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Corridor Variance Swaps

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It is widely recognised that delta-hedged positions in options can be used to trade volatility. To facilitate volatility trading for their clients, several institutions routinely offer variance swaps. A variance swap is a financial contract that upon expiry pays the difference between a standard historical estimate of daily return variance and a fixed rate determined at inception. As in any swap, the fixed rate is initially chosen so that a variance swap has zero cost to enter. See Demeterfi *et al* (1999) or Carr & Madan (1998) for pricing and Chriss & Morokoff (1999) for risk management issues.

Over the past few years, several institutions have also begun offering corridor variance swaps. These differ from standard variance swaps only in that the underlying's price must be inside a specified corridor in order for its squared return to be included in the floating part of the variance swap payout. As in a standard variance swap, the fixed payment is made at maturity and is initially chosen so that the corridor variance swap has zero cost to enter. In the corridor variance swap considered in this chapter, the fixed payment is independent of the occupation time of the corridor. However, variations exist in which the fixed payment accrues over time at a constant rate only while the underlying is in the corridor.

A corridor variance swap is a generalisation of a standard variance swap in that the latter results from the former when the corridor is extended to all possible price levels. An upside variance swap uses a corridor extending from a fixed barrier up to infinity, while a downside variance swap uses a corridor extending from a fixed barrier down to zero. From the speculator's perspective, the

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advantage of a corridor variance swap over a variance swap is that it allows the expression of a view on volatility that is contingent upon the price level. For example, an investor who thinks that returns are likely to be more negatively skewed than predicted by the market might buy a downside variance swap and sell an upside variance swap. From the hedger's perspective, the advantage of a corridor variance swap over a standard variance swap is that the hedge involves fewer initial positions and less frequent revision over time.

Carr & Madan (1998) showed how to synthesise continuously monitored variance and corridor variance swaps when the underlying price process is assumed to be continuous. The purpose of this chapter is to show how to accurately approximate the payout to discretely monitored variance and corridor variance swaps under no assumptions about the underlying price process. Given the increasing recognition of the importance of jumps and that all swaps are monitored daily in practice, these extensions are long overdue. We show that with frictionless markets and deterministic interest rates, the payout to a corridor variance swap can be accurately approximated by combining at most daily trading in the underlying with static positions in standard European-style options maturing with the swap and struck inside the corridor. In particular, the approximation error is third order and, hence, the strategy replicates well if third and higher powers of daily returns sum to a negligible amount.

The structure of this chapter is as follows. In the next section we define the payouts to upside and downside variance swaps. Then we give a section showing how to approximate the payouts to these swaps by combining static positions in options struck inside the corridor with at most daily trading in the underlying futures. In the subsequent section we show the results of a Monte Carlo simulation of our payout definition and hedging strategy. In the penultimate section we discuss corridor variance swaps when the corridor is defined by two positive finite constants. We summarise and suggest extensions in the final section.

STRUCTURING UPSIDE AND DOWNSIDE VARIANCE SWAPS

Here, we define the payouts to upside and downside variance swaps. For an upside variance swap, the corridor needed to define the payout is the semi-infinite interval (L, ∞), where the fixed constant $L \geq 0$ denotes the lower bound. Upside variance swaps can be used to create other corridor variance swaps. For example, when $L = 0$, the payout to an upside variance swap will degenerate into the payout from a standard variance swap. For a downside variance swap, the payout will be given by the difference between the payout to a standard variance swap and an upside variance swap. To create a corridor variance swap whose supporting corridor has a positive lower bound and a finite upper bound, we can take the difference of two upside variance swaps with different lower bounds, as will be discussed in the penultimate section.

Consider a finite set of discrete times $\{t_0, t_1, \ldots, t_n\}$ at which the path of some underlying is monitored. In what follows, we use a futures price as the underlying and we take the monitoring times to be daily closings. Let F_0 denote the known initial futures price and let $F_i \geq 0$ denote the random futures price at the close of day i, for $i = 1, 2, \ldots, n$. For an upside variance swap, the futures price is said to start in the corridor on day i if $F_{i-1} > L$ and it is said to stop in the corridor on day i if $F_i > L$. The opposite inequalities hold for downside variance swaps. For an upside variance swap, the futures price is said to enter the corridor on day i if $F_{i-1} \leq L$ and $F_i > L$. In contrast, it is said to exit the corridor on day i if $F_{i-1} > L$ and $F_i \leq L$. For a downside variance swap, entry occurs on days when the futures price exits the upside corridor. Likewise, exit occurs on days when the futures price enters the upside corridor.

