

# Forward Equations for American and Path-Dependent Options

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## Motivation

Propagating option prices in the maturity and strike directions

Enhances computational efficiency of calibration

Promotes computational efficiency in marking

We always work in a one factor Markov setting

We consider various additional process assumptions (e.g. continuity, stationarity, independent increments)

## Outline

Single Factor Markovian Stock Price Process

Review of the Dupire PDE for European options

Forward PDE for Down-and-Out Call Values

Forward PDE for Up-and-Out Call Values

Four PIDE's for American Put Values

## Markovian Stock Price Process

We always assume that under a risk neutral measure  $\mathbb{Q}$ , the stock price  $s_t$  solves the following stochastic differential equation:

$$ds_t = [r(t) - q(t)] s_{t-} dt + \sigma(s_{t-}, t) s_{t-} dW_t + \int_{-\infty}^{\infty} s_{t-} (e^x - 1) [\mu(dx, dt) - \nu(x, s_{t-}, t) dx dt],$$

for all  $t \in [0, \bar{T}]$ . Thus, the change in the stock price decomposes into the risk-neutral drift, the diffusion part, and the jump part.

The random measure  $\mu(dx, dt)$  counts the number of jumps of size  $x$  in the log price at time  $t$ .

## Markovian Stock Price Process (cont'd)

The Hunt Lévy density  $\{\nu(x, s, t), s > 0, x \in \mathbb{R}, t \in [0, \bar{T}]\}$  compensates the jump process

$$J_t \equiv \int_0^t \int_{-\infty}^{\infty} s_t (e^x - 1) \mu(dx, dt)$$

so that the last term is the increment of a  $\mathbb{Q}$  jump martingale.

Thus

$$\mathbb{E}_{\mathbb{Q}}[s_t | s_0] = s_0 e^{\int_0^t [r(u) - q(u)] du}.$$

## The Dupire PDE

Assuming no jumps, Dupire derives the following forward PDE for European call prices

$$\frac{\sigma^2(K, T)}{2} K^2 \frac{\partial^2 c}{\partial K^2} - [r(T) - q(T)] K \frac{\partial c}{\partial K} - q(T) c(K, T) = \frac{\partial c}{\partial T}$$

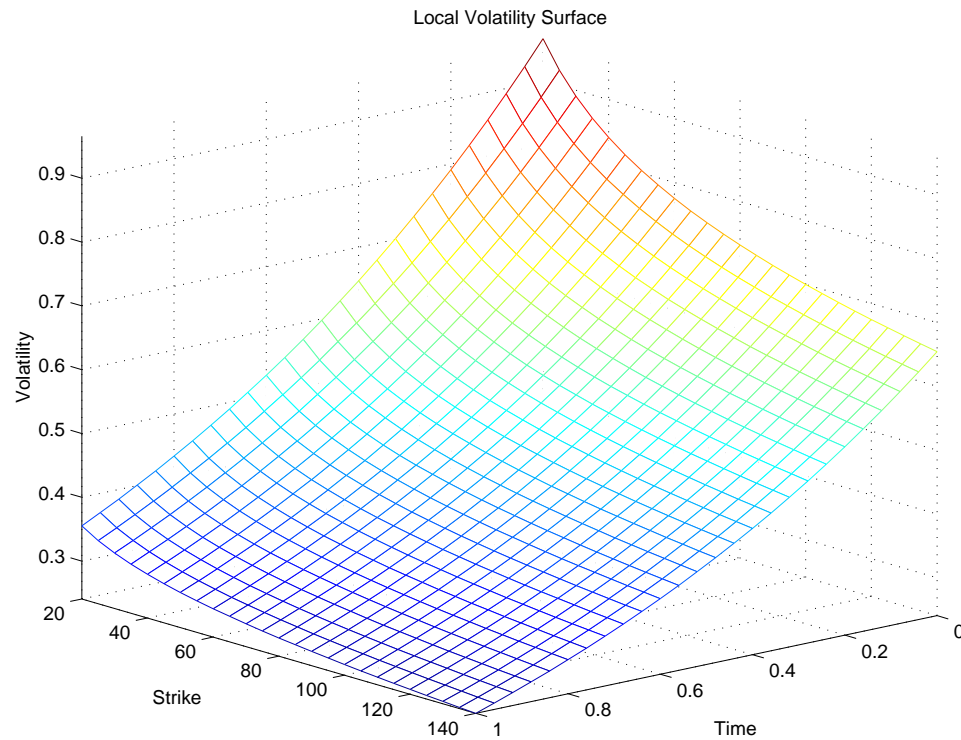
By having the local volatility surface,  $\sigma(K, T)$ , one can compute call prices,  $c(K, T)$ , for all strikes and maturities.

