \frac{1}{2} \frac{\partial^2 s}{\partial x^2}(W_t^a, t) \right] dt + \frac{\partial s}{\partial x}(W_t^a, t) dW_t^a, \quad t \in [0, T').$$

- Equating coefficients on  $dt$  gives a partial differential equation:

$$\frac{\partial s}{\partial t}(x, t) + \frac{1}{2} \frac{\partial^2 s}{\partial x^2}(x, t) = (r - q)s(x, t), \quad t \in [0, T'), x > L,$$

which we assume is subject to the following boundary conditions:

$$\lim_{x \downarrow L} s(x, t) = 0, \quad t \in [0, T'),$$

$$\lim_{x \uparrow \infty} s(x, t) = o(e^x), \quad t \in [0, T').$$

- Once we obtain a “payoff” at some future time such as  $T$  or  $T'$ , the stock pricing function is determined.

## Implied Stock Payoff Function

- Assume that one can observe the continuum of arbitrage-free  $T$ -maturity European put prices  $\{P_0(K), K > 0\}$ .
- Let  $P_k(K) \equiv \lim_{\Delta K \downarrow 0} \frac{P_0(K+\Delta K) - P_0(K)}{\Delta K}$  denote the observed slope in strike at  $K$ . Breeden and Litzenberger show that  $e^{rT} P_k(K)$  is the risk-neutral probability that  $S_T < K$ , or under path-independence, that  $s(W_T^a, T) \equiv f(W_T^a) < K$ , or equivalently that  $W_T^a < f^{-1}(K)$ .

- Thus, for all  $K > 0$ :

$$e^{rT} P_k(K) = \int_L^{f^{-1}(K)} \frac{1}{\sqrt{2\pi T}} \left\{ \exp \left[ -\frac{1}{2} \left( \frac{z}{\sqrt{T}} \right)^2 \right] - \exp \left[ -\frac{1}{2} \left( \frac{z - 2L}{\sqrt{T}} \right)^2 \right] \right\} dz.$$

- Letting  $x = f^{-1}K$ :

$$e^{rT} P_k(f(x)) = \int_L^x \frac{1}{\sqrt{2\pi T}} \left\{ \exp \left[ -\frac{1}{2} \left( \frac{z}{\sqrt{T}} \right)^2 \right] - \exp \left[ -\frac{1}{2} \left( \frac{z - 2L}{\sqrt{T}} \right)^2 \right] \right\} dz, x > L$$

- Solving the LHS for the implied payoff function  $f(x)$  yields:

$$f(x) = P_k^{-1} \left\{ e^{-rT} \int_L^x \frac{1}{\sqrt{2\pi T}} \left\{ \exp \left[ -\frac{1}{2} \left( \frac{z}{\sqrt{T}} \right)^2 \right] - \exp \left[ -\frac{1}{2} \left( \frac{z - 2L}{\sqrt{T}} \right)^2 \right] \right\} dz \right\},$$

where  $x > L$ .

- Since  $P_k$  is a positive increasing function,  $f(x)$  is a positive increasing function. If  $f(x) = 0$  for  $x$  less than some point, then we identify this point as the lower liquidation level  $L$ .

## Stock Pricing Function on $[0, T)$

- Recall the PDE for the stock pricing function:

$$\frac{\partial s}{\partial t}(x, t) + \frac{1}{2} \frac{\partial^2 s}{\partial x^2}(x, t) = (r - q)s(x, t), \quad t \in [0, T'), x > L,$$

subject to the boundary conditions:

$$\lim_{x \downarrow L} s(x, t) = 0, \quad t \in [0, T'),$$

$$\lim_{x \uparrow \infty} s(x, t) = o(e^x), \quad t \in [0, T').$$

$$s(x, T) = P_k^{-1} \left\{ \int_L^x \frac{e^{-rT}}{\sqrt{2\pi T}} \left\{ \exp \left[ -\frac{1}{2} \left( \frac{z}{\sqrt{T}} \right)^2 \right] - \exp \left[ -\frac{1}{2} \left( \frac{z - 2L}{\sqrt{T}} \right)^2 \right] \right\} dz \right\}, x >$$

- The Feynman-Kac theorem expresses the solution as:

$$s(x, t) = e^{-(r-q)(T-t)} E_{x,t}^Q [f(W_T^a)], \quad x > L, t \in [0, T],$$

where  $\{W_u; u \in [t, T]\}$  is a Q-SBM starting at  $x$  and absorbing at  $L < 0$ .

- Since  $f(\cdot)$  is a positive increasing function of  $x$ , so is  $s(x, t)$  for each  $t$ .