The exact specification of the payout to a corridor variance swap differs from firm to firm. Our specification of the payout to a corridor variance swap is chosen so that the hedging error can be made third order without imposing a model for price dynamics. We also insist that the payouts to upside and downside variance swaps be defined so that they sum to the payout of a standard variance swap. To begin specifying the payout of a corridor variance swap, let $\mathbb{1}_{F_{i-1} \in R_{i-1}, F_i \in R_i}$ denote the indicator function, which is one when F_{i-1} is in region R_{i-1} and F_i is in region R_i, but is zero otherwise. An upside variance contract is defined to be a financial security that has the following non-negative payout at the fixed time t_n:

$$Q_n^u(L) \equiv \sum_{i=1}^{n} \left\{ \mathbb{1}_{F_{i-1}>L, F_i>L} \left(\ln \frac{F_i}{F_{i-1}} \right)^2 + \mathbb{1}_{F_{i-1}\leq L, F_i>L} \left(\ln \frac{F_i}{L} \right)^2 \right.$$
$$\left. + \mathbb{1}_{F_{i-1}>L, F_i\leq L} \left[\left(\ln \frac{F_i}{F_{i-1}} \right)^2 - \left(\ln \frac{F_i}{F_L} \right)^2 \right] \right\} \quad (1.1)$$

The first term in the summand is due to the days in which the futures price starts and stops inside the upper corridor, while the last two terms are due to the entry and exit of the corridor, respectively. If the futures price starts and stops below the upper corridor on day i, then that day's move is ignored. If the futures price starts and stops in the corridor on day i, then that day's percentage change is squared. If the futures price enters the corridor on day i, then only the percentage change from L is squared. If the futures price exits the corridor on day i, then the square of the percentage change outside the corridor is subtracted from the square of the total percentage change.

Our formulation in (1.1) treats entry and exit asymmetrically. Under our asymmetric formulation, there exists a model-free hedging strategy whose error is only third order, as shown in the next section. In contrast, suppose that the exit payout was defined symmetrically to the entry payout to be $(\ln L/F_{i-1})^2$. Then, there does not exist a model-free hedging strategy whose error is only third order.

A second reason for our asymmetric treatment of entry and exit is that we want the sum of the payouts from an upside variance contract and a downside variance contract with the same barrier L to equal the payout from a standard variance contract. A downside variance contract is defined to be a financial security that has the following non-negative payout at the fixed time t_n:

$$Q_n^d(L) \equiv \sum_{i=1}^{n} \left\{ \mathbb{1}_{F_{i-1}\leq L, F_i\leq L} \left(\ln \frac{F_i}{F_{i-1}} \right)^2 + \mathbb{1}_{F_{i-1}>L, F_i\leq L} \left(\ln \frac{F_i}{L} \right)^2 \right.$$
$$\left. + \mathbb{1}_{F_{i-1}\leq L, F_i>L} \left[\left(\ln \frac{F_i}{F_{i-1}} \right)^2 - \left(\ln \frac{F_i}{F_L} \right)^2 \right] \right\} \quad (1.2)$$

The payouts in (1.1) and (1.2) sum to the following payout of a standard variance contract:

$$Q_n(L) \equiv \sum_{i=1}^{n} \left(\ln \frac{F_i}{F_{i-1}} \right)^2 \quad (1.3)$$

From (1.2), our treatment of entry and exit on the downside variance contract is also asymmetric. In contrast, suppose that the exit payout for a downside variance contract was defined symmetrically with the entry payout to be $(\ln L/F_{i-1})^2$. Then the payouts to upside and downside variance contracts with symmetrically defined exit and entry would not sum to the payout (1.3) of a variance contract. The reason is that while the total return decomposes into the returns to and from the barrier,

$$\ln \frac{F_i}{F_{i-1}} = \ln \frac{F_i}{L} + \ln \frac{L}{F_{i-1}} \tag{1.4}$$

the squared total return differs from the sum of squared returns to and from the barrier by twice the product of these returns,

$$\left(\ln \frac{F_i}{F_{i-1}}\right)^2 = \left(\ln \frac{F_i}{L}\right)^2 + \left(\ln \frac{L}{F_{i-1}}\right)^2 + 2\left(\ln \frac{F_i}{L}\right) \ln\left(\frac{L}{F_{i-1}}\right) \tag{1.5}$$

Hence, if entry and exit are defined symmetrically for both upside and downside variance contracts, the payout to a portfolio of an upside and downside variance contract would miss the payout to a variance contract by the last term in (1.5).

The asymmetry of our payout definition in (1.1) vanishes if one assumes continuous price processes, continuous path monitoring and the ability to trade the underlying continuously. Under these assumptions, the hedging strategy we propose in the next section works perfectly. It should not be too surprising that the relaxation of these idealised conditions necessarily introduces replication error. What is perhaps surprising is that the replication error can be kept to third order provided that one is willing to treat entry and exit asymmetrically.

By definition, a corridor variance swap is a swap with a single payment at maturity given by the difference between a floating part and a fixed part:

$$VS_n^c(L) = Q_n^c(L) - F_0^c(L) \tag{1.6}$$

The floating part is the payout $Q_n^c(L)$ to a corridor variance contract, where the superscript c takes on the value u for an upside variance swap and the value d for a downside variance swap. The fixed payment $F_0^c(L), c = u, d$ is chosen at time t_0 so that the corridor variance swap has zero initial cost of entry. Suppose that $Q_0^c(L)$

is the known initial cost of creating the terminal random payout $Q_n^c(L)$, where $c = u, d$. Then the fair fixed payment to initially charge on the corridor variance swap is simply given by

$$F_0^c(L) = \frac{Q_0^c(L)}{B_0(t_n)}, \quad c = u, d \tag{1.7}$$

where $B_0(t_n)$ denotes the initial price of a pure discount bond paying US\$1 at t_n. As a result, the next section focuses on determining an accurate approximation to $Q_0^c(L)$.