Or by having the call prices,  $c(K, T)$ , the local volatility surface,  $\sigma(K, T)$ , can be calculated.

## Local Volatility Surface

In our numerical examples, we consider the following local volatility surface

$$\sigma(K, T) = 0.7e^{-T} (100/K)^{0.2}$$



## Forward PDE for Down-and-Out Calls

Also assuming no jumps, the forward PDE for DOC with  $K > H$  is the same as the Dupire PDE

$$\frac{\sigma^2(K, T)}{2} K^2 \frac{\partial^2 D}{\partial K^2} - [r(T) - q(T)] K \frac{\partial D}{\partial K} - q(T) D = \frac{\partial D}{\partial T}$$

with initial condition

$$D(K, 0) = (S_0 - K)^+, \text{ for } K > H, \text{ and } S_0 > H.$$

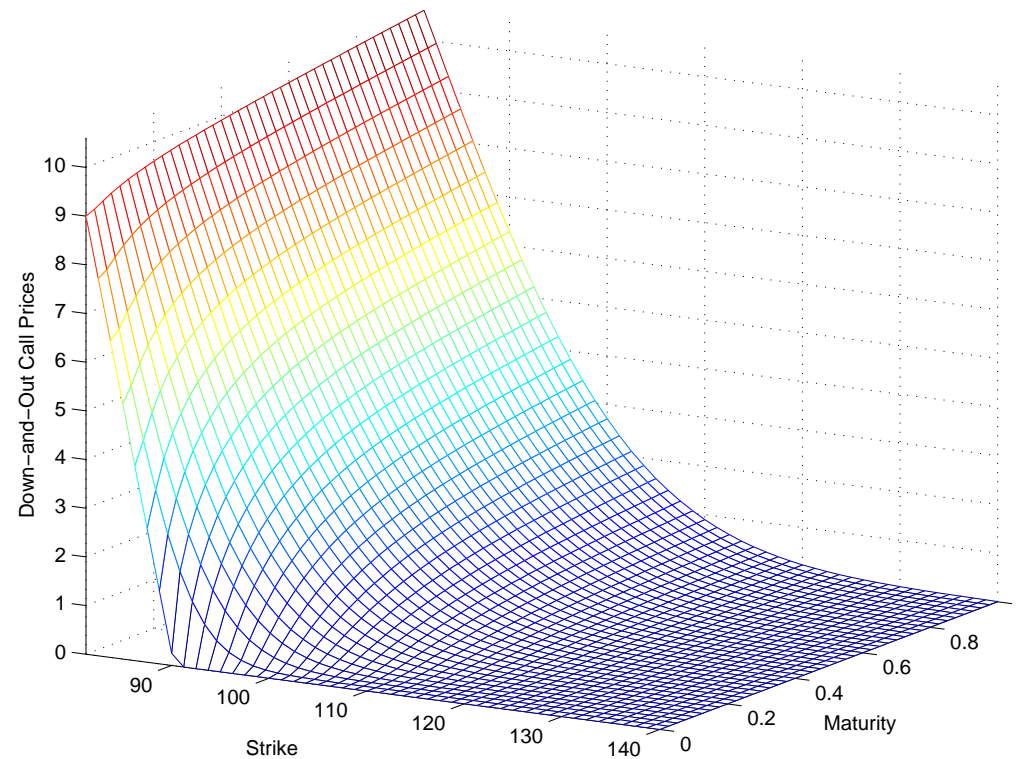
Boundary conditions are

$$D_{KK}(H, T) = 0$$

$$\lim_{K \rightarrow \infty} D_{KK}(K, T) = 0$$

## Down-and-Out Call Prices

In this illustration, the variables are: barrier  $H = 80$ , spot  $S_0 = 90$ , risk-free rate  $r = 0.05$ , and dividend rate  $q = .02$ .