## Stock Pricing Function on $[0, T')$

- If we can obtain the horizon payoff at  $T' > T$ , then the Feynman-Kac theorem yields the stock pricing function over  $(0, T')$ .

- It is tempting to implicitly define the horizon payoff  $h(z)$  for  $x > L$  by:

$$f(x) = e^{-(r-q)(T'-T)} \int_L^\infty \frac{\exp\left[-\frac{1}{2}\left(\frac{z-x}{\sqrt{T'-T}}\right)^2\right] - \exp\left[-\frac{1}{2}\left(\frac{z-x-2L}{\sqrt{T'-T}}\right)^2\right]}{\sqrt{2\pi(T'-T)}} h(z) dz.$$

- This formulation may be ill-posed. For example,  $f(x)$  might describe the value at  $T$  of a call on the SBM paying  $(W_M - K)^+$  at some date  $M$  fixed strictly between  $T$  and  $T'$ .
- On the other hand, the situation is not hopeless since the Black Scholes case  $f(x) = e^{\sigma x - \sigma^2 T/2}$  engenders a horizon payoff  $h(z) = e^{\sigma z - \sigma^2 T'/2}$ , for any  $T' \geq T$ .
- Is there a class of stock price processes “between” pure geometric Brownian motion and the process followed by an  $M$ -maturity call?

## Positive Stock Pricing Functions on $[0, \infty)$

- Robbins and Siegmund (1973) showed that the *entire* set of functions  $f(x, t)$  which are strictly positive, increasing in  $x \in \mathfrak{R}$ , and which satisfy  $f(0, 0) = 1$  and  $\frac{\partial f}{\partial t}(x, t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x, t) = 0$  for  $x \in \mathfrak{R}$  and  $t \in (0, \infty)$  are given by:

$$f(x, t) = \int_0^\infty e^{\theta x - \frac{\theta^2}{2} t} dG(\theta), \quad x \in \mathfrak{R}, t \in (0, \infty),$$

where  $G(\theta)$  is a distribution function with  $G(\infty) = 1$  and  $G(0+) = 0$ .

- Thus, the entire set of positive increasing solutions to the backward diffusion equation on  $(0, \infty)$  is given by randomizing the volatility of a geometric Brownian motion.
- We can treat  $f$  as the forward price relative,  $\frac{F_t}{F_0}$  and solve for the stock price.

## Stock Pricing Function on $[0, \infty)$

- The paper proves that the *entire* set of stock pricing functions  $s(x, t)$  which vanish at  $x = L$ , are positive, increasing, and unbounded in  $x$  on  $x > L$ , and which satisfy  $s(0, 0) = S_0$  and  $\frac{\partial s}{\partial t}(x, t) + \frac{1}{2}\frac{\partial^2 s}{\partial x^2}(x, t) = (r - q)s(x, t)$  for  $x > L$  and  $t \in (0, \infty)$  are given by:

$$s(x, t) = S_0 e^{(r-q)t} \int_0^\infty \frac{\sinh[\theta(x - L)]}{\sinh[-\theta L]} e^{-\frac{\theta^2}{2}t} dG(\theta), \quad x > L, t \in (0, T'),$$

where  $\sinh(x) \equiv \frac{e^x - e^{-x}}{2}$  and  $G(\theta)$  is a distribution function with  $G(\infty) = 1$  and  $G(0+) = 0$ .

- Evaluating at  $t = T$  relates the implied stock payoff  $f(x)$  to the distribution function  $G(\theta)$ :

$$s(x, T) \equiv f(x) = S_0 e^{(r-q)T} \int_0^\infty \frac{\sinh[\theta(x - L)]}{\sinh[-\theta L]} e^{-\frac{\theta^2}{2}T} dG(\theta), \quad x > L.$$

- Assuming that  $G(\cdot)$  is the distribution function of a continuous random variable with a probability density function  $G'(\theta)$ , the paper inverts this equation to obtain the density function  $G'(\theta)$  from the known implied payoff  $f(x)$ :

$$G'(\theta) = \begin{cases} \frac{\sqrt{T} \sinh(-\theta L)}{S_0 e^{(r-q)T} \pi \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\theta\sqrt{T} + \frac{\omega^2}{2}} \int_{-\infty}^{\infty} e^{i\omega z - \frac{z^2}{2}} f(L + z\sqrt{T}) dz d\omega, & \theta \geq 0; \\ 0, & \theta < 0. \end{cases}$$

- Substituting  $G'(\theta)$  in  $s(x, t)$  and simplifying gives:

$$s(x, t) = \frac{e^{(r-q)(t-T)}}{2\pi/\sqrt{T}} \int_{-\infty}^{\infty} e^{\frac{\omega^2 + ty_t^2}{2}} \int_{-\infty}^{\infty} e^{i\omega z - \frac{z^2}{2}} f(L + z\sqrt{T}) [N(y_t) - N(-y_t)] dz d\omega,$$

for  $x > L, t \in (0, T')$ , where  $y_t \equiv \frac{x - L - i\omega\sqrt{T}}{\sqrt{t}}$ .