APPROXIMATE REPLICATION

Assumptions

For the rest of this chapter, we assume frictionless markets and deterministic interest rates. We also assume that one can trade futures at the same frequency (for example, daily) with which marking-to-market and swap monitoring occur. Finally, we assume that one can take static positions in the continuum of European-style futures options with strikes inside the supporting corridor and maturing with the swap. Note that we make no assumptions regarding the stochastic process followed by futures or option prices. In particular, jumps are allowed, volatility can be stochastic and the process parameters do not need to be known. Under the above assumptions, this section shows that the payouts on upside and downside variance swaps can be well approximated by combining static positions in standard options with at most daily trading in the underlying futures.

The market that probably best approaches the above idealised conditions is that for S&P500 derivatives, where one has liquid trading in both futures and in European-style options. Although the S&P500 index options are written on the cash index, they often mature with the futures and, hence, in those cases can be regarded as European-style futures options. Furthermore, as our replication error will be related to third and higher moments of the underlying's return, the reduction in these moments arising from diversification in the index is attractive. We next review the approximate replication of a variance swap payout before tackling the harder problem of approximately replicating the payout to a corridor variance swap.

Variance swap

It is well known that the geometric mean of a set of positive numbers is never greater than the arithmetic mean. It is also well known that the larger the variation in the set of numbers, the greater the disparity between the two means. The approximate replication of the payout to a variance swap exploits this basic property.

By Taylor series expansion of $\ln F$ about $F = F_{i-1}$, we note that

$$\ln F_i - \ln F_{i-1} = \frac{1}{F_{i-1}}\Delta F_i - \frac{1}{2F_{i-1}^2}(\Delta F_i)^2 + O\left(\frac{\Delta F_i}{F_{i-1}}\right)^3, \quad i = 1, \ldots, n \tag{1.8}$$

where $\Delta F_i \equiv F_i - F_{i-1}$ denotes the change in the futures price over day i. Rearranging implies that the squared daily return is just twice the difference between the daily compounded return and the continuously compounded return, up to terms of order $O(\Delta F_i/F_{i-1})^3$:

$$\left(\frac{\Delta F_i}{F_{i-1}}\right)^2 = 2\left[\frac{\Delta F_i}{F_{i-1}} - \ln\left(\frac{F_i}{F_{i-1}}\right)\right] + O\left(\frac{\Delta F_i}{F_{i-1}}\right)^3, \quad i = 1, \ldots, n \tag{1.9}$$

Squaring both sides of (1.8) implies:

$$\left(\ln \frac{F_i}{F_{i-1}}\right)^2 = \left(\frac{\Delta F_i}{F_{i-1}}\right)^2 + O\left(\frac{\Delta F_i}{F_{i-1}}\right)^3, \quad i = 1, \ldots, n \tag{1.10}$$

Substituting (1.10) in (1.9) implies that the squared continuously compounded return is just twice the difference between the daily compounded return and the continuously compounded return, up to terms of order $O(\Delta F_i/F_{i-1})^3$:

$$\left(\ln \frac{F_i}{F_{i-1}}\right)^2 = 2\left[\frac{\Delta F_i}{F_{i-1}} - \ln\left(\frac{F_i}{F_{i-1}}\right)\right] + O\left(\frac{\Delta F_i}{F_{i-1}}\right)^3, \quad i = 1, \ldots, n \tag{1.11}$$

Summing over i gives a decomposition of the sum of squared returns:

$$\begin{aligned}\sum_{i=1}^{n}\left(\ln \frac{F_i}{F_{i-1}}\right)^2 &= \sum_{i=1}^{n} \frac{2}{F_{i-1}}\Delta F_i - 2\sum_{i=1}^{n}(\ln F_i - \ln F_{i-1}) + \sum_{i=1}^{n} O\left(\frac{\Delta F_i}{F_{i-1}}\right)^3 \\ &= \sum_{i=1}^{n} \frac{2}{F_{i-1}}\Delta F_i - 2\ln F_n + 2\ln F_0 + \sum_{i=1}^{n} O\left(\frac{\Delta F_i}{F_{i-1}}\right)^3\end{aligned} \tag{1.12}$$

due to telescoping. Thus, up to third-order terms, the sum of squared returns decomposes into the payout from a dynamic futures strategy and a function $f(F_n) = -2\ln F_n + 2\ln F_0$ of just the final futures price. As a static position in bonds and options can be used to create this final payout function, approximate replication is feasible.