## Forward PDE for Up-and-Out Calls

Under no jumps, the forward PDE for UOC with  $K < H$  is\*:

$$\frac{\sigma^2(K, T)}{2} K^2 \frac{\partial^2 U}{\partial K^2} - [r(T) - q(T)] K \frac{\partial U}{\partial K} - q(T) U = \frac{\partial U}{\partial T} + \left[ \frac{\sigma^2(H, T)}{2} H^2 \frac{\partial^3 U}{\partial K^3}(H, T) \right] (K - H)$$

with initial condition

$$U(K, 0) = (S_0 - K)^+, \text{ for } K < H, \text{ and } S_0 < H.$$

Boundary conditions are

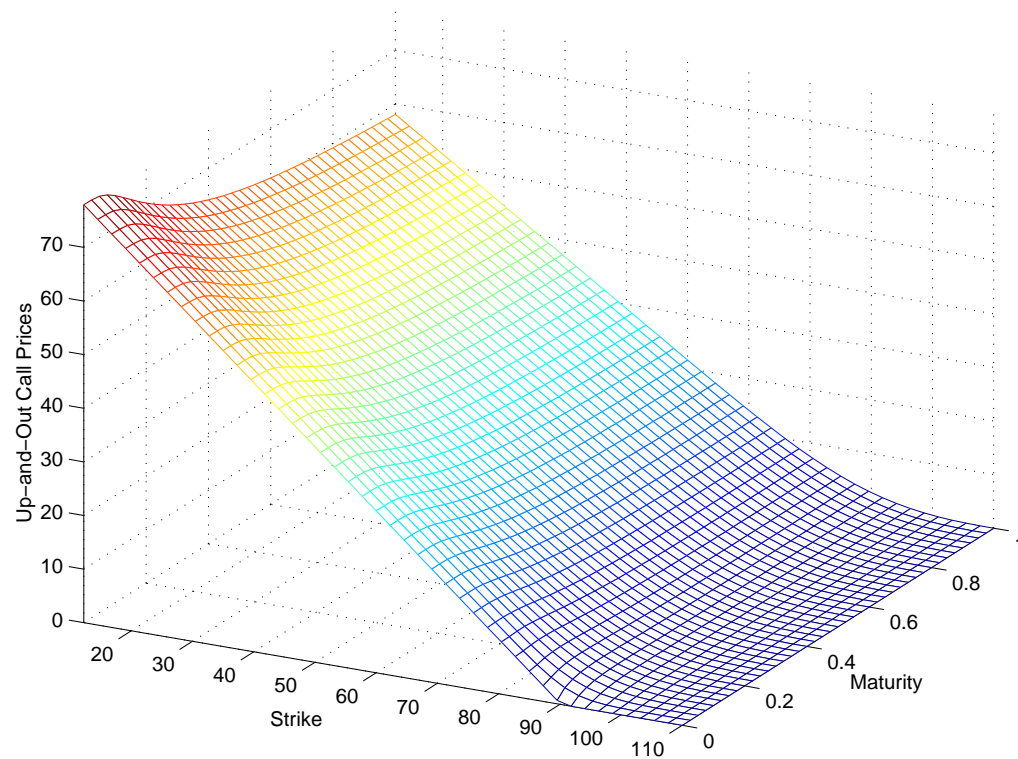
$$U_{KK}(0, T) = 0$$

$$U_{KK}(H, T) = 0$$

\*The new term is due to the fact that the intrinsic value of the vanilla call is positive at the barrier

## Up-and-Out Call Prices

In this illustration, the variables are: barrier  $H=110$ , spot  $S_0=90$ , risk-free rate  $r = 0.05$ , and dividend rate  $q = .02$ .



