Forward Equations for American and Path-Dependent Options

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Motivation

Propagating option prices in the maturity and strike directions

Enhances computational efficiency of calibration

Promotes computational efficiency in marking

We always work in a one factor Markov setting

We consider various additional process assumptions (e.g. continuity, stationarity, independent increments)
Outline

Single Factor Markovian Stock Price Process

Review of the Dupire PDE for European options

Forward PDE for Down-and-Out Call Values

Forward PDE for Up-and-Out Call Values

Four PIDE’s for American Put Values
We always assume that under a risk neutral measure $\mathbb{Q}$, the stock price $s_t$ solves the following stochastic differential equation:

$$d s_t = \left[ r(t) - q(t) \right] s_t \, dt + \sigma(s_t, t) s_t \, dW_t + \int_{-\infty}^{\infty} s_t - (e^x - 1) \left[ \mu(dx, dt) - \nu(x, s_t, t)dxdt \right],$$

for all $t \in [0, \bar{T}]$. Thus, the change in the stock price decomposes into the risk-neutral drift, the diffusion part, and the jump part.

The random measure $\mu(dx, dt)$ counts the number of jumps of size $x$ in the log price at time $t$. 
The Hunt Lévy density \( \{ \nu(x, s, t), s > 0, x \in \mathbb{R}, t \in [0, \bar{T}] \} \) compensates the jump process

\[
J_t \equiv \int_0^t \int_{-\infty}^\infty s_t (e^x - 1) \mu(dx, dt)
\]

so that the last term is the increment of a \( \mathbb{Q} \) jump martingale. Thus

\[
\mathbb{E}_\mathbb{Q}[s_t | s_0] = s_0 e^{\int_0^t [r(u) - q(u)] du}.
\]
The Dupire PDE

Assuming no jumps, Dupire derives the following forward PDE for European call prices

\[ \frac{\sigma^2(K,T)}{2} K^2 \frac{\partial^2 c}{\partial K^2} - \left[ r(T) - q(T) \right] K \frac{\partial c}{\partial K} - q(T) c(K,T) = \frac{\partial c}{\partial T} \]

By having the local volatility surface, \( \sigma(K,T) \), one can compute call prices, \( c(K,T) \), for all strikes and maturities.

Or by having the call prices, \( c(K,T) \), the local volatility surface, \( \sigma(K,T) \), can be calculated.
In our numerical examples, we consider the following local volatility surface

\[ \sigma(K, T) = 0.7e^{-T}(100/K)^{0.2} \]
Forward PDE for Down-and-Out Calls

Also assuming no jumps, the forward PDE for DOC with \( K > H \) is the same as the Dupire PDE

\[
\frac{\sigma^2(K, T)}{2} K^2 \frac{\partial^2 D}{\partial K^2} - [r(T) - q(T)] K \frac{\partial D}{\partial K} - q(T) D = \frac{\partial D}{\partial T}
\]

with initial condition

\[
D(K, 0) = (S_0 - K)^+, \text{ for } K > H, \text{ and } S_0 > H.
\]

Boundary conditions are

\[
D_{KK}(H, T) = 0
\]

\[
\lim_{K \to \infty} D_{KK}(K, T) = 0
\]
In this illustration, the variables are: barrier $H = 80$, spot $S_0 = 90$, risk-free rate $r = 0.05$, and dividend rate $q = 0.02$. 
Under no jumps, the forward PDE for UOC with $K < H$ is*:

$$
\frac{\sigma^2(K, T)}{2} K^2 \frac{\partial^2 U}{\partial K^2} - [r(T) - q(T)] K \frac{\partial U}{\partial K} - q(T)U = \\
\frac{\partial U}{\partial T} + \left[ \sigma^2(H, T) \right] H^2 \frac{\partial^3 U}{\partial K^3}(H, T) (K - H)
$$

with initial condition

$$
U(K, 0) = (S_0 - K)^+, \text{ for } K < H, \text{ and } S_0 < H.
$$

Boundary conditions are

$$
U_{KK}(0, T) = 0
$$

$$
U_{KK}(H, T) = 0
$$

*The new term is due to the fact that the intrinsic value of the vanilla call is positive at the barrier
Up-and-Out Call Prices