There is some flexibility in choosing the composition of the replicating portfolio since any linear function added to f can be offset by the appropriate position in bonds and futures. As our ultimate goal is to approximately replicate the payout to a corridor variance swap, we will add a linear function to f so that it becomes U-shaped with the minimum occurring at L. Hence, for any $L > 0$, suppose that we subtract and add $2\ln L + 2/L \times (F_n - F_0)$ to the right-hand side of (1.12):

$$\sum_{i=1}^{n}\left(\ln \frac{F_i}{F_{i-1}}\right)^2 = \sum_{i=1}^{n} \frac{2}{F_{i-1}}\Delta F_i - \frac{2}{L}(F_n - F_0) + u(F_n) - u(F_0) + \sum_{i=1}^{n} O\left(\frac{\Delta F_i}{F_{i-1}}\right)^3 \tag{1.13}$$

where

$$u(F) \equiv 2\left[\frac{F-L}{L} - \ln\frac{F}{L}\right] \tag{1.14}$$

As shown in Carr & Madan (1998), any continuous payout at t_n of just the final futures price can be spanned by the payouts from a static position in bonds and European-style options maturing at t_n. To determine the replicating portfolio for the payout $u(F_n)$, note that the function $u(F)$ is U-shaped with zero value and slope at $F = L$. The second derivative $u''(F) = 2/F^2 > 0$. Hence, a Taylor series expansion with second-order remainder of $u(F_n)$ about $F_n = L$ implies

$$u(F_n) = \int_0^L \frac{2}{K^2}(K - F_n)^+\, dK + \int_L^\infty \frac{2}{K^2}(F_n - K)^+\, dK \tag{1.15}$$

Now,

$$F_n - F_0 = \sum_{i=1}^{n} \Delta F_i$$

and substituting this and (1.15) into (1.13) implies

$$\sum_{i=1}^{n}\left(\ln\frac{F_i}{F_{i-1}}\right)^2=\sum_{i=1}^{n}\left(\frac{2}{F_{i-1}}-\frac{2}{L}\right)\Delta F_i+\int_0^L\frac{2}{K^2}(K-F_n)^+\,dK\\ +\int_L^\infty\frac{2}{K^2}(F_n-K)^+\,dK-u(F_0)+\sum_{i=1}^{n}O\left(\frac{\Delta F_i}{F_{i-1}}\right)^3 \tag{1.16}$$

Thus, the payout to a variance swap is well approximated by summing the payouts from a dynamic position in futures and a static position in options and bonds. For the dynamic component, (1.16) indicates that one holds $e^{-y_{in}(t_n-t_i)}(2/F_{i-1}-2/L)$ futures contracts from day $i-1$ to day i, where y_{in} is the continuously compounded yield on day t_i to maturity t_n. For the static component, (1.16) indicates that one holds $(2/K^2)\,dK$ puts at all strikes below L, $(2/K^2)\,dK$ calls at all strikes above L and one shorts $u(F_0)$ bonds. If the initial cost of the approximate hedge is financed by borrowing, then the repayment at t_n is

$$\int_0^L\frac{2}{K^2}\frac{P_0(K,t_n)}{B_0(t_n)}\,dK+\int_L^\infty\frac{2}{K^2}\frac{C_0(K,t_n)}{B_0(t_n)}\,dK-u(F_0) \tag{1.17}$$

where $P_0(K,t_n)$ and $C_0(K,t_n)$, respectively, denote the initial prices of puts and calls struck at K and maturing at t_n. If there is no charge for the third-order approximation error, then (1.17) is the fair (non-annualised) fixed payment for a variance swap on US$1 of notional. This fixed payment is actually independent of the choice of L, since it only depends on the convexity of the payout.

One can interpret the dynamic component of our approximate replicating strategy as a Black (1976) model dynamic hedge to the static portfolio described above. By the Black model dynamic hedge, we have in mind that the hedger trades futures continuously under the belief that the futures price process is continuous with constant volatility σ. As is well known, the number of futures held at any time in this model is given by the first partial derivative of the value function with respect to the futures price. To show that the dynamic component of our hedge can be interpreted as a Black model dynamic hedge, let

$$
\begin{aligned}
U(F_n) &\equiv \int_0^L \frac{2}{K^2}(K - F_n)^+ \, dK + \int_L^\infty \frac{2}{K^2}(F_n - K)^+ \, dK - u(F_0) \\
&= u(F_n) - u(F_0) = 2\left[\frac{F_n - F_0}{L} - \ln\frac{F_n}{F_0}\right] \qquad (1.18)
\end{aligned}
$$

be the U-shaped payout created by the static position in bonds and options. Since $u(L) = 0$ by (1.14), U takes its minimum value of $-u(F_0)$ at $F_n = L$. The Black model value at time t_{i-1} of this payout is given by

$$
\begin{aligned}
V(F_{i-1}, t_{i-1}) \equiv e^{-y_{i-1,n}(t_n - t_{i-1})} 2\left[\frac{F_{i-1} - F_0}{L} - \ln\frac{F_{i-1}}{F_0}\right] \\
- \sigma^2(t_n - t_{i-1})
\end{aligned}
$$

Hence, the Black model delta at time t_{i-1} is

$$
\frac{\partial}{\partial F} V(F_{i-1}, t_{i-1}) = e^{-y_{i-1,n}(t_n - t_{i-1})}\left(\frac{2}{F_{i-1}} - \frac{2}{L}\right) \qquad (1.19)
$$

which differs from the number of futures needed to hedge a variance swap only by a small present value factor. Surprisingly, the Black model delta in (1.19) is actually independent of σ, that is, $\partial^2 V / \partial\sigma\partial F = 0$. Put another way, the static portfolio is chosen so that its Black model vega is independent of the futures price. Of course, the Black model dynamic hedge of this portfolio only works perfectly under continuous trading, continuous price paths and constant volatility. Since the approximate replicating strategy actually involves only discrete trading at prices that can reflect jumps and stochastic volatility, one would anticipate that this attempt at a Black model hedge would fail. However, for the particular static portfolio described above, the hedging error approximates the payout to a variance swap with fixed payment $\sigma^2(t_n - t_0)$.