- It can be shown that  $s(x, t)$  is a positive increasing function of  $x$  on  $x > L$  and  $t \in (0, T')$ .

## Example

- Suppose that for  $K > 0$ , the initial  $T$ -maturity put prices are such that vertical spreads have the form:

$$e^{rT} P_k(K) = \int_L^{\frac{\sinh^{-1}(K/\beta) + L}{\alpha}} \frac{1}{\sqrt{2\pi T}} \left\{ \exp \left[ -\frac{1}{2} \left( \frac{z}{\sqrt{T}} \right)^2 \right] - \exp \left[ -\frac{1}{2} \left( \frac{z - 2L}{\sqrt{T}} \right)^2 \right] \right\} dz.$$

- Since both sides give  $Q\{W_T^a < f^{-1}(K)\}$ :

$$f^{-1}(S) = \begin{cases} \frac{\sinh^{-1}(S/\beta) + L}{\alpha} & \text{if } S > 0; \\ 0 & \text{if } S < 0. \end{cases}$$

- Inverting this function:

$$f(x) = \begin{cases} \beta \sinh[\alpha(x - L)] & \text{if } x > L; \\ 0 & \text{if } x < L. \end{cases}$$

- Figure 0.1 graphs the implied payoff function  $f(\cdot)$  and its inverse.

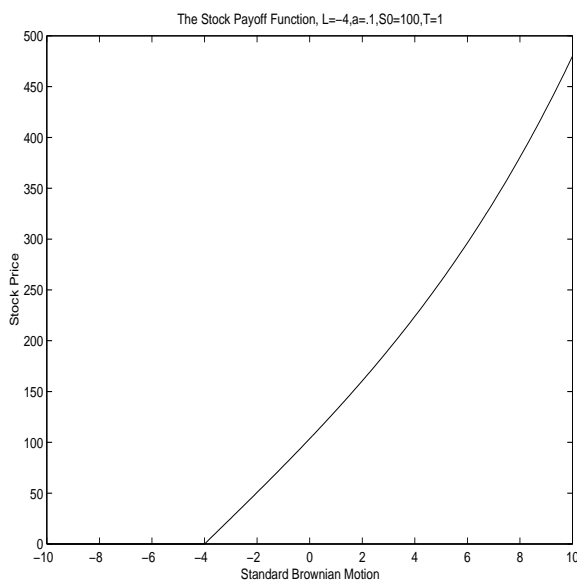


Figure 0.1: The Implied Stock Payoff Function and Its Inverse

## Example(Con'd)

- Using the inversion formula, the density function  $G'(\theta)$  evaluates to:

$$G'(\theta) = \delta(\theta - \alpha),$$

for  $\alpha > 0$ , where  $\delta(\cdot)$  is Dirac's delta function.

- Evaluating the stock pricing function at  $t = T'$  and substituting in  $G'$  gives:

$$s(x, T') \equiv g(x) = \beta' \sinh[\alpha(x - L)].$$

- The solution of the PDE subject to this terminal condition and  $s(0, 0) = S_0$  is the stock pricing function:

$$s(x, t) = S_0 e^{\mu t} \operatorname{csch}(-\alpha L) \sinh[\alpha(x - L)], \quad t \in [0, T'],$$

where  $\mu \equiv r - q - \alpha^2/2$ . For future use, note that this pricing function has an explicit inverse  $x = s^{-1}(S, t)$ .

- Figure 0.2 graphs this  $s(x, t)$  against the driving SBM and time.