In this illustration, the variables are: barrier $H = 110$, spot $S_0 = 90$, risk-free rate $r = 0.05$, and dividend rate $q = .02$. 
When the underlying stock price follows our single factor Markov process, the backward PIDE for pricing American puts is*:

\[
\frac{\partial p}{\partial t}(s, t) + \frac{\sigma^2(s, t)}{2} s^2 \frac{\partial^2 p}{\partial s^2}(s, t) + [r(t) - q(t)] s \frac{\partial p}{\partial s}(s, t) - r(t)p(s, t) \\
+ \int_{-\infty}^{+\infty} \left[ p(se^x, t) - p(s, t) - \frac{\partial p}{\partial s}(s, t)s(e^x - 1) \right] \nu(x, s, t) dx \\
+ 1_{s < s(t)} \left\{ r(t)K_0 - q(t)s - \int_{\ln(s(t)/s)}^{\infty} [p(se^x, t) - (K_0 - se^x)] \nu(x, s, t) dx \right\} = 0,
\]

*For the derivation and numerical solution of the PIDE in the special case of VG, see *Pricing American Options Under Variance Gamma* by Hirsa & Madan
The terminal condition is

$$p(s, T_0) = \max(K_0 - s, 0),$$

and the boundary conditions are

$$\lim_{s \to 0} p_{ss}(s, t) = \lim_{s \to \infty} p_{ss}(s, t) = 0.$$
Domain Extension in the Maturity Direction and Stationarity

To derive a forward FBP for American put values, we extend the domain to all $T \in [0, \bar{T}]$, keeping the strike fixed at $K_0$. Let $\pi(s, t; T)$ denote the American put value on this extended domain.

Now suppose stationarity, i.e. that $r(t)$, $q(t)$, $\sigma(s, t)$, and $\nu(x, s, t)$ are all independent of time $t$. Then theta is just the negative of the maturity derivative:

$$\frac{\partial}{\partial t} \pi(s, t; T) = -\frac{\partial}{\partial T} \pi(s, t; T)$$
The following relation holds in the extended domain:

\[- \frac{\partial \pi}{\partial T}(s, t; T) + \frac{\sigma^2(s)}{2} s^2 \frac{\partial^2 \pi}{\partial s^2}(s, t; T) + (r - q)s \frac{\partial \pi}{\partial s}(s, t; T) - r \pi(s, t; T)\]

\[+ \int_{-\infty}^{+\infty} \left[ \pi(se^x, t; T) - \pi(s, t; T) - \frac{\partial \pi}{\partial s}(s, t; T)s(e^x - 1) \right] \nu(x, s) dx\]

\[+ 1_{s < s(t; T)} \left\{ rK_0 - qs - \int_{\ln(s(t; T)/s)}^{\infty} [\pi(se^x, t; T) - (K_0 - se^x)] \nu(x, s) dx \right\} = 0,\]

We note that one can fix $t$ at $t_0$ and just solve the above problem in the $s, T$ plane if desired. In this case, the initial condition is

\[\pi(s, t_0; t_0) = \max(K_0 - s, 0),\]

and the boundary conditions are

\[\lim_{s \to 0} \pi_{ss}(s, t_0; T) = \lim_{s \to \infty} \pi_{ss}(s, t_0; T) = 0.\]
In the last slide, we assumed that the strike $K$ was fixed at $K_0$ and that $r(t)$, $q(t)$, $\sigma(s,t)$, and $\nu(x,s,t)$ are all independent of time $t$. To derive a new PIDE for American put values, we further extend the domain of the problem to all $K > 0$. We also restore the dependence on $t$.

The backward PIDE holding on the three dimensional domain holds on the larger four dimensional domain with $K_0$ replaced by all $K > 0$.

On this larger domain, let $s(t; T, K)$ be the function relating the exercise surface to $t$, $T$, and $K$. 
Now assume that the log price has independent increments, i.e. is an additive process. Hence, the local volatility $\sigma(s, t)$ and the jump arrival rate $\nu(x, s, t)$ are both independent of the stock price $s$.

Then for each fixed $t$ and $T$, the critical stock price $s(t; T, K)$ is proportional to $K$. It can be shown that for each fixed $s$, $t$, and $T$

$$s > s(t; T, K) \Rightarrow K < K(s, t; T)$$

where $K(s, t; T)$ relates the critical strike price to $s, t$ and $T$. By definition, the critical strike price is the lowest strike price at which an American put is exercised early for fixed $s, t, T$. 

\[ \text{Additivity} \]
The additivity of the log price process implies that the put value function $P(s,t;K,T)$ is linearly homogeneous in $s$ and $K$. By Euler’s theorem:

$$P(s,t;K,T) = s \frac{\partial}{\partial s} P(s,t;K,T) + K \frac{\partial}{\partial K} P(s,t;K,T).$$