Upside and downside variance swaps

The previous section showed that the payout to a variance swap

$$
\sum_{i=1}^{n}\left(\ln\frac{F_i}{F_{i-1}}\right)^2 - \sigma^2(t_n - t_0)
$$

could be approximately replicated by forming a static portfolio of options and bonds that has the U-shaped payout $U(F_n)$. This portfolio is delta-hedged daily with the Black model delta for each option calculated using the fixed volatility rate σ. It follows that if we just wish to create the payout to a variance contract

$$\sum_{i=1}^{n}\left(\ln\frac{F_i}{F_{i-1}}\right)^2$$

we could delta-hedge each option at $\sigma = 0$.

To approximate the payouts to upside and downside variance contracts, suppose that we guess that the approximate hedge just involves delta-hedging options struck above and below the barrier, respectively, where each option is delta-hedged at zero volatility. Hence, for the upside variance contract, the static component of the proposed hedge has a payout that is constant for $F_n \leq L$ and is given by the right half of the U-shaped payout $U(F_n)$ defined in (1.18) for $F_n > L$:

$$U_r(F_n) \equiv u_r(F_n) - u_r(F_0), \quad \text{where } u_r(F) \equiv u(F)\mathbb{1}_{F>L} \tag{1.20}$$

To create the payout $U_r(F_n)$, we would only hold the $(2/K^2)\,dK$ calls at all strikes K above the lower bound L and we would also short $u_r(F_0)$ bonds. No puts would be held.

Similarly, the proposed hedge for the downside variance contract has a static component payout that is constant for $F_n \geq L$ and given by the left half of the U-shaped payout $U(F_n)$ defined in (1.18) for $F_n < L$:

$$U_\ell(F_n) \equiv u_\ell(F_n) - u_\ell(F_0), \quad \text{where } u_\ell(F) \equiv u(F)\mathbb{1}_{F<L} \tag{1.21}$$

Hence, we hold $(2/K^2)\,dK$ puts at all strikes K below L and we short $u_\ell(F_0)$ bonds. No calls are held. We note that the sum of the static positions in options in the proposed hedges for the upside and downside variance contracts is just the static option position in the hedge for the variance contract.

For the dynamic components of the proposed hedges, recall that we delta-hedge each option at $\sigma = 0$. Hence, for the dynamic component of the proposed hedge for an upside variance contract, one holds $-e^{-y_{in}(t_n-t_i)(2/L-2/F_{i-1})^+}$ futures contracts from day $i-1$ to day i, since out-of-the-money call deltas vanish under zero volatility. For the dynamic component of the proposed hedge for a downside variance contract, one holds $-e^{-y_{in}(t_n-t_i)}(2/F_{i-1}-2/L)^+$ futures contracts from day $i-1$ to day i, since out-of-the-money put deltas also vanish under zero volatility. We again note that the sum of the dynamic futures positions in the proposed hedges for the upside and downside variance contracts is just the dynamic futures position in the hedge for the variance contract.

If the futures price opens and closes below L, then in the proposed hedge for the upside variance contract, no futures are held and there is no gain in the bonds or out-of-the-money calls marked at zero volatility. Likewise, the intrinsic value of the upside variance contract does not change in this case, so the proposed hedge works perfectly in this case. Furthermore, the proposed hedge for a downside variance contract must also work in this case since this hedge is just the difference in the successful hedges of a variance contract and an upside variance contract.

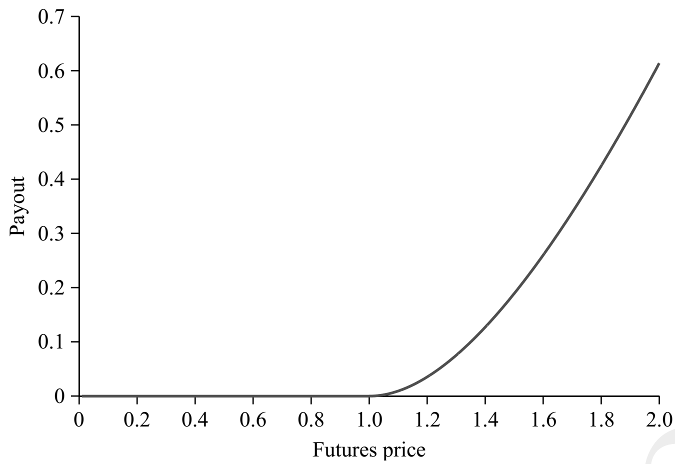
If the futures price opens below L and closes above, then no futures are held in the proposed hedge for the upside variance contract, and the intrinsic value of the call portfolio rises from zero to $u_r(F_i)$. Figure 1.1 shows u_r as a function of F when $L = 1$. The first derivative is $u_r'(F) = (2/L - 2/F)^+$, which is continuous at $F = L$. The second derivative is $u_r''(F) = \mathbb{1}_{F>L} 2/F^2$, which is discontinuous at $F = L$. By a Taylor series expansion of $u(F_i)$ about $F_i = L$:

$$u_r(F_i) = \left(\frac{\Delta F_i}{L}\right)^2 + O\left(\frac{\Delta F_i}{F_{i-1}}\right)^3 \tag{1.22}$$

for $F_i > L$ since $|F_i - L| \leq |\Delta F_i|$. On days when the futures price enters the upper corridor, the intrinsic value of the upside variance contract rises by $(\ln F_i/L)^2$. From (1.10), this differs from $u_r(F_i)$ by $O(\Delta F_i/F_{i-1})^3$, so the proposed hedge is sufficiently accurate in this case as well. Furthermore, the proposed hedge for a downside variance contract must also work in this second case for the same reason as in the first case.

If the futures price opens and closes above L, then the analysis of the previous section implies that the proposed hedge of the upside variance contract has a profit and loss of $(\Delta F_i/F_{i-1})^2 + O(\Delta F_i/F_{i-1})^3$, while the intrinsic value of the upside variance contract rises by $(\ln F_i/F_{i-1})^2$. From (1.10), the proposed hedge has

Figure 1.1 Function u_r with $L = 1$.



sufficient accuracy in this third case. Furthermore, the proposed hedge for a downside variance contract must also work in this case.

If the futures price opens above L but closes below it, then the analysis of the profit and loss from the proposed hedge of the upside variance contract is complicated by the fact that $u_r(F)$ defined in (1.20) is not an analytic function of F. However, we note that exit of the upper corridor is equivalent to entry of the lower corridor. If the futures price enters the lower corridor from above, then no futures are held in the proposed hedge to the downside variance contract and the intrinsic value of the puts held rises from zero to

$$u_\ell(F_i) = \left(\frac{\Delta F_i}{L}\right)^2 + O\left(\frac{\Delta F_i}{F_{i-1}}\right)^3, \quad F_i < L \tag{1.23}$$

from a Taylor series expansion of $u(F_i)$ about $F_i = L$. When the futures price enters the lower corridor, the intrinsic value of the downside variance contract rises by $(\ln F_i/L)^2$, which only differs from $u_\ell(F_i)$ by $O(\Delta F_i/F_{i-1})^3$. From (1.10), the proposed hedge to the downside variance contract has sufficient accuracy in this last case. It follows that the proposed hedge for the upside variance contract must also work in this case since this hedge is just the difference in the successful hedges of a variance contract and the downside variance contract.

We have just shown that the payout $Q_n^u(L)$ of the upside variance contract is well approximated in all cases:

$$Q_n^u(L) = \int_L^\infty \frac{2}{K^2}(F_n - K)^+ \, dK - u_r(F_0) \\ - \sum_{i=1}^{n} \left(\frac{2}{L} - \frac{2}{F_{i-1}} \right)^+ \Delta F_i + \sum_{i=1}^{n} O\left(\frac{\Delta F_i}{F_{i-1}} \right)^3 \tag{1.24}$$

If the initial cost of the approximate hedge is financed by borrowing, then the repayment at t_n is

$$\int_L^\infty \frac{2}{K^2} \frac{C_0(K, t_n)}{B_0(t_n)} \, dK - u_r(F_0) \tag{1.25}$$

If there is no charge for the third-order approximation error, then this is the fair (non-annualised) fixed payment for an upside variance swap on US\$1 of notional. The corresponding entity for a downside variance contract is

$$\int_0^L \frac{2}{K^2} \frac{P_0(K, t_n)}{B_0(t_n)} \, dK - u_\ell(F_0) \tag{1.26}$$

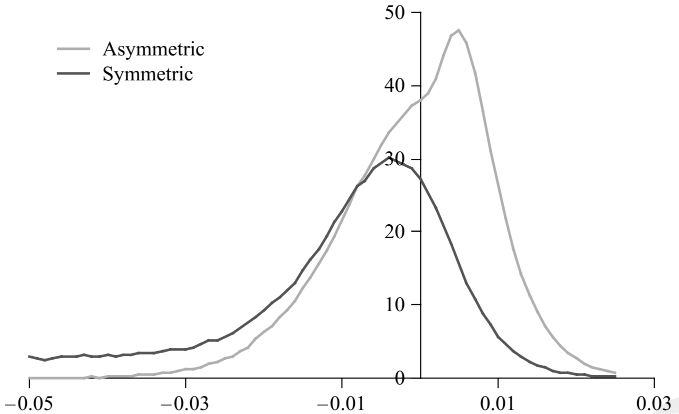
MONTE CARLO SIMULATION

This section reports the distribution of hedging errors arising from simulating the hedge of an upside variance swap over 250,000 paths. We considered the errors arising from hedging the sale of an upside variance swap with daily monitoring, a three-month term and a lower barrier of US\$90. We first assumed that the underlying futures price starts at US\$100 and follows geometric Brownian motion with 5% real-world drift and 30% volatility.