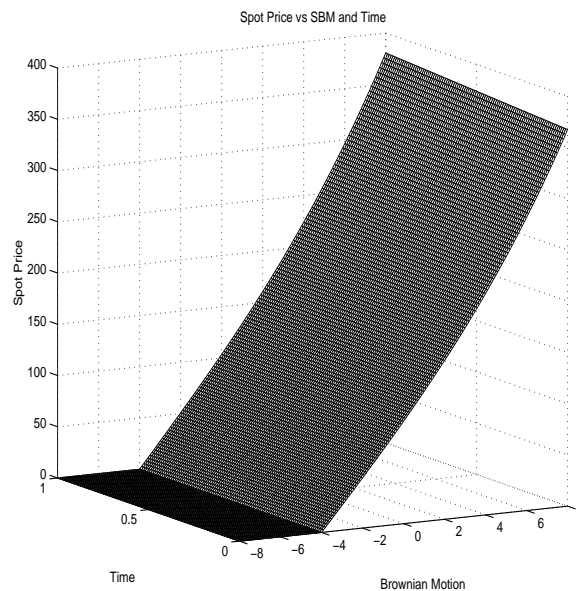


Figure 0.2: The Stock Pricing Function

## Part III

# The Local Volatility Surface

## Derivation of Volatility Function

- Since the stock pricing function  $s(x, t)$  is increasing in  $x$  on  $x > L$ , it can be inverted for  $x = s^{-1}(S, t)$ .
- Recall that under the risk-neutral measure  $Q$ :

$$dS_t = (r - q)S_t dt + \sigma(S_t, t)S_t dW_t^a, \quad t \in [0, T'],$$

where  $W_t^a$  is a  $Q$ -SBM absorbing at  $L < 0$ .

- By path-independence  $S_t = s(W_t^a, t)$ ,  $t \in [0, T']$  and by Itô's lemma:

$$dS_t = \left[ \frac{\partial s}{\partial t}(W_t^a, t) + \frac{1}{2} \frac{\partial^2 s}{\partial x^2}(W_t^a, t) \right] dt + \frac{\partial s}{\partial x}(W_t^a, t) dW_t^a, \quad t \in [0, T'].$$

- Equating coefficients on  $dW_t^a$  determines local volatility  $\sigma(S, t)$  in terms of  $\frac{\partial s}{\partial x}(x, t)$  and  $s^{-1}(S, t)$ :

$$\sigma(S, t) = \frac{1}{S} \frac{\partial s}{\partial x}(s^{-1}(S, t), t), \quad t \in [0, T'], x \in \mathfrak{R}.$$

- Since  $s(\cdot, t)$  is increasing in  $x$  for each  $t$ ,  $\sigma(S, t)$  is positive.

## Example

- Again suppose that option prices imply a stock pricing function given by:

$$s(x, t) = S_0 e^{\mu t} \operatorname{csch}(-\alpha L) \sinh[\alpha(x - L)], \quad t \in [0, T'],$$

where  $\mu \equiv r - q - \alpha^2/2$ .

- Recall that local volatility is given by:

$$\sigma(S, t) = \frac{1}{S} \frac{\partial s}{\partial x}(s^{-1}(S, t), t), \quad t \in [0, T'], x \in \mathfrak{R}.$$

- Differentiating and inverting the top equation gives the following local volatility surface:

$$\sigma(S, t) = \alpha \coth \left[ \sinh^{-1} \left( \frac{S}{S_0 e^{\mu(T'-t)}} \sinh(-\alpha L) \right) \right], \quad t \in [0, T').$$

- Figure 0.3 graphs this local volatility surface.

Figure 0.3: The Local Volatility Surface

## Intuition on Example

- To understand the behavior of this local volatility as a function of the stock price, recall that:

$$s(x, t) = S_0 e^{\mu t} \operatorname{csch}(-\alpha L) \sinh[\alpha(x - L)], \quad t \in [0, T'],$$

where  $\mu \equiv r - q - \alpha^2/2$ .

- Note that the stock pricing function is proportional to  $\sinh[\alpha(x - L)]$ , which behaves linearly in  $x$  for  $x$  near  $L$  and exponentially in  $\alpha x$  for  $x$  large. Thus at each future date, the volatility smile is approximately hyperbolic in  $S$  (“normal volatility”) for  $S$  near zero, while it is asymptoting to the constant  $\alpha$  (“lognormal volatility”) for  $S$  high.
- To understand the behavior of this local volatility as a function of time, recall:

$$\sigma(S, t) = \alpha \coth \left[ \sinh^{-1} \left( \frac{S}{S_0 e^{\mu(T'-t)}} \sinh(-\alpha L) \right) \right], \quad t \in [0, T').$$

- Thus, along the path  $S = S_0 e^{\mu(T'-t)}$ , volatility is constant at  $\alpha \coth(-\alpha L)$ . Thus, just as  $\alpha$  controls the asymptotic height of the volatility smile, the parameter  $L$  controls this “at-the-money” volatility.
- As the terminal time  $T'$  approaches infinity, then for any fixed level of  $S$ , the terminal volatility smile  $\sigma(S, T')$  asymptotes down to  $\alpha$  if  $\mu < 0$  and becomes unbounded above if  $\mu > 0$ .