## The Backward Free Boundary Problem for American Puts

When the underlying stock price follows our single factor Markov process, the backward PIDE for pricing American puts is\*:

$$\begin{aligned} & \frac{\partial p}{\partial t}(s, t) + \frac{\sigma^2(s, t)}{2} s^2 \frac{\partial^2 p}{\partial s^2}(s, t) + [r(t) - q(t)] s \frac{\partial p}{\partial s}(s, t) - r(t)p(s, t) \\ & \quad + \int_{-\infty}^{+\infty} \left[ p(se^x, t) - p(s, t) - \frac{\partial p}{\partial s}(s, t) s(e^x - 1) \right] \nu(x, s, t) dx \\ & + \mathbf{1}_{s < s(t)} \left\{ r(t)K_0 - q(t)s - \int_{\ln(s(t)/s)}^{\infty} [p(se^x, t) - (K_0 - se^x)] \nu(x, s, t) dx \right\} = 0, \end{aligned}$$

\*For the derivation and numerical solution of the PIDE in the special case of VG, see *Pricing American Options Under Variance Gamma* by Hirta & Madan

## The Backward Free Boundary Problem for American Puts (cont'd)

The terminal condition is

$$p(s, T_0) = \max(K_0 - s, 0),$$

and the boundary conditions are

$$\lim_{s \rightarrow 0} p_{ss}(s, t) = \lim_{s \rightarrow \infty} p_{ss}(s, t) = 0.$$

## Domain Extension in the Maturity Direction and Stationarity

To derive a forward FBP for American put values, we extend the domain to all  $T \in [0, \bar{T}]$ , keeping the strike fixed at  $K_0$ . Let  $\pi(s, t; T)$  denote the American put value on this extended domain.

Now suppose stationarity, i.e. that  $r(t)$ ,  $q(t)$ ,  $\sigma(s, t)$ , and  $\nu(x, s, t)$  are all independent of time  $t$ . Then theta is just the negative of the maturity derivative:

$$\frac{\partial}{\partial t} \pi(s, t; T) = -\frac{\partial}{\partial T} \pi(s, t; T)$$

## Stationarity and Domain Extension in the Maturity Direction (cont'd)

The following relation holds in the extended domain:

$$\begin{aligned}
 & -\frac{\partial \pi}{\partial T}(s, t; T) + \frac{\sigma^2(s)}{2} s^2 \frac{\partial^2 \pi}{\partial s^2}(s, t; T) + (r - q) s \frac{\partial \pi}{\partial s}(s, t; T) - r \pi(s, t; T) \\
 & \quad + \int_{-\infty}^{+\infty} \left[ \pi(se^x, t; T) - \pi(s, t; T) - \frac{\partial \pi}{\partial s}(s, t; T) s(e^x - 1) \right] \nu(x, s) dx \\
 & + \mathbf{1}_{s < s(t; T)} \left\{ rK_0 - qs - \int_{\ln(s(t; T)/s)}^{\infty} [\pi(se^x, t; T) - (K_0 - se^x)] \nu(x, s) dx \right\} = 0,
 \end{aligned}$$

We note that one can fix  $t$  at  $t_0$  and just solve the above problem in the  $s, T$  plane if desired. In this case, the initial condition is

$$\pi(s, t_0; t_0) = \max(K_0 - s, 0),$$

and the boundary conditions are

$$\lim_{s \rightarrow 0} \pi_{ss}(s, t_0; T) = \lim_{s \rightarrow \infty} \pi_{ss}(s, t_0; T) = 0.$$

## Domain Extension in the Strike Direction

In the last slide, we assumed that the strike  $K$  was fixed at  $K_0$  and that  $r(t)$ ,  $q(t)$ ,  $\sigma(s, t)$ , and  $\nu(x, s, t)$  are all independent of time  $t$ . To derive a new PIDE for American put values, we further extend the domain of the problem to all  $K > 0$ . We also restore the dependence on  $t$ .