Figure 1.2 shows the density function of the hedging errors. The units on the x-axis correspond to the fixed leg price of $900 = 10{,}000(30\%)^2$. The curve designated symmetric corresponds to an upside variance swap payout where exit and entry are treated symmetrically. The curve designated asymmetric uses the asymmetric upside variance swap payout proposed in this chapter. The asymmetric profit and loss has mean -0.0097 and standard deviation 0.01 while the symmetric profit and loss has mean -0.042 and standard deviation 0.069. Note that the symmetric profit and loss has a fat tail for losses.

We next assumed that the underlying futures price follows the jump diffusion process suggested by Merton (1976). The initial

Figure 1.2 Distribution of profit and loss under geometric Brownian motion.



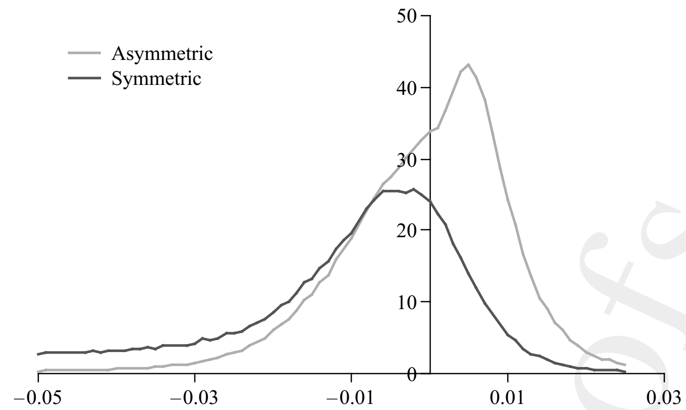
futures price start at US\$100, has 5% real-world drift and a 30% diffusion coefficient as before. We set the arrival rate equal to one, so that jumps arrive once a year on average. The jump in the log price is normally distributed with mean zero and standard deviation of 10%. Figure 1.3 summarises the results of the previous simulation but now using the Merton model. The hedges corresponding to the asymmetric payout definition continue to perform well, while the same hedges coupled to a symmetric payout definition are actually "short jumps". The asymmetric mean is -0.0037 with standard deviation 0.094 while the symmetric mean is -0.08 with standard deviation 0.3.

These simulation results clearly suggest that the complications arising from an asymmetric payout definition are outweighed by the improved hedge effectiveness.

INTERIOR CORRIDOR

In this section, we generalise our results on upside and downside variance swaps to corridor variance swaps with an interior corridor. Let $\mathcal{C}$ denote the interior corridor (L, H) where $L > 0$ and H is finite. The futures price is now said to enter the corridor on day i if $F_{i-1} \notin \mathcal{C}$ and $F_i \notin \mathcal{C}$. In this case, define the entry price N_{i-1} as $N_{i-1} \equiv L$ if $F_{i-1} < L$ and $N_{i-1} \equiv H$ if $F_{i-1} > H$. The futures price is now said to exit the corridor on day i if $F_{i-1} \in \mathcal{C}$ and $F_i \notin \mathcal{C}$. In this case, define

Figure 1.3 Distribution of profit and loss under jump diffusion.



the exit price X_i as $X_i \equiv L$ if $F_i < L$ and $X_i \equiv H$ if $F_i > H$. The payout on the corridor variance contract is now defined as

$$Q_n(L, H) \equiv \sum_{i-1}^{n} \Bigg\{ \mathbb{1}_{F_{i-1} \in \mathcal{C}, F_i \in \mathcal{C}} \left(\ln \frac{F_i}{F_{i-1}} \right)^2 + \mathbb{1}_{F_{i-1} \notin \mathcal{C}, F_i \in \mathcal{C}} \left(\ln \frac{F_i}{N_{i-1}} \right)^2 + \mathbb{1}_{F_{i-1} \in \mathcal{C}, F_i \notin \mathcal{C}} \left[\left(\ln \frac{F_i}{F_{i-1}} \right)^2 - \left(\ln \frac{F_i}{X_i} \right)^2 \right] + [\mathbb{1}_{F_{i-1} < L, F_i > H} + \mathbb{1}_{F_{i-1} > H, F_i < L}] \times \left| \left(\frac{F_i - L}{L} \right)^2 - \left(\frac{F_i - H)}{H} \right)^2 \right| \Bigg\} \tag{1.27}$$

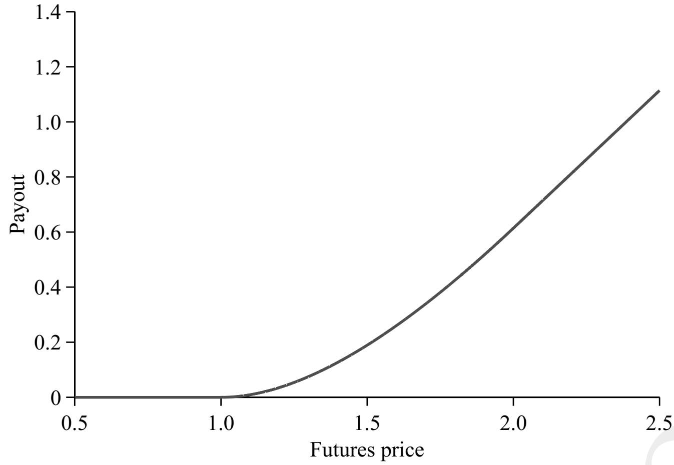
The first three terms in the summand correspond to the three terms in the summand in (1.1). The last term arises from the possibility of jumping over the corridor in either direction. In either case, the squared return from the exit price is subtracted from the squared return from the entry price.