## PDE and Converse

- Recall that the stock pricing function  $s(x, t)$  satisfies the linear PDE:

$$\frac{\partial s}{\partial t}(x, t) + \frac{1}{2} \frac{\partial^2 s}{\partial x^2}(x, t) = (r - q)s(x, t), \quad x > L, t \in (0, T').$$

- Also recall that local volatility is given by:

$$\sigma(S, t) = \frac{1}{S} \frac{\partial s}{\partial x}(s^{-1}(S, t), t), \quad t \in [0, T'], S > 0.$$

- Using the second equation as a change of variables in the first, the paper proves that  $\sigma(S, t)$  obeys the following nonlinear PDE on  $S > 0, t \in (0, T')$ :

$$\frac{\sigma^2(S, t)S^2}{2} \frac{\partial^2 \sigma(S, t)}{\partial S^2} + [r - q + \sigma^2(S, t)]S \frac{\partial \sigma}{\partial S}(S, t) + \frac{\partial \sigma}{\partial t}(S, t) = 0.$$

- Conversely, the paper proves that if a given local volatility function  $\sigma(S, t)$  satisfies this nonlinear PDE, then the stock price is path-independent.
- The paper also shows that the stock pricing function  $s(x, t)$  is just the conditional expectation at state  $x$  and time  $t$  of the implied payoff  $f(S)$ , which can be determined from the  $T$ -maturity local volatility smile  $v(S) \equiv \sigma(S, T)$  by:

$$f(x) = I_v^{-1}(x + c_0), \quad x \in \mathfrak{R},$$

where:

$$I_v(f) \equiv \int_{c_1}^f \frac{1}{v(S)S} dS, \quad f \geq 0,$$

and  $c_0$  and  $c_1$  are constants.

## Part IV

# Valuing Path-Independent Options

## Valuing Standard Options in terms of the SBM

- To value European calls of some strike  $K$  and intermediate maturity  $M \in [0, T']$  in terms of the contemporaneous spot price, we first determine the function  $\gamma(x, t)$  relating the call value to the SBM  $W_t$  and time  $t$ . By Itô's lemma, this pricing function solves:

$$\frac{1}{2} \frac{\partial^2 \gamma}{\partial x^2}(x, t) + \frac{\partial \gamma}{\partial t}(x, t) = r\gamma(x, t), \quad x > L, t \in (0, M),$$

subject to the boundary conditions:

$$\lim_{x \downarrow L} \gamma(x, t) = 0, \quad \lim_{x \uparrow \infty} \gamma(x, t) = s(x, t)e^{-q(M-t)} - Ke^{-r(M-t)}, \quad t \in [0, M],$$

where  $s(x, t)$  is the known stock pricing function, and subject to the terminal condition:

$$\gamma(x, M) = [s(x, M) - K]^+, \quad x > L.$$

- By the Feynman-Kac theorem, the continuous solution to this BVP is:

$$\gamma(x, t) = e^{-r(M-t)} E_{x,t}^Q [s(W_M^a, M) - K]^+, \quad x > L, t \in [0, M),$$

where  $\{W_u^a, u \in [t, M]\}$  is a Q-SBM starting at  $x$  at time  $t$  and absorbing at  $L < 0$ .

## Valuing Standard Options in Terms of the Stock Price

- Recall that the function linking the value of the European call struck at  $K$  and maturing at  $M$  to the SBM is:

$$\gamma(x, t) = e^{-r(M-t)} E_{x,t}^Q [s(W_M^a, M) - K]^+, \quad x > L, t \in [0, M),$$

where  $\{W_u, u \in [t, M]\}$  is a Q-SBM starting at  $x$  at time  $t$  and absorbing at  $L < 0$ .

- To instead relate the call value to the stock price and time, let  $c(S, t) = \gamma(x, t)$  where  $x = s^{-1}(S, t)$ . Then  $c(S, t) =$ :

$$e^{-r(M-t)} \int_L^\infty \frac{[s(z, M) - K]^+}{\sqrt{2\pi(M-t)}} \left\{ \exp \left[ -\frac{1}{2} \left( \frac{z - s^{-1}(S, t)}{\sqrt{M-t}} \right)^2 \right] - \exp \left[ -\frac{1}{2} \left( \frac{z + s^{-1}(S, t) - 2L}{\sqrt{M-t}} \right)^2 \right] \right\} dz,$$

for  $S \geq 0, t \in [0, M \wedge \tau]$ , where  $\tau$  is the first passage time of the stock price to the origin.

- This solution will be an explicit function of  $S$  and  $t$  if  $s^{-1}(S, t)$  can be written explicitly in terms of its arguments.