The backward PIDE holding on the three dimensional domain holds on the larger four dimensional domain with  $K_0$  replaced by all  $K > 0$ .

On this larger domain, let  $s(t; T, K)$  be the function relating the exercise surface to  $t$ ,  $T$ , and  $K$ .

## Additivity

Now assume that the log price has independent increments, i.e. is an additive process. Hence, the local volatility  $\sigma(s, t)$  and the jump arrival rate  $\nu(x, s, t)$  are both independent of the stock price  $s$ .

Then for each fixed  $t$  and  $T$ , the critical stock price  $s(t; T, K)$  is proportional to  $K$ . It can be shown that for each fixed  $s$ ,  $t$ , and  $T$

$$s > s(t; T, K) \Rightarrow K < K(s, t; T)$$

where  $K(s, t; T)$  relates the *critical strike price* to  $s, t$  and  $T$ . By definition, the critical strike price is the lowest strike price at which an American put is exercised early for fixed  $s, t, T$ .

## Additivity and Domain Extension in the Strike Direction (cont'd)

The additivity of the log price process implies that the put value function  $P(s, t; K, T)$  is linearly homogeneous in  $s$  and  $K$ . By Euler's theorem:

$$P(s, t; K, T) = s \frac{\partial}{\partial s} P(s, t; K, T) + K \frac{\partial}{\partial K} P(s, t; K, T).$$

Differentiation w.r.t.  $s$  and  $K$  establishes that:

$$s^2 \frac{\partial^2}{\partial s^2} P(s, t; K, T) = K^2 \frac{\partial^2}{\partial K^2} P(s, t; K, T).$$

## Additivity and Domain Extension in the Strike Direction (cont'd)

After substitution and some straightforward calculations, we obtain the following hybrid relation:

$$\begin{aligned}
 & \frac{\partial P(s, t; K, T)}{\partial t} + \frac{\sigma^2(t)}{2} K^2 \frac{\partial^2 P(s, t; K, T)}{\partial K^2} - [r(t) - q(t)] K \frac{\partial P(s, t; K, T)}{\partial K} - q(t) P(s, t; K, T) \\
 & + \int_{-\infty}^{+\infty} \left[ P(s, t; K e^{-x}, T) - P(s, t; K, T) - \frac{\partial P(s, t; K, T)}{\partial K} K (e^{-x} - 1) \right] e^x \nu(x, t) dx \\
 & + \mathbf{1}_{K > \bar{K}(s, t; T)} \left\{ r(t) K - q(t) s - \int_{\ln(K/\bar{K}(s, t; T))}^{\infty} [P(s, t; K e^{-x}, T) - (K e^{-x} - s)] e^x \nu(x, t) dx \right\} \\
 & = 0.
 \end{aligned}$$

We note that one can fix  $s$  and  $T$  at say  $s_0$  and  $T_0$  and just solve the PIDE in the  $K, t$  plane if desired.

## Additivity and Domain Extension in the Strike Direction (cont'd)

In this case, the terminal condition is:

$$P(s_0, T_0; K, T_0) = \max(K - s_0, 0),$$

and the boundary conditions are

$$\lim_{K \rightarrow 0} P_{KK}(s_0, t; K, T_0) = \lim_{K \rightarrow \infty} P_{KK}(s_0, t; K, T_0) = 0.$$

Note that eliminating jumps reduces the PIDE to a PDE arising in the special case of the time-dependent Black Scholes model.

## The Forward Free Boundary Problem

We now assume that we have both stationarity and additivity. In other words, the log price is a Lévy process and  $r(t)$ ,  $q(t)$ ,  $\sigma(s, t)$ , and  $\nu(x, s, t)$  are all independent of both time  $t$  and the stock price  $s$ .