To create the payout in (1.27), consider the following function:

$$\phi(F) \equiv \begin{cases} \mathbb{1}_{F > L} u(F) & \text{if } F \leq H \\ 2\left[\dfrac{H}{L} - 1 - \ln\left(\dfrac{H}{L} \right) \right] + \left(\dfrac{2}{L} - \dfrac{2}{H} \right)(F - H) & \text{if } F > H \end{cases} \tag{1.28}$$

where $u(F)$ is defined in (1.14). Thus, $\phi(F) = u_r(F)$ for $F \leq H$ and is the tangent to u_r at $F = H$ for $F > H$. Figure 1.4 graphs ϕ against F.

Figure 1.4 Function ϕ with $L = 1$ and $H = 2$.



The function ϕ is continuous and differentiable everywhere, but it is not twice differentiable at L and at H.

We can use calls maturing at t_n to create the payout $\phi(F_n)$:

$$\phi(F_n) = \int_L^H \frac{2}{K^2}(F_n - K)^+ \, dK \tag{1.29}$$

Using an analysis similar to that in the last section, we conclude that

$$\begin{aligned} Q_n(L, H) = {} & \int_L^H \frac{2}{K^2}(F_n - K)^+ \, dK - \phi(F_0) \\ & - \sum_{i=1}^{n} \left(\frac{2}{L} - \frac{2}{F_{i-1 \wedge H}} \right)^+ \Delta F_i + \sum_{i=1}^{n} O\left(\frac{\Delta F_i}{F_{i-1}} \right)^3 \end{aligned} \tag{1.30}$$

Thus, the desired payout is again well approximated by the sum of the payout from a static position in calls and bonds with the payouts from a dynamic position in futures. For the static component, one holds $(2/K^2)dK$ calls at all strikes in the corridor (L, H). One also borrows $\phi(F_0)$ pure discount bonds paying US\$1 at t_n. For the dynamic component, one holds $-e^{-y_{in}(t_n - t_i)}(2/L - 2/F_{i-1} \wedge H)^+$ futures contracts from day $i-1$ to day i. When $F_{i-1} \leq L$, no futures are held, as in the last section. When $F_{i-1} > H$, the number of futures contracts held is independent of F_{i-1}, so that the number of contracts held hardly changes while the underlying remains above the corridor. If the initial cost of the approximate hedge is financed by borrowing, then the repayment at t_n is

$$\int_L^H \frac{2}{K^2} \frac{C_0(K, t_n)}{B_0(t_n)} \, dK - \phi(F_0) \tag{1.31}$$

If there is no charge for the third-order approximation error, then this is the fair (non-annualised) fixed payment for the interior corridor variance swap on US$1 of notional.

SUMMARY AND EXTENSIONS

We defined the payout to a corridor variance swap in such a way that the payout could be well approximated by the payout from combining static positions in options and bonds with at most daily trading in the underlying futures. Although our payout definition treats entry and exit asymmetrically, it treats entry for a downside variance swap symmetrically with entry for an upside variance swap. Exit for the two swaps is also treated symmetrically. As a result, the sum of a downside variance swap and an upside variance swap is a standard variance swap, which does not remain true when exit and entry are treated symmetrically.

There are at least seven extensions to this work. First, one can further analyse our small approximation error to see whether it can be at least partially spanned. For example, a simple linear regression of the error on a constant and the change in the futures price can be used as a guide to how to account for this error in determining the fixed rate for the corridor variance swap and the dynamic component of the hedge. Using ordinary least squares, one finds that the return skewness affects the fixed rate, while the return kurtosis affects the futures position. Second, one can try to relax our model assumptions such as continuum of strikes, deterministic interest rates and frictionless futures trading, or at least try to determine their effect. Third, one can attempt to determine the effect of small perturbations in our definitions. For example, (1.10) makes it clear that the hedge has the same order error if daily returns are discretely compounded rather than continuously compounded. One can also try to determine the effect of demeaning the daily returns as some (corridored) variance swaps have this feature. Fourth, one can adapt our approach to make it more applicable to the corridor variance realised from individual stock returns.

If stocks replace single-name futures as the underlying, then stock dividends become relevant. Also, if listed options are used in the hedge, then American-style options must be handled. Fifth, one can supplement the approximation developed here by also developing bounds on the fair fixed payment via super- and sub-replication of the corridor variance. Sixth, it would be interesting to extend this work by characterising the entire class of path-dependent payouts that can be approximated or bounded in this way. Finally, one can also try to develop a theory of model-free approximate hedging that would in general allow semi-dynamic trading in both futures and options. In the interests of brevity, these extensions are best left for future research.

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