## Risk-Neutral Density of the Stock Price

- Recall the expression for the call price, i.e.  $c(S, t) =$ :

$$e^{-r(M-t)} \int_L^\infty \frac{[s(z, M) - K]^+}{\sqrt{2\pi(M-t)}} \left\{ \exp \left[ -\frac{1}{2} \left( \frac{z - s^{-1}(S, t)}{\sqrt{M-t}} \right)^2 \right] - \exp \left[ -\frac{1}{2} \left( \frac{z + s^{-1}(S, t) - 2L}{\sqrt{M-t}} \right)^2 \right] \right\} dz,$$

- Differentiating twice w.r.t. strike  $K$  gives the risk-neutral probability density of the stock price at  $K$  at time  $M$ , given that the stock is at  $S$  at time  $t$ .
- Alternatively, the change of variables  $S = s(x, t)$  for the absorbing SBM transition density expresses this probability as:

$$q_{S,t}(Z, M) = \frac{1}{\sqrt{2\pi(M-t)}} \left\{ \exp \left\{ -\frac{1}{2} \left[ \frac{s^{-1}(Z, M) - s^{-1}(S, t)}{\sqrt{M-t}} \right]^2 \right\} - \exp \left\{ -\frac{1}{2} \left[ \frac{s^{-1}(Z, M) + s^{-1}(S, t)}{\sqrt{M-t}} \right]^2 \right\} \right\} \frac{\partial s^{-1}}{\partial S}(Z, M),$$

- The density will be positive only if  $s^{-1}(S, M)$  is increasing in  $S$  and it will be explicit in  $Z$  only if  $s^{-1}(Z, M)$  is explicit in  $Z$ .

## Example: The Arcsinormal Density Function

- Continuing with the previous example, the risk-neutral stock pricing density works out to:

$$q_{S,t}(Z, M) = \frac{1}{\sqrt{2\pi(M-t)}} \left\{ \exp \left\{ -\frac{1}{2} \left[ \frac{\sinh^{-1} \left( \frac{Z}{\beta'} e^{\mu(T'-M)} \right) - \sinh^{-1} \left( \frac{S}{\beta'} e^{\mu(T'-t)} \right)}{\alpha \sqrt{M-t}} \right]^2 \right\} - \exp \left\{ -\frac{1}{2} \left[ \frac{\sinh^{-1} \left( \frac{Z}{\beta'} e^{\mu(T'-M)} \right) + \sinh^{-1} \left( \frac{S}{\beta'} e^{\mu(T'-t)} \right)}{\alpha \sqrt{M-t}} \right]^2 \right\} \right\} \frac{1}{\alpha \sqrt{Z^2 + (\beta')^2 e^{-2\mu(T'-M)}}},$$

for  $S > 0, t \in [0, M \wedge \tau)$ .

- Figure 0.4 graphs this density (termed the arcsinormal) against the future spot price and time. The downward sloping volatility surface cancels much of the positive skewness of the lognormal density leading to a close approximation of a Gaussian density.

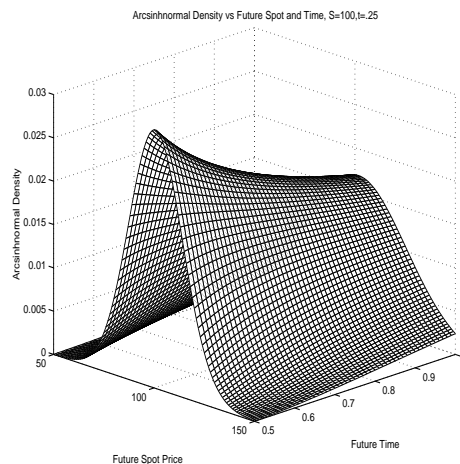


Figure 0.4: The Arcsinormal Probability Density Function

## Example: The Arcsinhnormal Call Pricing Function

- Integrating the call's payoff against the risk neutral density yields the following pricing formula:

$$\begin{aligned}
 C(S, t) = & \frac{e^{-q(M-t)}}{2} \left( S + \sqrt{S^2 + \beta^2 e^{-2\mu(T-t)}} \right) [N(d_+ + \alpha\sqrt{M-t}) \\
 & + N(d_- - \alpha\sqrt{M-t})] \\
 & - \frac{\beta^2 e^{-q(M-t)}}{2e^{2\mu(T-t)}} \frac{1}{S + \sqrt{S^2 + \beta^2 e^{-2\mu(T-t)}}} [N(d_+ - \alpha\sqrt{M-t}) \\
 & + N(d_- + \alpha\sqrt{M-t})] \\
 & - Ke^{-r(M-t)} [N(d_+) - N(d_-)],
 \end{aligned}$$

where:

$$d_{\pm} \equiv \frac{\pm \sinh^{-1} \left( \frac{S}{\beta} e^{\mu(T-t)} \right) - \sinh^{-1} \left( \frac{K}{\beta} e^{\mu(T-M)} \right)}{\alpha\sqrt{M-t}}.$$

- Figure 0.5 graphs the call value and time values of this model against the corresponding values in the Black Scholes model with the same at-the-money implied volatility.