Stationarity implies that the put value function  $P(s, t; K, T)$  depends on  $t$  and  $T$  only through  $T - t$ . It again follows that:

$$\frac{\partial}{\partial t} P(s, t; K, T) = -\frac{\partial}{\partial T} P(s, t; K, T)$$

## The Forward Free Boundary Problem (cont'd)

Substituting in the hybrid relation yields the following:

$$\begin{aligned} & \frac{\partial P(s, t; K, T)}{\partial T} - \frac{\sigma^2}{2} K^2 \frac{\partial^2 P(s, t; K, T)}{\partial K^2} + (r - q) K \frac{\partial P(s, t; K, T)}{\partial K} + q P(s, t; K, T) \\ & - \int_{-\infty}^{+\infty} \left[ P(s, t; K e^{-x}, T) - P(s, t; K, T) - \frac{\partial P(s, t; K, T)}{\partial K} K (e^{-x} - 1) \right] e^x \nu(x) dx \\ & - \mathbf{1}_{K > \bar{K}(s, t; T)} \left\{ rK - qs - \int_{\ln(K/\bar{K}(s, t; T))}^{\infty} [P(s, t; K e^{-x}, T) - (K e^{-x} - s)] e^x \nu(x) dx \right\} = 0, \end{aligned}$$

where  $\nu(x)$  is the *Lévy density*.

## The Forward Free Boundary Problem (cont'd)

We note that one can fix  $s$  and  $t$  at say  $s_0$  and  $t_0$  and just solve the forward PIDE in the  $K, T$  plane if desired. In this case, the initial condition is:

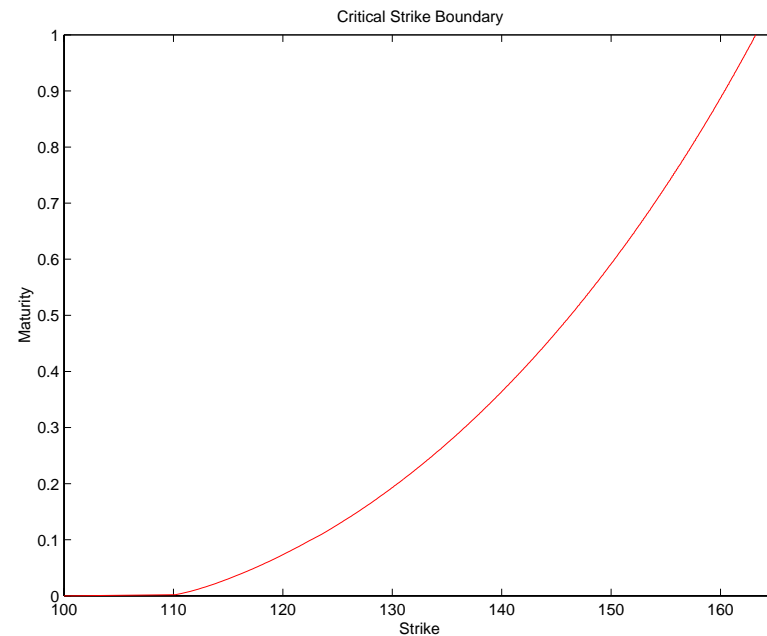
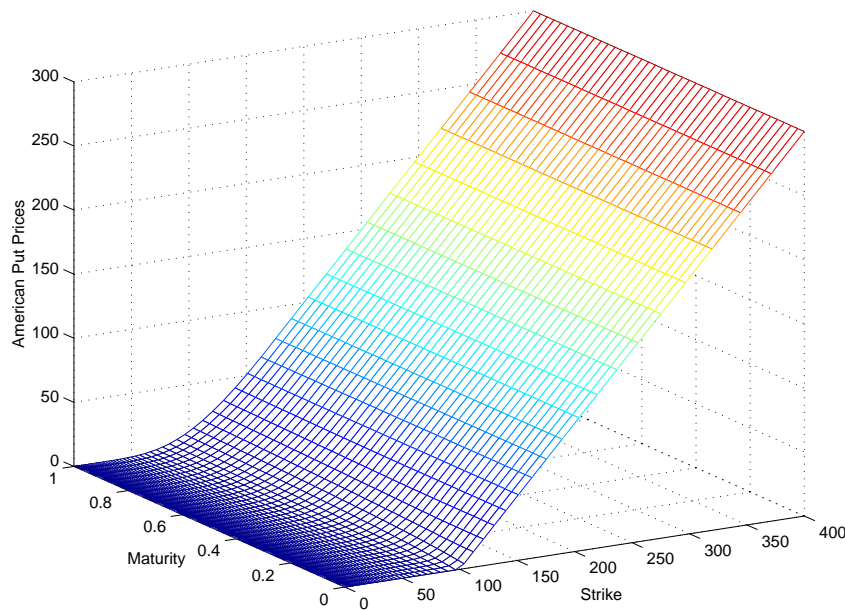
$$P(s_0, t_0; K, t_0) = \max(K - s_0, 0),$$