Differentiation w.r.t. $s$ and $K$ establishes that:

$$s^2 \frac{\partial^2}{\partial s^2} P(s,t;K,T) = K^2 \frac{\partial^2}{\partial K^2} P(s,t;K,T).$$
After substitution and some straightforward calculations, we obtain the following hybrid relation:

\[
\frac{\partial P(s,t;K,T)}{\partial t} + \frac{\sigma^2(t)}{2} K^2 \frac{\partial^2 P(s,t;K,T)}{\partial K^2} - [r(t) - q(t)] \frac{K \partial P(s,t;K,T)}{\partial K} - q(t) P(s,t;K,T) \\
+ \int_{-\infty}^{+\infty} \left[ P(s,t;Ke^{-x},T) - P(s,t;K,T) - \frac{\partial P(s,t;K,T)}{\partial K} \right] K(e^{-x} - 1) e^{x \nu(x,t)} dx \\
+ 1_{K > \bar{K}(s,t;T)} \left\{ r(t) K - q(t) s - \int_{\ln(K/\bar{K}(s,t;T))}^{\infty} \left[ P(s,t;Ke^{-x},T) - (Ke^{-x} - s) \right] e^{x \nu(x,t)} dx \right\} = 0.
\]

We note that one can fix \( s \) and \( T \) at say \( s_0 \) and \( T_0 \) and just solve the PIDE in the \( K,t \) plane if desired.
In this case, the terminal condition is:

\[ P(s_0, T_0; K, T_0) = \max(K - s_0, 0), \]

and the boundary conditions are

\[ \lim_{K \to 0} P_{KK}(s_0, t; K, T_0) = \lim_{K \to \infty} P_{KK}(s_0, t; K, T_0) = 0. \]

Note that eliminating jumps reduces the PIDE to a PDE arising in the special case of the time-dependent Black Scholes model.
We now assume that we have both stationarity and additivity. In other words, the log price is a Lévy process and \( r(t), q(t), \sigma(s, t), \) and \( \nu(x, s, t) \) are all independent of both time \( t \) and the stock price \( s \).

Stationarity implies that the put value function \( P(s, t; K, T) \) depends on \( t \) and \( T \) only through \( T - t \). It again follows that:

\[
\frac{\partial}{\partial t} P(s, t; K, T) = -\frac{\partial}{\partial T} P(s, t; K, T)
\]
Substituting in the hybrid relation yields the following:

\[
\frac{\partial P(s, t; K, T)}{\partial T} - \frac{\sigma^2}{2} K^2 \frac{\partial^2 P(s, t; K, T)}{\partial K^2} + \frac{(r - q) K}{2} \frac{\partial P(s, t; K, T)}{\partial K} + q P(s, t; K, T) \\
- \int_{-\infty}^{+\infty} \left[ P(s, t; Ke^{-x}, T) - P(s, t; K, T) - \frac{\partial P(s, t; K, T)}{\partial K} Ke^{-x} (1 - e^{-x}) \right] e^x \nu(x) dx \\
- 1_{K > \bar{K}(s, t; T)} \left\{ r K - q s - \int_{\ln(K/\bar{K}(s, t; T))}^{\infty} \left[ P(s, t; Ke^{-x}, T) - (Ke^{-x} - s) \right] e^x \nu(x) dx \right\} = 0,
\]

where \( \nu(x) \) is the Lévy density.
We note that one can fix $s$ and $t$ at say $s_0$ and $t_0$ and just solve the forward PIDE in the $K,T$ plane if desired. In this case, the initial condition is:

$$P(s_0, t_0; K, t_0) = \max(K - s_0, 0),$$

and the boundary conditions are

$$\lim_{K \to 0} P_{KK}(s_0, t_0; K, T) = \lim_{K \to \infty} P_{KK}(s_0, t_0; K, T) = 0.$$
American Put Prices and Critical Strike Boundary

In this example, the variables are: spot $S_0=100$, risk-free rate $r = 0.05$, dividend rate $q = 0.02$, volatility $\tilde{\sigma} = 0.20$, and VG parameters $\sigma = 0.3$, $\nu = 0.25$, $\theta = -0.3$. 
Asymptotic Behavior of the Critical Strike Boundary
Future work and Conclusions

Forward PIDEs for Down-and-Out and Up-and-Out options when the underlying is a jump diffusion with a rebate

Derivation of a forward PIDE for double barrier options

Derivation of a forward PIDE for American options with local volatility surface

Draw your own *forward evolution* conclusions!