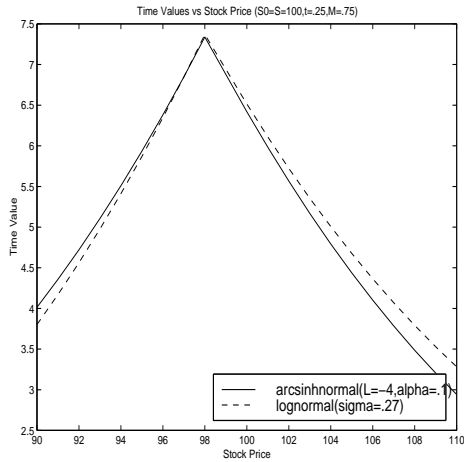


Figure 0.5: The Arcsinhnormal Call Value and Time Value vs. Black Scholes

- Figure 0.6 plots the arcsinhnormal call value against the current stock price and time.

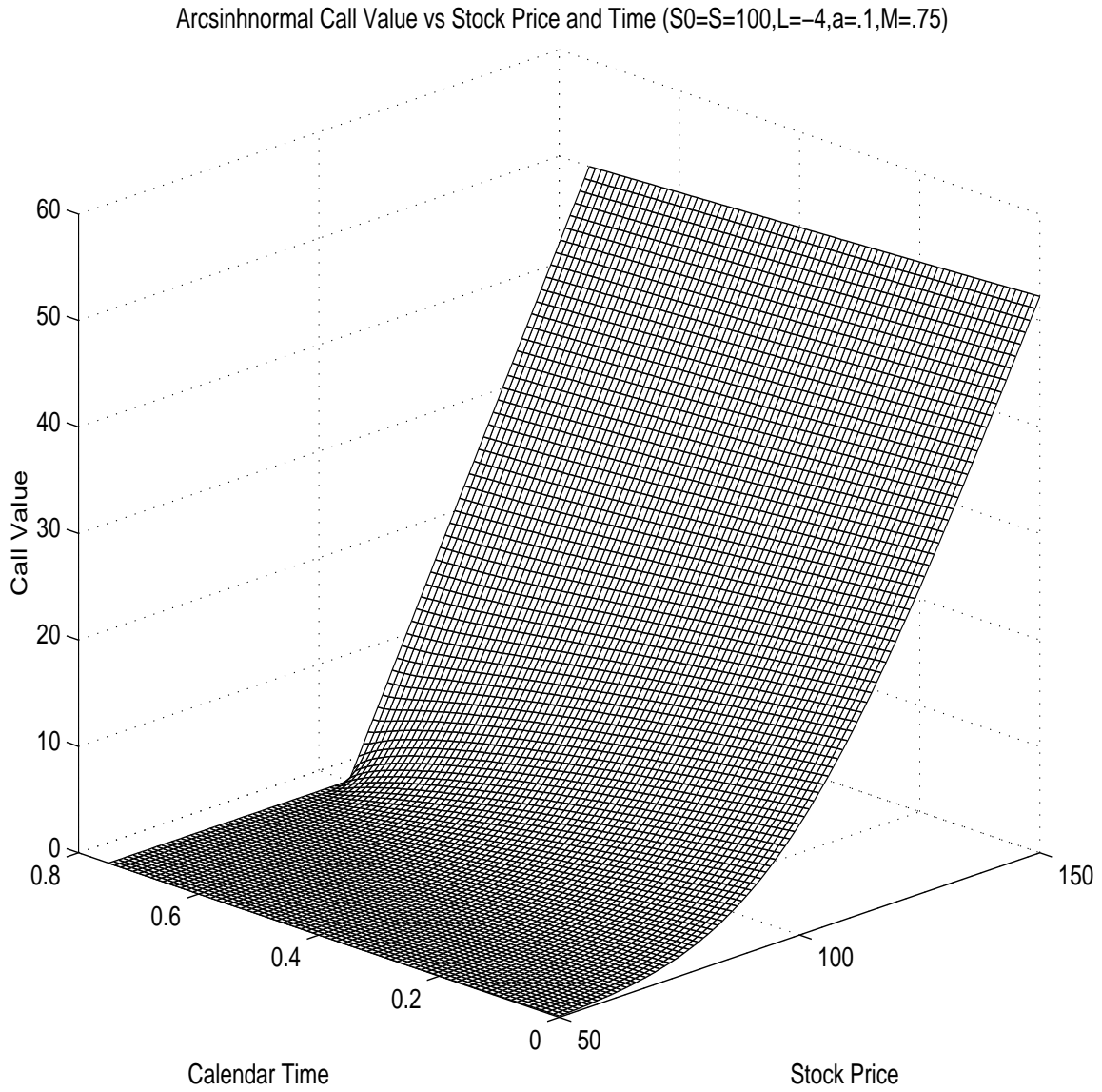


Figure 0.6: The Arcsinhnormal Call Valuation Function

## The SBM as a Path-Independent Exotic Option

- The SBM  $W_t$  is the forward price of a path-independent exotic option with the payoff  $g^{-1}(S_{T'})$  at  $T'$ , where  $g^{-1}(\cdot)$  is the inverse of the time  $T'$  implied payoff  $g(x)$ .
- This follows from the observation that deferring the payment of this exotic's premium to  $T'$  induces zero drift under  $Q$  and the payoff induces unit volatility.
- Thus,  $s(x, t)$  is properly called the spot pricing function, since it relates the spot price of the underlying to the (forward) value of an asset.
- Similarly,  $s^{-1}(S_t, t)$  is a standard pricing function relating the time  $t$  forward price of the Brownian exotic paying  $g^{-1}(S_{T'})$  at  $T'$  to the time  $t$  spot price and time.

## Local Volatility as a Path-Independent Exotic Option

- The local volatility  $\sigma(S_t, t)$  can also be interpreted as the price process for an exotic equity derivative. To determine the payoff, note that the  $T'$ -maturity *absolute volatility*  $a(S, T') \equiv \sigma(S, T')S$  is the following function of the time  $T'$  spot price  $S_{T'}$ :

$$a(S_{T'}, T') = g'(W_{T'}^a) = \frac{1}{\frac{\partial g^{-1}}{\partial S}(S_{T'})} = \lim_{h \downarrow 0} \frac{h}{g^{-1}(S_{T'} + h) - g^{-1}(S_{T'})}.$$

- The  $T'$ -maturity (relative) volatility smile arises from quantoing this dollar payoff into shares i.e. the payoff in shares is:

$$\sigma(S_{T'}, T') = \frac{g'(W_{T'}^a)}{S_{T'}} = \frac{1}{S_{T'}} \lim_{h \downarrow 0} \frac{h}{g^{-1}(S_{T'} + h) - g^{-1}(S_{T'})}.$$