and the boundary conditions are

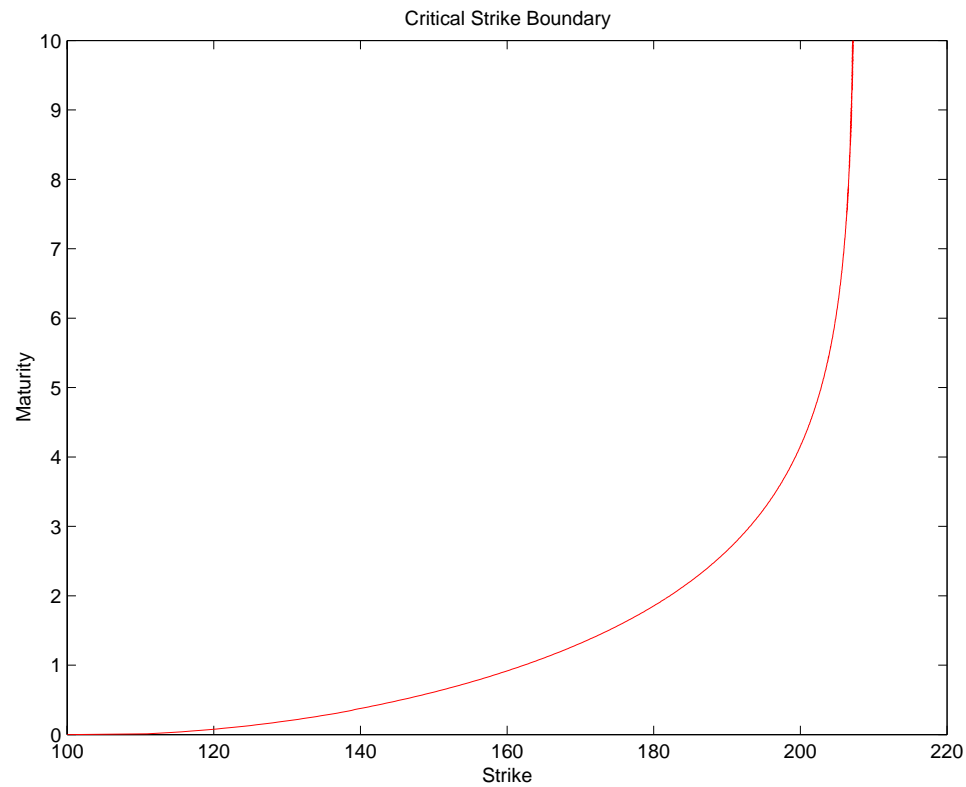
$$\lim_{K \rightarrow 0} P_{KK}(s_0, t_0; K, T) = \lim_{K \rightarrow \infty} P_{KK}(s_0, t_0; K, T) = 0.$$

## American Put Prices and Critical Strike Boundary

In this example, the variables are: spot  $S_0=100$ , risk-free rate  $r = 0.05$ , dividend rate  $q = .02$ , volatility  $\hat{\sigma} = .20$ , and VG parameters  $\sigma = .3$ ,  $\nu = .25$ ,  $\theta = -.3$ .



# Asymptotic Behavior of the Critical Strike Boundary



## Future work and Conclusions

Forward PIDEs for Down-and-Out and Up-and-Out options when the underlying is a jump diffusion with a rebate

Derivation of a forward PIDE for double barrier options

Derivation of a forward PIDE for American options with local volatility surface

Draw your own *forward evolution* conclusions!