- If we also assume that the premium on this exotic is specified in shares and paid at  $T'$ , then this share-denominated forward price matches the local volatility  $\sigma(S_t, t)$  at all times up to  $T'$ . The appearance of  $\sigma^2(S, t)$  in the middle term of the nonlinear PDE:

$$\frac{\sigma^2(S, t)S^2}{2} \frac{\partial^2 \sigma(S, t)}{\partial S^2} + [r - q + \sigma^2(S, t)]S \frac{\partial \sigma}{\partial S}(S, t) + \frac{\partial \sigma}{\partial t}(S, t) = 0$$

is now easily interpreted as a quanto correction. The absence of the usual discounting term is due to the deferral of the premium payment to maturity.

- As usual, the PDE for volatility is a consequence of the martingale property, which holds for volatility since it is the price of an asset.

## Part V

# Summary and Extensions

## Summary

- By interpreting the stock as essentially a path-independent derivative on the driving Brownian motion (or conversely), we were able to develop a PDE for the stock pricing function.
- Adding the information from a complete strike structure of option prices and the economic restrictions implied by no arbitrage, limited liability, and viability over an arbitrarily large horizon allowed us to uniquely determine this stock pricing function.
- Similarly, by interpreting local volatility as an exotic derivative on the stock, we were able to develop a nonlinear PDE for the volatility function, which we solved analytically using the known stock pricing function.
- Conversely, if local volatilities are assumed to satisfy the nonlinear PDE, then stock prices are path-independent and the function linking the SBM to the stock price can be determined analytically.
- Finally, we derived closed form formulas for option prices and risk-neutral densities consistent with a wide class of local volatility functions.

## Extensions

- path-independence after a deterministic time change (trivial)
- valuing American and other path-dependent options
- calibrating the smile to a matrix of initial option prices.
- different drivers eg. Bessel squared or symmetric VG